

QUASIEQUATIONAL THEORIES OF FLAT ALGEBRAS

J. JEŽEK, Praha, M. MARÓTI and R. MCKENZIE, Nashville

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Abstract. We prove that finite flat digraph algebras and, more generally, finite compatible flat algebras satisfying a certain condition are finitely q -based (possess a finite basis for their quasiequations). We also exhibit an example of a twelve-element compatible flat algebra that is not finitely q -based.

Keywords: quasiequation, flat algebra

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1. INTRODUCTION

For a finite directed graph (V, E) one can define an algebra with the underlying set $V \cup E \cup \{0\}$, one constant 0 and two binary operations \wedge, \cdot in this way: $a \wedge a = a$ and $a \wedge b = 0$ whenever $a \neq b$; $ab = c$ whenever $a, c \in V$ and $b = (a, c) \in E$; $ab = 0$ in all other cases. Algebras obtained from finite directed graphs in this way are called finite *flat digraph algebras*. One particular six-element flat digraph algebra (inherently non-finitely based for equations) played a significant role in the proof of undecidability of the existence of a finite basis for the equational theory of a finite algebra ([2], [3] and [4]). It was plausible to expect that it could serve a similar purpose in an attempt to prove that also the existence of a finite basis for the quasiequations of a finite algebra is undecidable. However, in this paper we are going to show that all finite flat digraph algebras are finitely q -based (possess a finite basis for their uasiequations), which makes them unsuitable. We will investigate a more

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general class of finite compatible flat algebras, in which (under a modest assumption on the signature) every algebra can be embedded both into a finitely q -based and into a non-finitely q -based algebra.

For the terminology and basic concepts of universal algebra the reader is referred to the monograph [5]. For the literature on quasiequational theories see, e.g., [1] and [6].

2. COMPATIBLE 0-SEMILATTICE ALGEBRAS

Let σ be a finite signature containing (among other symbols) a binary symbol \wedge (the meet) and a nullary symbol 0 .

By a *0-semilattice algebra* we mean a σ -algebra satisfying the equations

- (1) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$,
- (2) $x \wedge y = y \wedge x$,
- (3) $x \wedge x = x$,
- (4) $f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) = 0$ for every n -ary operation f of σ and every $i \in \{1, \dots, n\}$.

A 0-semilattice algebra is said to be *compatible* if it satisfies the equations

- (5) $f(z_1, \dots, z_{i-1}, x \wedge y, z_{i+1}, \dots, z_n) = f(z_1, \dots, z_{i-1}, x, z_{i+1}, \dots, z_n) \wedge f(z_1, \dots, z_{i-1}, y, z_{i+1}, \dots, z_n)$ for every n -ary operation f of σ and every $i \in \{1, \dots, n\}$.

So, the class of compatible 0-semilattice σ -algebras is a variety.

For a variable x , *basic x -terms* of depth n are defined as follows. The term x is the only basic x -term of depth 0. For $n > 0$, basic x -terms of depth n are the terms $f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)$ such that f is an n -ary operation of σ , $1 \leq i \leq n$, t is a basic x -term of depth $n - 1$ and x_1, \dots are variables different from x . A basic x -term t will be usually denoted by $t(x)$, in which case $t(u)$ stands for the term resulting from t by substituting u for x (where u is any term).

For a σ -algebra B and a basic x -term t of depth n , any interpretation of the variables different from x by elements of B gives rise to a unary polynomial of B . The unary polynomials obtained in this way will be called the *basic polynomials* of B of depth n .

Lemma 2.1. *Let A be a compatible 0-semilattice algebra. Then $p(a \wedge b) = p(a) \wedge p(b)$ for all basic polynomials p of A and all elements $a, b \in A$.*

Proof. It is easy. (Observe that the statement is not true for all unary polynomials p .) □

Lemma 2.2. *Let A be a compatible 0-semilattice algebra and F be a proper filter of A (i.e., a nonempty subset closed under meet, not containing 0 and such that $b \in F$ whenever $a \in F$ and $a \leq b$). Then for every basic polynomial p of A , $p^{-1}(F)$ is either empty or a proper filter of A .*

Proof. It follows easily from Lemma 2.1. □

By a *flat* algebra we mean a 0-semilattice algebra A such that $a \wedge b = 0$ for all pairs of distinct elements $a, b \in A$. Observe that a flat algebra is monotonic, i.e., satisfies $x \leq y \rightarrow f(z_1, \dots, z_{i-1}, x, z_{i+1}, \dots, z_n) \leq f(z_1, \dots, z_{i-1}, y, z_{i+1}, \dots, z_n)$ for every n -ary operation f of σ and every $i \in \{1, \dots, n\}$.

One can easily see that a flat algebra is compatible if and only if

(5') $f(c_1, \dots, c_{i-1}, a, c_{i+1}, \dots, c_n) = f(c_1, \dots, c_{i-1}, b, c_{i+1}, \dots, c_n) \neq 0$ implies $a = b$ for every n -ary operation f of σ and every $i \in \{1, \dots, n\}$.

For every partial algebra G of a signature τ not containing \wedge and 0 we can define a flat $\tau \cup \{\wedge, 0\}$ -algebra A , called the *flat algebra over G* , by $A = G \cup \{0\}$, $f(a_1, \dots, a_n) = a$ in A whenever $f(a_1, \dots, a_n) = a$ in G , and $f(a_1, \dots, a_n) = 0$ otherwise. This flat algebra is not necessarily compatible. For example, if G is a finite groupoid, then the flat algebra over G is compatible if and only if G is a quasigroup. Finite flat digraph algebras are all compatible.

Observation 2.3. *For every finite compatible flat algebra A there exists a first-order sentence Φ such that the finite models of Φ are precisely the finite algebras belonging to the quasivariety generated by A .*

Proof. Put $K = |A|$. It is easy to see that the following are equivalent for a finite compatible 0-semilattice algebra B :

- (e1) B belongs to the quasivariety generated by A ;
- (e2) every two elements b_0, b_1 of B such that $b_0 < b_1$ can be separated by a congruence of B , the factor by which is isomorphic to a subalgebra of A ;
- (e3) for every $b_0, b_1 \in B$ with $b_0 < b_1$ there exist elements $c_1, \dots, c_r \in B$ for some $r < K$ such that the principal filters F_1, \dots, F_r generated by c_1, \dots, c_r are pairwise disjoint, $b_1 \in F_1$, b_0 belongs to the complement O of $F_1 \cup \dots \cup F_r$ in B , the equivalence R with blocks O, F_1, \dots, F_r is a congruence of B and the factor B/R is isomorphic to a subalgebra of A .

Clearly, the condition (e3) can be rewritten as a first-order sentence. □

3. THE QUASIVARIETY Q'_A

In the following let A be a finite compatible, flat algebra. Put $K = |A|$.

Denote by Q'_A the quasivariety determined by the equations (1)–(5) and the following quasiequations:

- (6) $x_0 \leq x_1 \ \& \ t(x) \geq x_1 \ \& \ u(x) \geq x_1 \ \& \ t(y) \geq x_1 \ \& \ u(y) \wedge x_1 \leq x_0 \rightarrow x_0 = x_1$ for every pair of basic x -terms t, u of depth $\leq K$;
- (7) $x_0 \leq x_1 \ \& \ H_{t_1, \dots, t_K} \rightarrow x_0 = x_1$ for every K -tuple of basic x -terms t_1, \dots, t_K of depth $\leq K$, where H_{t_1, \dots, t_K} is the conjunction of the following equations:

$$\begin{aligned} t_i(x_i) &\geq x_1 \quad (i = 1, \dots, K), \\ t_i(x_j) \wedge x_1 &\leq x_0 \quad (i, j = 1, \dots, K \text{ and } i \neq j). \end{aligned}$$

Lemma 3.1. Q'_A is a finitely q -based quasivariety containing A .

Proof. The set of quasiequations (6)–(7) is essentially finite, as it contains only finitely many quasiequations that differ by not only renaming their variables. Consequently, Q'_A is finitely q -based. It remains to prove that (6) and (7) are satisfied in A . Suppose that (6) fails in A by some interpretation $v \mapsto v'$ of variables. Then $x'_0 < x'_1$, so that $x'_0 = 0$; now $t(x') \geq x'_1$ implies $t(x') = x'_1$. Similarly we get $u(x') = x'_1$ and $t(y') = x'_1$. But A satisfies (5'), so $t(x') = t(y') \neq 0$ implies $x' = y'$; hence $x'_1 = u(x') \wedge x'_1 = u(y') \wedge x'_1 = 0$, a contradiction. Using the fact that A cannot contain K nonzero, pairwise distinct elements, one can similarly prove that A satisfies the quasiequations (7). □

Lemma 3.2. Let $B \in Q'(A)$ and $b_0, b_1 \in B$ be two elements such that $b_1 \not\leq b_0$; let F be a maximal filter of B such that $b_1 \in F$ and $b_0 \notin F$. For any two basic polynomials p, q of B of depth $\leq K$, the sets $p^{-1}(F)$ and $q^{-1}(F)$ are either disjoint or equal.

Proof. The two basic polynomials p and q correspond to two basic terms t and u of depth $\leq K$. Suppose that there exist elements x', y' such that $p(x') \in F$, $p(y') \in F$, $q(x') \in F$ and $q(y') \notin F$. It follows from the maximality of F that there exists an element $e \in F$ with $q(y') \wedge e \leq b_0$. Put $x'_1 = p(x') \wedge p(y') \wedge q(x') \wedge e$, so that $x'_1 \in F$. Put $x'_0 = b_0 \wedge x'_1$, so that $x'_0 < x'_1$. But the quasiequation (e6) interpreted by $x \mapsto x', y \mapsto y', x_0 \mapsto x'_0, x_1 \mapsto x'_1$ gives $x'_0 = x'_1$, a contradiction. □

Lemma 3.3. *Let $B \in Q'(A)$ and $b_0, b_1 \in B$ be two elements such that $b_1 \not\leq b_0$; let F be a maximal filter of B such that $b_1 \in F$ and $b_0 \notin F$. There are at most $K - 1$ nonempty subsets of B that can be expressed as $q^{-1}(F)$ for a basic polynomial q of B , and they can be arranged into a sequence F_1, \dots, F_r (for some $r < K$) in such a way that $F_1 = F$ and for every $i \in \{2, \dots, r\}$ there are an index $j \in \{1, \dots, i-1\}$ and a basic polynomial p_i of B of depth 1 with $F_i = p_i^{-1}(F_j)$. The collection F_1, \dots, F_r , together with the complement of their union, is a partition and the corresponding equivalence is a congruence of B .*

Proof. Let us define a (finite or infinite) sequence $F_1, p_1, F_2, p_2, \dots$ of filters F_i and basic polynomials p_i of depth ≤ 1 by induction in this way: $F_1 = F$ and p_1 is the identity on B ; if F_i, p_i have been defined and if there exist an element $a \notin F_1 \cup \dots \cup F_i$ and a basic polynomial p of depth 1 such that $p(a) \in F_j$ for some $j \leq i$, take one such pair a, p and put $p_{i+1} = p$ and $F_{i+1} = p_{i+1}^{-1}(F_j)$; if there is no such pair a, p , the sequence already constructed will have no continuation. Clearly (by induction on i), $F_i = q_i^{-1}(F)$ for a basic polynomial q_i of B of depth $< i$. The sets F_i are pairwise disjoint filters according to Lemmas 2.2 and 3.2.

Suppose that the sequence has at least K members F_1, \dots, F_K . For any $i = 1, \dots, K$ take an element $x'_i \in F_i$, so that $q_i(x'_i) \in F$. For every $i \neq j$ we have $x'_j \notin F_i$, i.e., $q_i(x'_j) \notin F$, so that there exists an element $e_{i,j} \in F$ with $q_i(x'_j) \wedge e_{i,j} \leq b_0$. There is an element $x'_1 \in F$ such that $x'_1 \leq q_i(x'_i)$ for all i and $x'_1 \leq e_{i,j}$ for all $i \neq j$. Put $x'_0 = b_0 \wedge x'_1$, so that $x'_0 < x'_1$. But the quasiequation (e7), interpreted in the obvious way, says that $x'_0 = x'_1$, a contradiction.

So, the sequence F_1, p_1, \dots ends with F_r, p_r for some $r \leq K - 1$. Clearly, every subset of the form $q^{-1}(F)$ for a basic polynomial q can be found among F_1, \dots, F_r . Put $O = B - (F_1 \cup \dots \cup F_r)$, so that $0 \in O$ and F_1, \dots, F_r, O is a partition of B . It remains to prove that the corresponding equivalence is a congruence of B .

Suppose that there exist an n -ary operation f in σ and an n -tuple a_1, \dots, a_n of elements of B such that $a_j \in O$ for some j but $f(a_1, \dots, a_n) \in F_i$ for some i . Then $p(a_j) \in F_i$ where $p(x) = f(a_1, \dots, a_{j-1}, x, a_{j+1}, \dots, a_n)$ is a basic polynomial of depth 1 and $a_j \notin F_1 \cup \dots \cup F_r$, so that $(q_i p)^{-1}(F)$ is nonempty and different from all F_1, \dots, F_r , a contradiction. We have proved that if at least one of the elements a_1, \dots, a_n belongs to O , then $f(a_1, \dots, a_n) \in O$.

Now it remains to show that if f is n -ary, $f(a_1, \dots, a_n) \in F_j$ and $a_i, a'_i \in F_k$ for some $j, k \in \{1, \dots, r\}$ and $i \in \{1, \dots, n\}$, then $f(a_1, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_n) \in F_j$. Put $q(x) = q_j(f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n))$, so that q is a basic polynomial of B of depth at most K . We have $q(a_i) \in F$ and $q_k(a_i) \in F$, so that $q^{-1}(F) = q_k^{-1}(F)$. Since a'_i belongs to this set, we get $q(a'_i) \in F$, i.e., $f(a_1, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_n) \in F_j$. \square

Theorem 3.4. *Let A be a finite compatible, flat algebra with K elements. Then Q'_A is a finitely q -based and locally finite quasivariety containing A ; every algebra in Q'_A is isomorphic to a subdirect product of algebras of cardinality at most K . Consequently, A is not inherently nonfinitely q -based.*

Proof. Let $B \in Q'_A$. For every pair b_0, b_1 of distinct elements of B (we can assume that $b_1 \not\leq b_0$) there exists a maximal filter of B containing b_1 but not b_0 , so that by Lemma 3.3 these two elements can be separated by a congruence with at most K blocks. It follows that every algebra from B is isomorphic to a subdirect product of algebras of cardinality at most K . Thus Q'_A is contained in a finitely generated variety and hence it is locally finite. According to Lemma 3.1, Q'_A is finitely q -based and contains A . \square

4. FINITELY q -BASED COMPATIBLE FLAT ALGEBRAS

Let A be a finite compatible flat algebra. By a *segment* of A we will mean a nonempty subset of A , the elements of which can be arranged into a finite sequence $0, c_1, \dots, c_r$ in such a way that $c_1 \neq 0$ and for every $i = 2, \dots, r$ there exists a basic polynomial p of A of depth 1 with $p(c_i) = c_j$ for some $j \in \{1, \dots, i-1\}$.

Let S be a segment of A . The algebra obtained from S , considered as a partial subalgebra of A , by setting all the undefined operations to 0 will be called the *0-completion* of S .

Let S be a segment of A and S' be the subalgebra of A generated by S . The segment S is said to be *regular* if the equivalence on S' with the only non-singleton block $\{0\} \cup (S' - S)$ is a congruence of S' . In that case, the factor of S' by this congruence is isomorphic to the 0-completion of S .

Theorem 4.1. *Let A be a finite compatible flat algebra such that the 0-completion of every regular segment of A belongs to the quasivariety generated by A . Then A is finitely q -based.*

Proof. Denote by Q''_A the subquasivariety of Q'_A determined by the quasi-equations (1)–(7) and all quasiequations in at most K variables that are satisfied in A . (Here $K = |A|$.) Since Q'_A is locally finite by Theorem 3.4, Q''_A is locally finite. Since only finitely many equations are needed to reduce the terms in at most K variables to a finite set T_0 of such terms, and then quasiequations in at most K variables correspond to subsets of T_0^2 with distinguished elements, Q''_A is finitely q -based. Of course, $A \in Q''_A$. We are going to prove that Q''_A is the quasivariety generated by A . It is sufficient to show that every finite algebra from Q''_A belongs to the quasivariety generated by A .

Let B be a finite algebra from Q''_A ; let $b_0, b_1 \in B$ be such that $b_1 \not\leq b_0$. By 3.3 there is a congruence with at most K blocks O, F_1, \dots, F_r , yielding a quotient algebra C , such that F_1, \dots, F_r are filters (now they are principal filters), $F_1 = F$, $b_1 \in F_1$, $b_0 \in O$, and for every $i \in \{2, \dots, r\}$ there exist an index $j < i$ and a basic polynomial p_i of length 1 with $F_i = p_i^{-1}(F_j)$. But all the coefficients occurring in p_i belong to $F_1 \cup \dots \cup F_r$, so there exists a basic x -term $u_i(x, x_1, \dots, x_r)$ of depth 1 such that $u_i(F_i, F_1, \dots, F_r) \subseteq F_j$. Now we can combine these terms u_i together to obtain, for each i , a basic x -term $t_i(x, x_1, \dots, x_r)$ such that $t_i(F_i, F_1, \dots, F_r) \subseteq F$, i.e., $t_i^C(F_i, F_1, \dots, F_r) = F_1$. (We take $t_1 = x$.) For any term u denote by $t_i(u)$ the term obtained from t_i by replacing the only occurrence of x with u . Now consider the quasiequation

$$x_0 \leq x_1 \ \& \ D \rightarrow x_0 = x_1$$

where D is the conjunction of all these equations:

- (i) $t_i(x_i) \geq x_1$, for any $i = 1, \dots, r$;
- (ii) $t_i(x_j) \wedge x_1 \leq x_0$, for any $i, j \in \{1, \dots, r\}$ with $i \neq j$;
- (iii) $t_i(f(x_{i_1}, \dots, x_{i_n})) \geq x_1$, for any n -ary operation f of σ and any i, i_1, \dots, i_n with $f^C(F_{i_1}, \dots, F_{i_n}) = F_i$;
- (iv) $t_i(u) \wedge x_1 \leq x_0$, for any $i = 1, \dots, r$ and any term u in variables x_1, \dots, x_r containing a subterm $f(x_{i_1}, \dots, x_{i_n})$ with $f^C(F_{i_1}, \dots, F_{i_n}) = O$ (it is possible to consider only finitely many such terms u).

Clearly, this quasiequation fails in B ; since it is a quasiequation in at most K variables x_0, \dots, x_r , it must fail in A by some elements a_0, a_1, \dots, a_r . But then the subset $\{a_0, a_1, \dots, a_r\}$ is a regular segment of A , and the 0-completion of this subset is isomorphic to C . Since C belongs to the quasivariety generated by A , the elements b_0, b_1 were separated by a congruence, the factor by which belongs to the quasivariety. \square

Corollary 4.2. *Every finite flat digraph algebra is finitely q -based.*

Proof. In this case, all segments are subalgebras. \square

Corollary 4.3. *The flat algebra over any finite quasigroup (considered as a groupoid) is finitely q -based.*

Proof. In this case, all regular segments are subalgebras. \square

Corollary 4.4. *If σ is the signature containing only one unary symbol in addition to \wedge and 0, then every finite compatible flat σ -algebra is finitely q -based.*

Proof. In this case, the 0-completion of every segment is isomorphic to a subalgebra. \square

5. THE EMBEDDING THEOREM

Theorem 5.1. *Let σ be a finite signature containing, in addition to \wedge and 0 , at least two unary symbols f and g (and, possibly, some other operation symbols). Then every finite compatible flat σ -algebra can be embedded into two finite compatible flat σ -algebras, one finitely q -based and the other one not finitely q -based.*

Proof. Let G be a finite compatible flat algebra.

Denote by S_1, \dots, S_r all the segments of G . (It would be sufficient to take just those with the 0 -completions not belonging to the quasivariety generated by G .) For every $i = 1, \dots, r$ let us take an isomorphic copy T_i of the partial algebra $S_i - \{0\}$, in such a way that the sets G, T_1, \dots, T_r are pairwise disjoint. Denote by G' the flat algebra with the underlying set $G \cup T_1 \cup \dots \cup T_r$, with the operations evaluated to 0 in all cases except when needed to define them in such a way that G is a subalgebra and T_i are partial subalgebras. It follows from Theorem 4.1 that G' is finitely q -based.

Next we are going to construct a non-finitely q -based extension of G . Let us take one fixed positive integer k such that $k \geq 2$ and there is no sequence u_0, u_1, \dots, u_k of pairwise distinct elements of $G - \{0\}$ such that $g(u_{i-1}) = u_i$ for $i = 1, \dots, k$. Denote by A the flat algebra, with G as a subalgebra, containing $k + 10$ additional elements $u_0, u_1, \dots, u_k, a, b, c, v_2, a_2, b_2, v_3, a_3, c_3$ with all operations not inside G evaluated to 0 except for

$$\begin{aligned} g(u_{i-1}) &= u_i \quad \text{for } i = 1, \dots, k, \\ f(u_0) &= a, \quad f(a) = b, \quad g(a) = c, \\ f(v_2) &= a_2, \quad f(a_2) = b_2, \quad f(v_3) = a_3, \quad g(a_3) = b_3. \end{aligned}$$

(Fig. 1, in which the elements not belonging to G are pictured for $k = 2$, may help to understand this definition.)

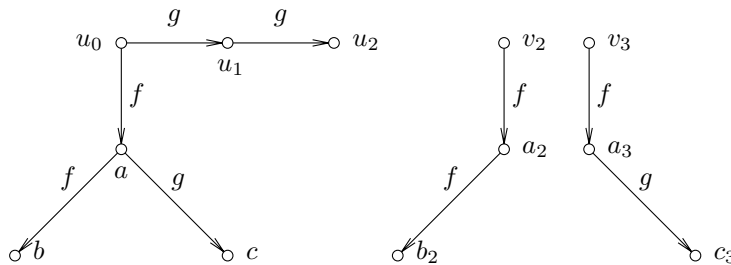


Fig. 1

Denote by Q the quasivariety generated by A . A σ -algebra B belongs to Q if and only if every two distinct elements of B can be separated by a homomorphism of B into A .

For every positive integer n let A_n be the σ -algebra with elements $0, u_0, \dots, u_k, \alpha_{i,j}, \beta_i, \gamma_j$ ($0 \leq i \leq n, 0 \leq j \leq n-1, i-1 \leq j \leq i$) and with operations defined in this way: A_n is a semilattice with the only comparabilities $0 < u_i$ ($i = 0, \dots, k$), $0 < \beta_n < \beta_{n-1} < \dots < \beta_0$, $0 < \gamma_{n-1} < \gamma_{n-2} < \dots < \gamma_0$, $0 < \alpha_{n,n-1} < \alpha_{n-1,n-1} < \alpha_{n-1,n-2} < \dots < \alpha_{1,0} < \alpha_{0,0}$; the other operations evaluate to 0 except that $g(u_{i-1}) = u_i$ ($i = 1, \dots, k$), $f(u_0) = \alpha_{0,0}$, $f(\alpha_{i,j}) = \beta_i$, $g(\alpha_{i,j}) = \gamma_j$. (Fig. 2, in which the situation is illustrated for $k = 2$ and $n = 3$, may help to understand this definition. In the picture lines with arrows indicate unary operations, while the other lines represent coverings but the covers between 0 and the elements u_i are not indicated.)

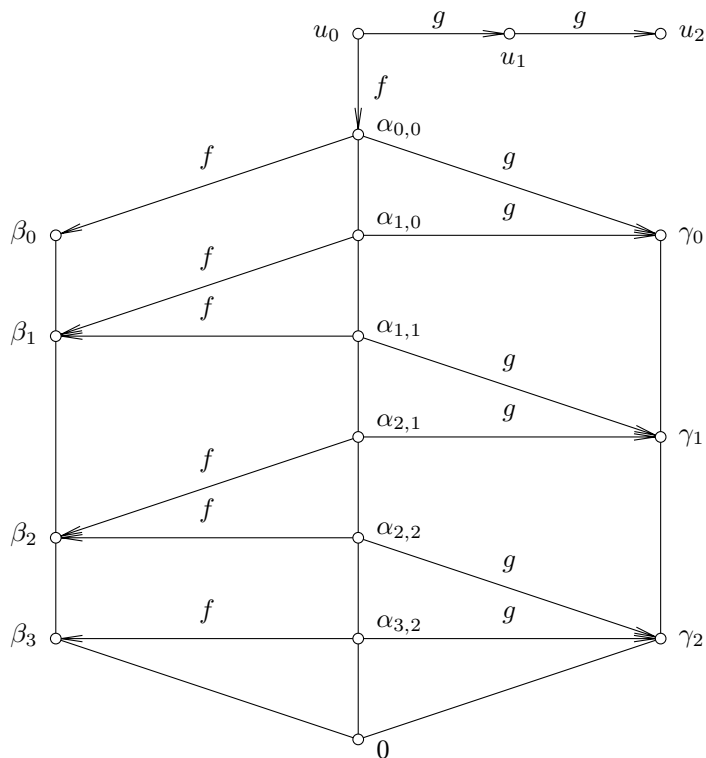


Fig. 2

Denote by r_n the equivalence on A_n with the only non-singleton block $\{0, \beta_n\}$. Clearly, r_n is a congruence of A_n . Denote the factor A_n/r_n by B_n . For $a \in A_n - \{0, \beta_n\}$, the element a/r_n will be identified with a .

Suppose that there exists a homomorphism $H: B_n \rightarrow A$ such that $H(u_k) \neq H(0/r_n)$, i.e., $H(u_k) \neq 0$. Since $g^k(u_0) = u_k$ in B_n and there is no other element e

in A with $g^k(e) \neq 0$ and $g^{k+1}(e) = 0$ other than u_0 , we get $H(u_0) = u_0$ and then $H(u_i) = H(g^i(u_0)) = g^i(H(u_0)) = g^i(u_0) = u_i$ for all i . Now $H(\alpha_{0,0}) = H(f(u_0)) = f(H(u_0)) = f(u_0) = a$. Consequently, $H(\beta_0) = b$ and $H(\gamma_0) = c$. Since $g(\alpha_{1,0}) = \gamma_0$ and a is the only element of A with $g(a) = c$, it follows that $H(\alpha_{1,0}) = a$. If $H(\alpha_{i,i-1}) = a$ for some $i < n$, then using f in a similar way we can show that $H(\alpha_{i,i}) = a$, and then using g to show that $H(\alpha_{i+1,i}) = a$. By induction we get $H(\alpha_{n,n-1}) = a$. But then $H(0/r_n) = H(\beta_n/r_n) = H(f(\alpha_{n,n-1})) = f(a) = b$, a contradiction.

Since the element u_k cannot be separated from $0/r_n$ by a homomorphism of B_n into A , we conclude that B_n does not belong to Q .

Let $\alpha_{m,m'}$ be an element of B_n such that $0 < m < n$. Clearly, the set $C = B_n - \{\alpha_{m,m'}\}$ is a subalgebra of B_n . We are going to prove that C belongs to Q . For this purpose, it is sufficient to show that whenever e, e' are two elements of C such that e is covered by e' , then e, e' can be separated by a homomorphism of C into A .

For every $i \leq n-1$ define a mapping ψ_i of B_n into A by $\psi_i(u_0) = v_2$, $\psi_i(e) = a_2$ for $e \geq \alpha_{i,i}$, $\psi_i(e) = b_2$ for $e \geq \beta_i$ and $\psi_i(e) = 0$ for all other elements e . Also, for every $i \leq n-1$ define a mapping χ_i of B_n into A by $\chi_i(u_0) = v_3$, $\chi_i(e) = a_3$ for $e \geq \alpha_{i+1,i}$, $\chi_i(e) = c_3$ for $e \geq \gamma_i$ and $\chi_i(e) = 0$ for all other elements e . It is easy to check that both ψ_i and χ_i are homomorphisms. Consequently, their restrictions to C are homomorphisms of C into A . The only pairs of covers not separated by any of these homomorphisms are the pairs $(0, u_1), \dots, (0, u_k)$. So, it remains to separate these pairs of elements.

If $m = m'$, then these pairs are separated by the homomorphism φ defined in this way: $\varphi(u_0) = u_0, \dots, \varphi(u_k) = u_k$, $\varphi(e) = a$ for $e \geq \alpha_{m,m-1}$, $\varphi(e) = b$ for $e \geq \beta_m$, $\varphi(e) = c$ for $e \geq \gamma_{m-1}$ and $\varphi(e) = 0$ for all other elements e . If $m' = m-1$, then they are separated by the homomorphism φ' defined in this way: $\varphi'(u_0) = u_0, \dots, \varphi'(u_k) = u_k$, $\varphi'(e) = a$ for $e \geq \alpha_{m',m'}$, $\varphi'(e) = b$ for $e \geq \beta_{m'}$, $\varphi'(e) = c$ for $e \geq \gamma_{m'}$ and $\varphi'(e) = 0$ for all other elements e .

We have proved that C belongs to Q . Since every subalgebra of B_n generated by at most $n-k$ elements is contained in at least one such C , it follows that every subalgebra generated by at most $n-k$ elements belongs to Q . Consequently, there is no base for the quasiequations of Q that would contain only quasiequations in at most $n-k$ variables. Since k was fixed while n was arbitrary, there is no finite base at all. \square

Remark 5.2. In the above construction of the algebra A it was not essential that the elements b_2 and c_3 are distinct.

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Authors' addresses: J. Ježek, MFF UK, Sokolovská 83, 186 00 Praha 8, Czech Republic, e-mail: jezek@karlin.mff.cuni.cz; M. Maróti and R. McKenzie, Department of Mathematics, Vanderbilt University, 1326 Stevenson Center, Nashville, TN 37240, e-mails: miklos.maroti@vanderbilt.edu, ralph.n.mckenzie@vanderbilt.edu.