# Problem Set 1

**Problem 1**.

(a) Find all operations on a given set $A$ which are polymorphisms of every relational structure with universe $A$.

(b) Assume that binary operations $\circ, \star$ are polymorphisms of a relational structure $\mathbb{A} = (A; \dots)$. Let $f$ be a ternary operation on $A$ defined by $f(x,y,z) = (x \circ (y \star z)) \star x$. Is it always the case that $f$ is a polymorphisms of $\mathbb{A}$? Can you generalize this?

(c) Find all relations on a fixed set $A$ which are compatible with all operations on the set $A$.

**Problem 2**. Find all polymorphisms of the following relational structures:

(a) $(\{0,1\}; \{0\}, \{1\}, R_1)$, where $R_1 = \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$

(b) $(\{0,1,2\}; \{0\}, \{1\}, R_2)$, where $R_2 = \{0,1,2\}^2 \setminus \{(0,0),(1,1),(2,2)\}$ (the inequality relation)

(c) $(\{0,1,2\}; R_2)$ (the same $R_2$ as above)

**Problem 3**. Let us call the polymorphisms from Problem 1, item (a) *trivial*.
Consider the relational structure:

(a) $\mathbb{A} = (\{0,1\}; \{0\}, \{1\}, S)$, where $S = \{0,1\}^3 \setminus \{(1,1,0)\}$

(b) $\mathbb{A} = (\{0,1\}; \{0\}, \{1\}, R_1, R_2, \dots, R_n)$, where $R_1, \dots, R_n$ is a list of all binary relations on $\{0,1\}$ (what is the value of $n$, by the way?)

(c) $\mathbb{A} = (\{0,1\}; \{0\}, \{1\}, R)$, where $R = \{(0,0,1),(0,1,0),(1,0,0),(1,1,1)\}$.

Does $\mathbb{A}$ admit any nontrivial polymorphism?
Is the answer different when we remove the unary relations $\{0\}, \{1\}$ from $\mathbb{A}$?.

**Problem 4**. Describe all relations compatible with the operation $f(x,y,z) = x + y + z$, where $+$ is the addition modulo 2 on the set $\{0,1\}$.

**Problem 5**. Let us say that an operation $f$ on the set $\{0,1\}$ is a *min* operation, if $f$ can be written in the form $f(x_1, \dots, x_n) = \min\{x_{i_1}, x_{i_2}, \dots x_{i_m}\}$ for some natural number $m$ and some $1 \leq i_1 < i_2 < \dots < i_m \leq n$.
Moreover, let us say that $f$ is a *max-min* operation, if
$f(x_1, \dots, x_n) = \max\{g_1(x_1, \dots, x_n), g_2(x_1, \dots, x_n), \dots, g_k(x_1, \dots, x_n)\}$, where $g_1, \dots, g_k$ are min operations.

(a) Show that the set of all min-max-max-min-max-min-max operations (defined in the obvious way) coincide with the set of all max-min operations.

(b) Find a relational structure $\mathbb{A} = \{\{0,1\}, \dots\}$ such that polymorphisms of $\mathbb{A}$ are precisely the max-min operations.

**Problem 6**. Let us call the polymorphisms from Problem 1, item (a) *trivial* (as in Problem 3).
Find the smallest digraph $\mathbb{A}$ such that $\mathbb{A}$ has only trivial polymorphisms. Where "smallest" is

(a) with respect to the number of vertices,

(b) with respect to the number of edges.

**Problem 7**. Find an infinite sequence $\mathbb{A}_1, \mathbb{A}_2, \dots$ of relational structures with universe $\{0,1\}$ such that, for all $i > j$, every polymorphism of $\mathbb{A}_i$ is a polymorphism of $\mathbb{A}_j$, and there exists a polymorphism of $\mathbb{A}_j$ which is not a polymorphism of $\mathbb{A}_i$.