Acta Sci. Math. (Szeged)
73 (2007), 463–470

On algebras with primitive positive clones

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Dedicated to Professor Béla Csákány on his 75th Birthday

Communicated by Á. Szendrei

Abstract. We determine the primitive positive clones F on finite sets A with at least three elements for which $(A; F)$ is simple and idempotent, and the primitive positive clones F having all constant operations for which $(A; F)$ either generates a congruence distributive variety or is a simple algebra that is not strongly abelian.

In the investigation of the structure of algebras the analyses based on the properties of their related structures play an essential role. Several important properties of an algebra are determined by one or more of its related structures: group of automorphims, monoid of endomorphisms, lattice of congruences, etc. The better we know the related sructures, the more effective these arguments are. One of the notable related structures of an algebra is its centralizer clone which is primitive positive. A clone is *primitive positive* if it contains all operations defined by primitive positive formulas over the clone. A. I. Kuznecov proved in [5] (see also [8]) that a set of operations F on a finite set A is a primitive positive clone if and only if $F = G^*$ for a set of operations G, where G^* (the *centralizer* of G) consists of all operations on A which are homomorphisms from $(A^n; G)$ to $(A; G)$ for some $n \geq 1$. The set of primitive positive clones $\mathcal{L}_A$ on a set A forms a lattice $(\mathcal{L}_A; \vee, \wedge)$ where $F \vee G = (F \cup G)^{**}$ and $F \wedge G = F \cap G$ for all $F, G \in \mathcal{L}_A$. The mapping $\mathcal{L}_A \to \mathcal{L}_A$, $F \mapsto F^*$ is a dual automorphism of $\mathcal{L}_A$. S. Burris and R.

Received January 15, 2007, and in final form August 19, 2007.
AMS Subject Classification (2000): 08A40.
* Research partially supported by Hungarian NFSR grants no. T48809 and K60148.

Willard proved in [1] that there are finitely many primitive positive clones on any finite set. If $|A| = 2$ then $|\mathcal{L}_A| = 25$ (see [9]), and A. F. Danil'cenko proved that if $|A| = 3$ then $|\mathcal{L}_A| = 2986$ (see e.g. [2] and [3]).

The aim of this paper is to determine in some classes of algebras all finite algebras whose clones are primitive positive. First we introduce some terminology and notation.

Let A be a nonempty set. For any positive integer n let $\boldsymbol{O}_A^{(n)}$ denote the set of all n-ary operations on A and let $\boldsymbol{O}_A = \bigcup_{n=1}^{\infty} \boldsymbol{O}_A^{(n)}$. The full symmetric group and the set of all unary constant operations on A will be denoted by S_A and C_A, respectively. An operation $f \in \boldsymbol{O}_A$ is *trivial* if it is a projection and f is *idempotent* if $f(a, \ldots, a) = a$ for all $a \in A$. By a *clone* we mean a subset of $\boldsymbol{O}_A$ which is closed under superpositions and contains all projections. If $F \subseteq \boldsymbol{O}_A$ then $[F]$ denotes the clone generated by F.

Let f be an n-ary and g be an m-ary operation on a set A. We say that f and g *commute* if

$$g(f(x_{11}, \ldots, x_{1n}), \ldots, f(x_{m1}, \ldots, x_{mn})) = f(g(x_{11}, \ldots, x_{m1}), \ldots, g(x_{1n}, \ldots, x_{mn}))$$

is an identity of the algebra $(A; f, g)$. Clearly, f and g commute if and only if f is a homomorphism from $(A^n; g)$ to $(A; g)$ (g is a homomorphism from $(A^m; f)$ to $(A; f)$). If F is a set of operations on A then F^* denotes the set of all operations commuting with every operation in F. As we have mentioned above a set of operations F on a finite set A is a primitive positive clone if and only if $F = G^*$ for a set of operations G on A. It is easy to see that the latter condition is equivalent to requiring that $F = F^{**}$.

The clone of all term operations and the clone of all polynomial operations of an algebra $\boldsymbol{A}$ are denoted by $\operatorname{Clo} \boldsymbol{A}$ and $\operatorname{Pol} \boldsymbol{A}$, respectively. For every $n \geq 1$ we put $\operatorname{Clo}_n \boldsymbol{A} = \operatorname{Clo} \boldsymbol{A} \cap \boldsymbol{O}_A^{(n)}$ and $\operatorname{Pol}_n \boldsymbol{A} = \operatorname{Pol} \boldsymbol{A} \cap \boldsymbol{O}_A^{(n)}$. Two algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ with a common base set are called *term equivalent* (*polynomially equivalent*) if $\operatorname{Clo} \boldsymbol{A} = \operatorname{Clo} \boldsymbol{B}$ ($\operatorname{Pol} \boldsymbol{A} = \operatorname{Pol} \boldsymbol{B}$). Two algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ are also called *term equivalent* (*polynomially equivalent*) if $\boldsymbol{A}$ is term equivalent (polynomially equivalent) to an algebra isomorphic to $\boldsymbol{B}$.

An algebra is *trivial (idempotent)* if it has only trivial (idempotent) fundamental operations. The set of all congruence relations and the set of all endomorphisms of an algebra $\boldsymbol{A}$ are denoted by $\operatorname{Con} \boldsymbol{A}$ and $\operatorname{End} \boldsymbol{A}$, respectively. An algebra $\boldsymbol{A}$ is *semi-affine* with respect to an elementary Abelian p-group $\bar{\boldsymbol{A}}$ (p prime), if $\boldsymbol{A}$ and $\bar{\boldsymbol{A}}$ have a common base set A and every fundamental operation of $\boldsymbol{A}$ commutes with the ternary operation $x - y + z$; if, in addition, $x - y + z$ is a term operation of $\boldsymbol{A}$ then $\boldsymbol{A}$ is said to be *affine* with respect to $\bar{\boldsymbol{A}}$.

By a diagonal algebra we mean an algebra of the form $(A_1 \times \cdots \times A_n; d)$ where $A_1, \ldots, A_n$ are nonempty sets and d is the n-ary operation defined as follows:

$$d((a_{11}, \ldots, a_{1n}), (a_{21}, \ldots, a_{2n}), \ldots, (a_{n1}, \ldots, a_{nn})) = (a_{11}, a_{22}, \ldots, a_{nn})$$

for any $(a_{11}, \ldots, a_{1n}), (a_{21}, \ldots, a_{2n}), \ldots, (a_{n1}, \ldots, a_{nn}) \in A_1 \times \cdots \times A_n$.

First we determine all finite simple idempotent algebras with primitive positive clones.

Theorem 1. *Let F be a primitive positive clone on a finite set A with at least three elements. If $(A; F)$ is a simple idempotent algebra then one of the following two conditions holds:*

(1.1) *$F = (G \cup C_A)^*$ for a permutation group G on A.*

(1.2) *There is a vector space ${}_K\boldsymbol{A} = (A; +, -, 0, K)$ over a finite field K such that*

$$F = \Big\{ \sum_{i=1}^{m} r_i x_i \colon\ m \geq 1,\ r_1, \ldots, r_m \in \mathrm{End}_K \boldsymbol{A},\ \sum_{i=1}^{m} r_i = 1 \Big\}.$$

Proof. Let F be a primitive positive clone on A, $|A| \geq 3$, and suppose that $(A; F)$ is a simple idempotent algebra. Consider the algebra $(A; F^*)$. Since every operation in F is idempotent, we have $C_A \subseteq F^*$. If $f \in F^*$ is a unary operation then $\mathrm{Ker}\, f \in \mathrm{Con}(A; F)$ is the equality or the full relation on A. Therefore $f \in S_A \cup C_A$. If F^* has no operation depending on at least two variables then $F^* = [G \cup C_A]$ for some permutation group G on A and $F = F^{**} = (G \cup C_A)^*$. If F^* has an operation depending on at least two variables then, by Pálfy's theorem [6], there is a vector space ${}_K\boldsymbol{A} = (A; +, -, 0, K)$ over a finite field K such that

$$F^* = \Big\{ \sum_{i=1}^{m} r_i x_i + a \colon\ m \geq 1,\ r_1, \ldots, r_m \in K,\ a \in A \Big\}.$$

Then we have

$$\begin{aligned} F = F^{**} &= \Big\{ \sum_{i=1}^{m} r_i x_i + a \colon\ m \geq 1,\ r_1, \ldots, r_m \in K,\ a \in A \Big\}^* \\ &= \Big\{ \sum_{i=1}^{m} r_i x_i \colon\ m \geq 1,\ r_1, \ldots, r_m \in \mathrm{End}_K \boldsymbol{A},\ \sum_{i=1}^{m} r_i = 1 \Big\}. \end{aligned}$$

For the last equality see e.g. [11, Exercise 2.11]. ∎

The next theorem describes, up to polynomial equivalence, all finite algebras $\boldsymbol{A}$ in congruence distributive varieties for which $\operatorname{Pol}\boldsymbol{A}$ is primitive positive. To formulate the theorem we need a notation. For sets $A_1, \ldots, A_n$ and m-ary operations $f_i \in \boldsymbol{O}_{A_i}^{(m)}$, $i = 1, \ldots, n$, the n-tuple $(f_1, \ldots, f_n)$ denotes the operation on $A_1 \times \cdots \times A_n$ defined as follows:

$$(f_1, \ldots, f_n)((a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn})) \\ = (f_1(a_{11}, \ldots, a_{m1}), \ldots, f_n(a_{1n}, \ldots, a_{mn}))$$

for all $(a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn}) \in A_1 \times \cdots \times A_n$.

Theorem 2. *Let F be a primitive positive clone on a finite set A containing all constant operations. If $(A; F)$ generates a congruence distributive variety then either $F = \boldsymbol{O}_A$ or $(A; F^*)$ is term equivalent to a diagonal algebra $(A_1 \times \cdots \times A_n; d)$, $n \geq 2$, and $(A; F)$ is term equivalent to the algebra*

$$(A_1 \times \cdots \times A_n; \{(f_1, \ldots, f_n) \colon m \geq 1,\ f_i \in \boldsymbol{O}_{A_i}^{(m)},\ i = 1, \ldots, n\}).$$

Proof. Let F be a primitive positive clone on a finite set A containing all constant operations, and suppose that $(A; F)$ generates a congruence distributive variety. Consider the algebra $(A; F^*)$. Since $C_A \subseteq F$ we have that $(A; F^*)$ is an idempotent algebra. If $(A; F^*)$ is a trivial algebra then $F = F^{**} = \boldsymbol{O}_A$.

Now suppose that $(A; F^*)$ is nontrivial and let $f \in F^*$ be an n-ary operation depending on all of its variables. Then f is a homomorphism from the direct power $(A^n; F)$ to $(A; F)$ and therefore $\operatorname{Ker} f \in \operatorname{Con}(A^n; F)$. Since $(A; F)$ generates a congruence distributive variety, $\operatorname{Ker} f$ is a product congruence on A^n. Thus there are congruences $\Theta_1, \ldots, \Theta_n$ of $(A; F)$ such that

$$((a_1, \ldots, a_n), (b_1, \ldots, b_n)) \in \operatorname{Ker} f \text{ if and only if } (a_i, b_i) \in \Theta_i,\ i = 1, \ldots, n.$$

Let k and k_i denote the numbers of classes of $\operatorname{Ker} f$ and Θ_i, $i = 1, \ldots, n$, respectively. Since f depends on all of its variables we have $k_i \geq 2$, $i = 1, \ldots, n$. It follows that

$$n \leq \prod_{i=1}^{n} k_i = k = |f(A^n)| \leq |A|.$$

Hence $(A; F^*)$ is a nontrivial idempotent algebra such that every term operation depends on at most $|A|$ variables. Therefore, by [12, Theorem 3], $(A; F^*)$ is term equivalent to a diagonal algebra $(A_1 \times \cdots \times A_n; d)$. To complete the proof we have to prove that

$$F = F^{**} = \{d\}^* = \{(f_1, \ldots, f_n) \colon m \geq 1,\ f_i \in \boldsymbol{O}_{A_i}^{(m)},\ i = 1, \ldots, n\}).$$

It is easy to check that every operation of the form $(f_1,\dots,f_n)$, $f_i \in O_{A_i}^{(m)}$, $i = 1,\dots,n$, belongs to $\{d\}^*$. If $f \in \{d\}^*$ is an m-ary operation then there are mn-ary operations $f_1,\dots,f_n$ on A such that for any $(a_{11},\dots,a_{1n}),\dots,(a_{m1},\dots,a_{mn}) \in A_1 \times \dots \times A_n$ we have

$$f((a_{11},\dots,a_{1n}),\dots,(a_{m1},\dots,a_{mn})) = (f_1(a_{11},\dots,a_{mn}),\dots,f_n(a_{11},\dots,a_{mn})).$$

Since f and d commute

$$d(f(\underline{x}_{11},\dots,\underline{x}_{1m}),\dots,f(\underline{x}_{n1},\dots,\underline{x}_{nm})) = f(d(\underline{x}_{11},\dots,\underline{x}_{n1}),\dots,d(\underline{x}_{1m},\dots,\underline{x}_{nm}))$$

for any $\underline{x}_{ij} = (x_{ij}^1,\dots,x_{ij}^n) \in A_1 \times \dots \times A_n$, $1 \le i \le n$, $1 \le j \le m$. Comparing the jth coordinates of both sides we get that

$$f_j(x_{j1}^1,\dots,x_{j1}^n,\dots,x_{jm}^1,\dots,x_{jm}^n) = f_j(x_{11}^1,\dots,x_{n1}^n,\dots,x_{1m}^1,\dots,x_{nm}^n),$$

$j = 1,\dots,n$. This is equivalent to the fact that f_j depends on the arguments $x_{j1}^j,\dots,x_{jm}^j$ only, $j = 1,\dots,n$. Hence there are m-ary operations $g_1,\dots,g_n$ on A such that

$$f_j(a_{11},\dots,a_{mn}) = g(a_{1j},\dots,a_{mj}),\ j = 1,\dots,n.$$

Summarizing the above computation we obtain that $f \in \{d\}^*$ if and only if $f = (g_1,\dots,g_n)$ for some m-ary operations $g_1,\dots,g_n$ on A. This completes the proof. ∎

Our last theorem describes, up to polynomial equivalence, all finite simple algebras $\boldsymbol{A}$ such that $\operatorname{Pol}\boldsymbol{A}$ is primitive positive, but $\boldsymbol{A}$ is not strongly abelian. The proof will assume familiarity with the basic facts of tame congruence theory [4].

Theorem 3. *Let F be a primitive positive clone on a finite set A containing all constant operations. If $(A;F)$ is a simple algebra then one of the following four conditions holds:*

(3.1) *$(A;F)$ is strongly abelian.*

(3.2) $F = \boldsymbol{O}_A$.

(3.3) *There is a semilattice $(A;\vee)$ such that $F = \{\vee\}^*$.*

(3.4) *There is a vector space ${}_K\boldsymbol{A} = (A;+,-,0,K)$ over a finite field K such that*

$$F = \Big\{\sum_{i=1}^{m} x_i r_i + a\colon\ m \ge 1,\ r_1,\dots,r_m \in \operatorname{End}_K \boldsymbol{A},\ a \in A\Big\}.$$

Proof. Let F be a primitive positive clone on a finite set A containing all constant operations, and suppose that $\boldsymbol{A} = (A; F)$ is simple. Taking into consideration [4, Theorem 5.7(1)], we have that $\boldsymbol{A}$ is tame. We define $\boldsymbol{U}$ to be the set of all sets of the form $f(A)$ where $f \in \mathrm{Pol}_1 \boldsymbol{A}$ and $|f(A)| \geq 2$. Furthermore, $\boldsymbol{M}$ is the set of minimal members of $\boldsymbol{U}$. For any $X \subseteq A$ by the non-indexed algebra induced by $\boldsymbol{A}$ on X we mean the algebra $\boldsymbol{A}|_X = (X; F|_X)$ where

$$F|_X = \{f|_X : f \in F,\ f(X, \ldots, X) \subseteq X\}.$$

Clearly, for any $M \in \boldsymbol{M}$ the algebra $\boldsymbol{A}|_M$ is minimal, i.e. every unary polynomial operation of $\boldsymbol{A}|_M$ is a permutation or a constant operation. Moreover, since $\boldsymbol{A}$ is simple, by [4, Lemma 2.3 and Theorem 2.8], $\boldsymbol{A}|_M$ is also simple.

Choose a set M from $\boldsymbol{M}$. Then M is a subuniverse of $(A; F^*)$ since for some unary operation $f \in F$ we have $f(A) = M$ and f is an endomorphism of $(A; F^*)$. Thus $(M; F^*|_M)$ coincides with the subalgebra $(M; F^*)$ of $(A; F^*)$.

Claim. *$(A; F^*)$ is in the variety generated by $(M; F^*)$.*

In order to prove the claim it is enough to show that every identity of $(M; F^*)$ holds in $(A; F^*)$. Let $f = g$ be an identity of $(M; F^*)$. We can assume without loss of generality that f and g have the same variables $x_1, \ldots, x_n$. Suppose that $f = g$ does not hold in $(A; F^*)$. Then there are $a_1, \ldots, a_n \in A$ such that

$$f(a_1, \ldots, a_n) = a \neq b = g(a_1, \ldots, a_n).$$

Since $\boldsymbol{A}$ is a tame algebra therefore, by [4, Theorem 2.8(4)], there is a unary operation $h \in F$ such that $h(A) = M$ and $h(a) \neq h(b)$. Then

$$\begin{aligned} f(h(a_1), \ldots, h(a_n)) &= h(f(a_1, \ldots, a_n)) = h(a) \neq h(b) \\ &= h(g(a_1, \ldots, a_n)) = g(h(a_1), \ldots, h(a_n)) \end{aligned}$$

which shows that the identity $f = g$ does not hold in $(M; F^*)$, contrary to our assumption.

Now we are ready to complete the proof of Theorem 3. If $(M; F^*)$ is a trivial algebra then, by the claim, $(A; F^*)$ is also trivial and we have $F = (F^*)^* = \boldsymbol{O}_A$. From now on suppose that $(M; F^*)$ is nontrivial. Since $(M; F|_M)$ is a simple minimal algebra, by [4, Corollary 4.11], we have one of the following possibilities:

(1) $(M; F|_M)$ is term equivalent to the algebra $(M; G \cup C_M)$ for some permutation group G on M;

(2) there is a 1-dimensional vector space ${}_K\boldsymbol{M} = (M; +, -, 0, K)$ over a finite field K such that

$$F|_M = \Big\{ \sum_{i=1}^{m} r_i x_i + a\colon\ m \geq 1,\ r_1, \dots, r_m \in K,\ a \in M \Big\};$$

(3) $(M; F|_M)$ is term equivalent to the algebra $(\{0,1\}; \vee, \wedge, ', 0, 1)$;
(4) $(M; F|_M)$ is term equivalent to the algebra $(\{0,1\}; \vee, \wedge, 0, 1)$;
(5) $(M; F|_M)$ is term equivalent to the algebra $(\{0,1\}; \vee, 0, 1)$,

where $x \vee y = \max(x, y)$, $x \wedge y = \min(x, y)$ and $x' = 1 - x$. In case (1), by [4, Theorem 5.7(3)], we have (3.1). It is easy to see that $F^*|_M \subseteq (F|_M)^*$. Using this fact and the list of primitive positive clones on a two element set given in [9], in case (3) or (4) we have that $(M; F^*)$ is a trivial algebra, contrary to our assumption. And in case (5) $(M; F^*)$ is term equivalent to the algebra $(\{0,1\}; \vee)$. In this case, by the claim, we have that $(A; F^*)$ is term equivalent to a semilattice $(A; \vee)$ and $F = (F^*)^* = \{\vee\}^*$.

Finally, suppose that there is a 1-dimensional vector space ${}_K\boldsymbol{M} = (M; +, -, 0, K)$ over a finite field K such that

$$F|_M = \Big\{ \sum_{i=1}^{m} r_i x_i + a\colon\ m \geq 1,\ r_1, \dots, r_m \in K,\ a \in M \Big\}.$$

Then taking into consideration [11, Exercise 2.11] and the fact that for a 1-dimensional vector space $\mathrm{End}_K \boldsymbol{M} = K$, we have

$$(F|_M)^* = \Big\{ \sum_{i=1}^{m} r_i x_i\colon\ m \geq 1,\ r_1, \dots, r_m \in K,\ \sum_{i=1}^{m} r_i = 1 \Big\}.$$

Taking into consideration [11, Proposition 2.9], in this case we have that every nontrivial subclone of $(F|_M)^*$ contains the operation $x - y + z$. Therefore since $(M; F^*)$ is nontrivial and $F^*|_M \subseteq (F|_M)^*$, we obtain that the algebra $(M; x - y + z)$ is a reduct of $(M; F^*)$. Taking into consideration the claim, it follows that the algebra $(A; x - y + z)$ is a reduct of $(A; F^*)$ for some elementary abelian p-group $(A; +)$. Thus $(A; F)$ is a semi-affine algebra. One can easily check that the restriction of the operation $x - y + z$ to M commutes with all operations of $F^*|_M \subseteq (F|_M)^*$. Applying again the claim, we obtain that the operation $x - y + z$ commutes with all operations of F^* and thus $x - y + z \in F = (F^*)^*$. Hence $(A; F)$ is a simple affine algebra and therefore, by [11, Proposition 2.10] we have (3.4). This completes the proof.

■

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