Problem Set 2

Problem 1. Find the clone generated by the given algebra $\mathbf{A}$:

(a) $\mathbf{A} = (\{0,1\}; \wedge)$

(b) $\mathbf{A} = (\{0,1\}; \wedge, \vee)$

(c) $\mathbf{A} = (\{0,1\}; \neg, \wedge, \vee)$

(d) $\mathbf{A} = (\{0,1\}; \neg, \wedge)$

(e) $\mathbf{A} = (\{0,1\}; f)$, where $f(x,y) = x + y \mod 2$

(f) $\mathbf{A} = (\{0,1\}; f)$, where $f(x,y,z) = x + y + z \mod 2$

(g) $\mathbf{A} = (\{0,1\}; \rightarrow)$ (Hint: Observe that the operation $f = \rightarrow$ has the following property: $f(x,y) = 0$ implies $y = 0$. Is some property like that preserved by composition?)

Problem 2. For every binary operation f on $\{0,1\}$, find the clone generated by the algebra $(\{0,1\}; f)$. (Thus we have 16 problems here.)

Problem 3. Prove that every clone on the set $\{0,1\}$, which has an operation which is not essentially unary, contains at least one of the following operations

(a) a constant unary operation (i.e. the operation $f(0) = f(1) = 0$, or the operation $f(0) = f(1) = 1$), or

(b) $\wedge$, or $\vee$, or

(c) the majority operation m (defined by $f(x,y,z) = 1$ iff at least two of the numbers x, y, z are equal to 1), or

(d) the ternary addition operation $p(x,y,z) = x + y + z \mod 2$.

A possible strategy for the proof:

- First assume that the only unary operation in the clone is the identity (the operation $f(x) = x$). (Observe, that then all the operations are *idempotent* – they satisfy $f(x, x, \ldots, x) = x$)
- Assume that none of the operations $\wedge, \vee, m, p$ are in the clone.
- Prove that the only binary operations are projections.
- Prove that the only ternary operations are projections.
- Take a smallest number n such that the clone contains a nontrivial operation of arity n and derive contradiction using the previous items.
- Get rid of the idempotency assumption.

Problem 4. For every natural number n find a finite number of operations $f_1, \ldots, f_m$ on the set $[n] = \{0, 1, \ldots, n-1\}$ so that the clone generated by the algebra $([n]; f_1, \ldots, f_m)$ contains all operations on $[n]$. Can you achieve $m = 1$?