Periodic Solutions of Dissipative Functional Differential Equations with Infinite Delay

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Abstract. We consider periodic, infinite delay differential equations. We investigate dissipativeness for these equations. Massat proved that dissipative, periodic, infinite delay equations have a periodic solution. For our purpose we need a weaker dissipativeness, so we prove Massat's theorem from this weak dissipativeness in an elementary way. Then we extend a theorem of Pliss giving a neccessary and sufficient condition for this weak dissipativeness. We also present a theorem using Liapunov functionals to show the weak dissipativeness and hence the existence of a periodic solution.

1. Introduction. Let $f : \mathbf{R}\times\mathbf{R}^d \to \mathbf{R}^d$ be continuous and locally Lipschitz in x with $f(t+T,x) = f(t,x)$ for all (t,x) and some $T > 0$. We say that the ordinary differential equation

$$x' = f(t,x) \tag{1}$$

is dissipative, if all solutions become bounded by a fixed constant at some time and remain bounded from that time on. Pliss [9, Theorem 2.1] showed that the ordinary differential equation is dissipative if and only if there is an $r > 0$ such that for each (t_0, x_0) there is a $\tau > t_0$ with $|x(\tau, t_0, x_0)| < r$. The author [7] generalized this result for finite delay differential equations stating that dissipativeness is equivalent to every solution becoming bounded by a fixed constant for an interval of length $2h$, where h is the retardation. The author also gave an elementary proof for a result of Hale and Lopes [3], who proved that dissipativeness implies the existence of a periodic solution for finite delay equations. The following Lyapunov-type theorem, which can also be found in [7], proves the existence of a periodic solution through dissipativeness.

Theorem A: Suppose there are a functional $V : \mathbf{R}\times\mathcal{C} \to \mathbf{R}$ and constants $a, b, M, U > 0$ such that

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(i) $0 \le V(t, \phi)$,

(ii) $V'(t, x_t) \le M$ and

(iii) $V'(t, x_t) \le -a|x'(t)| - b$ for $|x(t)| \ge U$.

Then the solutions of the finite delay differential equation are dissipative.

Stronger versions of the conditions asked in this theorem are used in the literature to prove uniform boundedness and uniform ultimate boundedness (see e.g. Theorem 4.2.11 in [1]). There naturally arises the question if these theorems can be generalized for infinite delay. By looking at Theorem A nothing seems to keep this theorem from being applicable to the infinite delay case. Also the theorem of Hale and Lopes can be stated for infinite delay, Massat [8] proved using an axiomatic approach that dissipativeness is enough to have a periodic solution for infinite delay differential equations. Hino and Murakami [4] also used axiomatic setup of the phase space to investigate infinite delay equations. Kato [6] summarized many boundedness-type properties and their connections, which are related to dissipativeness. For a good summary of the recent results concerning dissipativeness see Hale's book [2].

In this paper we consider an equation not satisfying some of the axioms used in the above mentioned papers. We will see that this paper is parallel to [7] although the properties used in this paper are different from the obvious generalizations of that paper. We prove the existence of a periodic solution from a property called weak dissipativity, which is weaker than dissipativity. With this weak dissipativity we prove a generalization of Pliss' Theorem and then use this result to get an exact counterpart of Theorem A.

2. Main results.

We now introduce a functional differential equation with infinite delay. Let $(\mathcal{C}, \|\cdot\|_g)$ be the Banach space of continuous functions $\phi : (-\infty, 0] \to \mathbf{R}^d$ with the so-called g-norm defined by

$$\|\phi\|_g := \sup_{s \le 0} \frac{|\phi(s)|}{g(s)},$$

where $g : (-\infty, 0] \to [1, \infty)$ is a continuous, decreasing function with $g(0) = 1$ and $g(s) \to \infty$ as $s \to -\infty$. If we talk about boundedness or compactness in the following we always mean it in the g-norm. Denote $x_t(s) = x(t+s)$ for $s \le 0$ and let $F : \mathbf{R} \times \mathcal{C} \to \mathbf{R}^d$ be continuous and locally Lipschitz in ϕ in the g-norm with $F(t+T,\phi)=F(t,\phi)$. Then

$$x' = F(t, x_t) \tag{2}$$

is a system of functional differential equations and for each $(t_0,\phi)\in\mathbf{R}\times\mathcal{C}$ there is a unique solution $x(t,t_0,\phi)$ which depends continuously on the initial data. The local Lipschitz condition together with the periodicity of F clearly implies that F takes bounded sets of ϕ into bounded sets. Let us denote by $L(M)$ the bound for F when ϕ is bounded by M and assume that L is a strictly increasing function of M.

We asked g to be decreasing in order to simplify our proofs. If one has a not decreasing $g(s)$, one can replace it by $\inf_{u\le s} g(u)$, which is decreasing, smaller than $g(s)$, and hence the conditions on F can be more easily satisfied.

We now show how to use the dissipativeness of the solutions in proving the existence of a periodic solution. We need a few quite technical lemmas, which we will use in the following.

LEMMA 1: *The set*

$$\begin{aligned}\mathcal{S}(R) := \{\psi\in\mathcal{C} : \textit{there is an } S>0 \textit{ such that } |\psi(s)|\le\sqrt{g(s)} \textit{ for all } s\le -S,\\ |\psi(u)-\psi(v)|\le L(\sqrt{g(\min\{u,v\})})|u-v| \textit{ for } u,v\le -S \textit{ and}\\ |\psi(s)|\le R, |\psi(u)-\psi(v)|\le L(R)|u-v| \textit{ for } s,u,v\in[-S,0]\}\end{aligned}$$

is compact in the g-norm.

Proof: Let $\psi_n\in\mathcal{S}(R)$ be an arbitrary sequence and let S_n denote the constant used for ψ_n. We have two cases:

Case 1. If S_n is bounded above, then take a subsequence, say S_n again, such that $S_n\to S$. Using Ascoli's Theorem, ψ_n has a subsequence, ψ_n again, which converges in the supremum norm on the interval $[-S,0]$. Now take an interval $[-Q,-S]$. On this interval ψ_n is bounded by $\sqrt{g(-Q)}$ and satisfies a Lipschitz condition with $L(\sqrt{g(-Q)})$ and hence we can apply Ascoli's theorem to prove that ψ_n has a subsequence, which converges in the supremum norm on the interval $[-Q,-S]$. Using this result for $-Q:=-S-m$ $(m\to\infty)$ and applying the diagonal method we can find a subsequence of ψ_n, say ψ_n again, which converges to a function ψ uniformly on any finite interval. Now we estimate the g-norm:

$$\begin{aligned}\|\psi_n - \psi\|_g &= \max\left\{ \sup_{s\le -Q} \frac{|\psi_n(s)-\psi(s)|}{g(s)}, \sup_{s\in[-Q,0]} \frac{|\psi_n-\psi|}{g(s)} \right\} \\ &\le \max\left\{ \sup_{s\le -Q} \frac{2\sqrt{g(s)}}{g(s)}, \sup_{s\in[-Q,0]} \frac{|\psi_n-\psi|}{g(s)} \right\}\end{aligned}$$

We can make the first argument small by taking Q large enough and using that $g(s)\to\infty$ as $s\to-\infty$. Also, the second term is small for large enough n, since ψ_n converges uniformly to ψ on the interval $[-Q,0]$. Hence, ψ_n converges to ψ in the g-norm.

Case 2. If S_n has a subseqence, say S_n again, so that $S_n\to\infty$, then using the usual diagonal method and Ascoli's Theorem we can find a subsequence of ψ_n (again ψ_n) so that it converges to a function ψ uniformly in the supremum norm on any finite interval, where ψ is bounded by R. We now estimate the g-norm:

$$\begin{aligned}\|\psi^n - \psi\|_g &= \max\left\{ \sup_{s\le -S_n} \frac{|\psi^n(s)-\psi(s)|}{g(s)}, \sup_{s\in[-S_n,0]} \frac{|\psi^n(s)-\psi(s)|}{g(s)} \right\} \\ &\le \max\left\{ \sup_{s\le -S_n} \frac{\sqrt{g(s)}+R}{g(s)}, \sup_{s\in[-S_n,0]} \frac{|\psi^n(s)-\psi(s)|}{g(s)} \right\}\end{aligned}$$

Since $g(s)\to\infty$ as $s\to-\infty$ and since $S_n\to\infty$ as $n\to\infty$, the first term tends to 0. Next,

$$\sup_{s\in[-S_n,0]} \frac{|\psi^n(s)-\psi(s)|}{g(s)} \le \max\left\{ \sup_{s\in[-S_n,-P]} \frac{2R}{g(s)}, \sup_{s\in[-P,0]} \frac{|\psi^n(s)-\psi(s)|}{g(s)} \right\}$$

is also small if we choose P independently of n but large enough to make the first term small, and then if n is large enough, the second term will also become small. This proves that ψ^n converges to ψ in the g-norm.

Note that $\|\phi\|_g \le R$ for all $\phi\in\mathcal{S}(R)$, if $R\ge 1$.

Lemma 2: *Let $R, r > 1$, Q constants be given and consider any function $x : (-\infty,\infty)\to\mathbf{R}^d$ with $x_Q\in\mathcal{S}(R)$ and $|x(s)|<r$ for $s\ge Q$. Then we can find an $H\ge Q$ independent of x so that $\|x_t\|_g < r$ for all $t\ge H$.*

Proof: We estimate $\|x_t\|_g$ in three parts. As $x_Q\in\mathcal{S}(R)$ there is an $S>0$ such that

$|x_Q(s)| \leq \sqrt{g(s)}$ for $s \leq -S$. Then

$$\begin{aligned}\|x_t\|_g &= \max\left\{\sup_{s\leq Q-S}\frac{|x(s)|}{g(s-t)}, \sup_{s\in[Q-S,Q]}\frac{|x(s)|}{g(s-t)}, \sup_{s\in[Q,t]}\frac{|x(s)|}{g(s-t)}\right\}\\ &\leq \max\left\{\sup_{s\leq -S}\frac{\sqrt{g(s)}}{g(s-t+Q))}, \sup_{s\in[Q-S-t,Q-t]}\frac{R}{g(s)}, \sup_{s\in[Q,t]}\frac{|x(s)|}{g(s-t)}\right\}\\ &\leq \max\left\{\sup_{s\leq -S}\frac{1}{\sqrt{g(s)}}, \sup_{s\leq Q-H}\frac{R}{g(s)}, \sup_{s\in[Q,t]}\frac{|x(s)|}{g(s-t)}\right\}.\end{aligned}$$

The first term is less than $1/\sqrt{g(s)} \leq 1 < r$ by our assumptions. Since we can have $g(s) > R/r$ for $s \leq Q - H$ for large H, we can also make the second expression smaller than r. The third argument is clearly smaller than r from $|x(s)| < r$. Hence we proved that for any large enough t we have $\|x_t\|_g < r$ independent of the x choosen from the given set.

DEFINITION: Equation (2) is *weakly dissipative (at $t = 0$)*, if there is an $r > 0$ such that

$$\limsup_{t\to\infty} |x(t, 0, \phi)| < r$$

for all $\phi \in \mathcal{S}(R)$, where $R > 0$ is arbitrary. For technical reasons we always assume in the following that $r > 1$.

LEMMA 3: *If (2) is weakly dissipative with r, and we start a solution from $\phi \in \mathcal{S}(R)$ (for some $R > 0$) then there is an $S > 0$ such that $x_t(\cdot, 0, \phi) \in \mathcal{S}(r)$ for all $t \geq S$.*

Proof: Let $x(s) := x(s, 0, \phi)$. Using the dissipativity we can find $Q > 0$ and $M > R$ such that $|x(s)| < M$ for $s \in [0, Q]$ and $|x(s)| < r$ for $s \geq Q$. Observe that $x_Q \in \mathcal{S}(M)$, and hence from Lemma 2 we find an $H > Q$ such that $\|x_s\|_g < r$ for $s \geq H$. Now let $\overline{H} > 0$ be large enough, so that $g(s) > M^2$ for $s \leq -\overline{H}$, and let $S := H + \overline{H}$. Then x_t is in $\mathcal{S}(r)$ by construction for all $t \geq S$. Note that we proved that x_t is in some sense in the inside of $\mathcal{S}(r)$. This means that if we start a solution from a function $\psi \in \mathcal{S}(R)$ close to ϕ and fix a $t \geq S$, then using the continuous dependence of the solutions on the initial data we also have $x_t(\cdot, 0, \psi) \in \mathcal{S}(r)$. We will use this remark in the following.

DEFINITION: Equation (2) is *weakly uniformly bounded (at $t = 0$)*, if for every $R > 0$ there is a $B > 0$ such that $|x(t, 0, \phi)| < B$ for all $\phi \in \mathcal{S}(R)$ and $t \geq 0$.

DEFINITION: Equation (2) is *weakly uniformly dissipative (at $t = 0$)*, if there is an $r > 0$ such that the for every $R > 0$ there is a $P > 0$ so that $|x(t, 0, \phi)| < r$ if $\phi \in \mathcal{S}(R)$ and $t \geq P$.

THEOREM 4: *If (2) is weakly dissipative, then it is weakly uniformly bounded.*

Proof: Let $r > 0$ be the number in the definition of weak dissipativeness. Suppose that the statement of the theorem does not hold. Then we find $R > r$ and sequences $\phi_n \in \mathcal{S}(R)$ and $t_n \geq 0$ such that $|x(t_n, 0, \phi_n)| \to \infty$. We assume that n is so large that $x_{t_n}(\cdot, 0, \phi_n)$ is not in $\mathcal{S}(R)$, because $\mathcal{S}(R)$ is bounded and $|x(t_n, 0, \phi_n)| \to \infty$. Since $\phi_n = x_0(\cdot, 0, \phi_n) \in \mathcal{S}(R)$, we can define τ_n so that $\psi_n := x_{\tau_n}(\cdot, 0, \phi_n) \in \mathcal{S}(R)$, but $x_t(\cdot, 0, \phi_n) \notin \mathcal{S}(R)$ for $t \in (\tau_n, t_n]$. Using a translation argument we find a $\overline{\tau}_n \in [0, T]$ such that $x(t, \overline{\tau}_n, \psi_n) = x(t + (\tau_n - \overline{\tau}_n), \tau_n, \psi_n) = x(t + (\tau_n - \overline{\tau}_n), 0, \phi_n)$. Since $\mathcal{S}(R)$ and $[0, T]$ are compact, there are subsequences, say ψ_n and $\overline{\tau}_n$ again, such that $\psi_n \to \psi \in \mathcal{S}(R)$ and $\overline{\tau}_n \to \overline{\tau} \in [0, T]$. Using Lemma 3 for this ψ we find a $t \geq \overline{\tau}$ such that $x_t(\cdot, \overline{\tau}, \psi) \in \mathcal{S}(r)$. Let $M := \sup_{x \in [\overline{\tau}, t]} |x(s, \overline{\tau}, \psi)|$. Now take any n large enough to have $|x(t_n, 0, \phi_n)| = |x(t_n - (\tau_n - \overline{\tau}_n), \overline{\tau}_n, \psi_n)| \geq M + 1$, $|x(s, \overline{\tau}_n, \psi_n)| < M + 1$ for $s \in [\overline{\tau}_n, t]$ and $x_t(\cdot, \overline{\tau}_n, \psi_n) \in \mathcal{S}(r) \subset \mathcal{S}(R)$ using the remark in Lemma 3. Then we must have $t_n - (\tau_n - \overline{\tau}_n) > t$, but this is a contradiction to the choice of τ_n, because we must have $x_t(\cdot, \overline{\tau}_n, \psi_n) \notin \mathcal{S}(R)$ for $t \in (\overline{\tau}_n, t_n - (\tau_n - \overline{\tau}_n)]$. This contradiction shows the required weak uniform boundedness.

THEOREM 5: *If (2) is weakly dissipative, then it is weakly uniformly dissipative.*

Proof: Let $r > 0$ be the number in the definition of the dissipativeness and $\overline{r} > 0$ be the number of weak uniform boundedness for $\mathcal{S}(r)$ from the previous theorem. We claim that (2) is weakly uniformly dissipative with $\overline{r}$. Suppose for contradiction that there is an $R > 0$ and sequences $\phi_n \in \mathcal{S}(R)$ and $t_n \to \infty$ such that $|x(t_n, 0, \phi_n)| \geq \overline{r}$. As $\mathcal{S}(R)$ is compact, there is a subsequence of ϕ_n, say ϕ_n again, such that $\phi_n \to \phi \in \mathcal{S}(R)$. For this ϕ we find a $m \geq 0$ such that $x_{mT}(\cdot, 0, \phi) \in \mathcal{S}(r)$ using Lemma 3. Now take n large enough to have $x_{mT}(\cdot, 0, \phi_n) \in \mathcal{S}(r)$ and $t_n > mT$. Using the weak uniform boundedness we find that $|x(s, 0, \phi_n)| = |x(s, mT, x_{mT}(\cdot, 0, \phi_n))| = |x(s - mT, 0, x_{mT}(\cdot, 0, \phi_n))| < \overline{r}$ for all $s \geq mT$, which is a contradiction to $t_n > mT$ and $|x(t_n, 0, \phi_n)| \geq \overline{r}$. This contradiction shows the required weak uniform dissipativeness.

THEOREM 6: *If (2) is weakly dissipative with $r > 0$ (and hence weakly uniformly dissipative with $\overline{r} > 0$ from the previous theorem), then for all $R > 0$ there is a $P > 0$ such* *that $x_t(\cdot,0,\phi)\in\mathcal{S}(\overline{r})$ for all $\phi\in\mathcal{S}(R)$ and $t\geq P$.*

Proof: This theorem states a stronger version of the weak uniform dissipativenes we will use in the following theorem. From Theorem 4 we have a constant $M>\max\{\overline{r},R\}$ such that $|x(t,0,\phi)|\leq M$ for all $\phi\in\mathcal{S}(R)$ and $t\geq 0$. We also have a $\overline{P}$ such that $|x(t,0,\phi)|<\overline{r}$ for all $\phi\in\mathcal{S}(R)$ and $t\geq\overline{P}$. Using Lemma 2 (since $x_{\overline{P}}(\cdot,0,\phi)\in\mathcal{S}(M)$ for all $\phi\in\mathcal{S}(R)$) take $H>\overline{P}$ large enough to have $\|x_t(\cdot,0,\phi)\|_g<\overline{r}$ for all $\phi\in\mathcal{S}(R)$ and $t\geq H$. Taking $\overline{H}>0$ so large that $g(s)\geq M^2$ for $s\leq-\overline{H}$ holds and defining $P:=H+\overline{H}$ we have $x_t(\cdot,0,\phi)\in\mathcal{S}(\overline{r})$ for all $\phi\in\mathcal{S}(R)$ and $t\geq P$, which was to be proven.

Theorem 7: *If (2) is weakly dissipative, then it has a T-periodic solution.*

Proof: From Theorems 4 and 5 we know that (2) is weakly uniformly bounded and weakly uniformly dissipative with $\overline{r}>0$. Since the proof of this theorem is very similar to the usual proof of the existence of a T-periodic solution assuming uniform boundedness and uniform ultimate boundedness (see [1, Theorem 4.2.2]), we will give only a sketch of the proof. Let $S_0:=\mathcal{S}(\overline{r})$ and define $P:\mathcal{C}\to\mathcal{C}$ by $P\phi:=x_T(\cdot,0,\phi)$. From the weak uniform boundedness we find $B_1>\overline{r}$ such that $|x(t,0,\phi)|<B_1$ for $t\geq 0$ and $\phi\in S_0$. Let $S_1:=\mathcal{S}(B_1)$. Once again using the weak uniform boundedness we define $B_2>B_1$ such that if $S_2:=\mathcal{S}(B_2)$, then $P^n(S_1)\subset S_2$ for all $n\geq 0$. Also, from Theorem 6 we find an $m>0$ such that $P^n(S_1)\subset S_0$ for $n\geq m$. Now all the conditions of Horn's fixed-point theorem (see [5] or [1, Section 3.4]) are satisfied, and hence there is a fixed point of P, which is (of course) a T-periodic solution of (2). The proof is complete.

Now we generalize a theorem of Pliss for this infinite delay case.

Theorem 8: *Equation (2) is weakly dissipative if and only if there exists an $r>0$ such that for all $\phi\in\mathcal{S}(R)$ ($R>0$) there is a $\tau>0$ such that $x_\tau(\cdot,0,\phi)\in\mathcal{S}(r)$.*

Proof: The implication follows from Lemma 3. To prove the opposite direction, suppose for contradiction that equation (2) is not weakly dissipative, i.e. there is a sequence $\phi_n\in\mathcal{S}(R)$ such that $\limsup_{t\to\infty}|x(t,0,\phi_n)|>r_n$, where $r_n\to\infty$. By our assumption, take $s_n>0$ so that $x_{s_n}(\cdot,0,\phi_n)\in\mathcal{S}(r)$. Let $t_n>s_n$ be any number with $|x(t_n,0,\phi_n)|>r_n$ and assume that n is large enough to have $r_n>r$. Let $\tau_n\in[s_n,t_n)$ be a number with $\psi_n:=x_{\tau_n}(\cdot,0,\phi_n)\in\mathcal{S}(r)$ and $x_t(\cdot,0,\phi_n)\notin\mathcal{S}(r)$ for $t\in(\tau_n,t_n]$. The proof from here on is the same as that of Theorem 4; we use the translation argument, take convergent subsequences of $\overline{\tau}_n$ and ψ_n and get a contradiction. This proves the weak dissipativity of equation (2).

Theorem 9: *Suppose there are a functional $V : \mathbf{R}\times\mathcal{C} \to \mathbf{R}$ and constants $a, b, M, U > 0$ such that*

(i) $0 \le V(t,\phi)$,

(ii) $V'(t,x_t) \le M$ *and*

(iii) $V'(t,x_t) \le -a|x'(t)| - b$ *for* $|x(t)| \ge U$.

Assume also, that $g(s) \ge c^2 s^4$ for all $s \le 0$ and some $c > 0$. Then the solutions of (2) are weakly dissipative.

Proof: By our previous theorem we need only to prove that there is an $r > 0$ such that for every $\phi \in \mathcal{S}(R)$ there is a $\tau \ge 0$ with $x_\tau(\cdot,0,\phi) \in \mathcal{S}(r)$. Fix $R > 0$ and $\phi \in \mathcal{S}(R)$, and let $x(t) := x(t,0,\phi)$, $V(t) := V(t,x_t)$ and $t_1 := 0$. Let $S > 0$ be the number for ϕ in the definition of $\mathcal{S}(R)$, and let $t_0 := -S$. Define L to be so large that if $Q \ge U + L > 1$ then $(Q-U)a - M(\sqrt{Q/c} + \sqrt[4]{Q/c^2}) > d_1 > 0$, where d_1 is a constant. Clearly we have such an $L > 0$. Define $r := U + L$ and $M_1 := R$. We will do an induction. Suppose, that $t_0, ..., t_n$ and $M_1, ..., M_n$ are defined so that $|x_{t_n}(s)| \le \sqrt{g(s)}$ and $|x_{t_n}(u) - x_{t_n}(v)| \le L(\sqrt{g(\min\{u,v\})})|u-v|$ for $s,u,v \le -(t_n - t_{n-1})$ and $|x_{t_n}(s)| \le M_n$ and $|x_{t_n}(u) - x_{t_n}(v)| \le L(M_n)|u-v|$ for $s,u,v \in [-(t_n - t_{n-1}), 0]$. This inductional assumption clearly holds for $n = 1$. Suppose, that it is true for some $n > 0$. Then we define $t_{n+1} := t_n + \sqrt{M_n/c} + \sqrt[4]{M_n/c^2}$. We have two cases:

Case I: If x is bounded by r on the interval $[t_n, t_{n+1}]$ then by construction we have $\|x_t\|_g \le r$ for $t \in [t_n + \sqrt[4]{M_n/c^2}, t_{n+1}]$, and hence x satisfies a Lipschitz condition with $L(r)$ in that interval. Also, for $s,u,v \le -\sqrt{M_n/c}$ we have $|x_{t_{n+1}}(s)| \le \sqrt{g(s)}$ and $|x_{t_{n+1}}(u) - x_{t_{n+1}}(v)| \le L(\sqrt{g(\min\{u,v\})})|u-v|$. Therefore $x_{t_{n+1}} \in \mathcal{S}(r)$ by construction, and the proof is finished (the induction is terminated).

Case II: Let $M_{n+1} > r$ be the maximum of x on the interval $[t_n, t_{n+1}]$. First, we estimate the decrease in V:

Case 1: If $|x(t)| \ge U$ for all $t \in [t_n, t_{n+1}]$ then using (iii) we have a $d_2 > 0$ such that

$$V(t_{n+1}) - V(t_n) \le -b(t_{n+1} - t_n) \le -d_2 \le -d_2 - (M_{n+1} - M_n)a$$

if $M_{n+1} \le M_n$ and

$$V(t_{n+1}) - V(t_n) \le -b(t_{n+1} - t_n) - (M_{n+1} - M_n)a \le -d_2 - (M_{n+1} - M_n)a$$

if $M_{n+1} > M_n$.

Case 2: If there is a $t \in [t_n, t_{n+1}]$ with $|x(t)| \le U$, then

$$\begin{aligned} V(t_{n+1}) - V(t_n) &\le M(t_{n+1} - t_n) - (M_{n+1} - U)a \\ &= M(\sqrt{M_n/c} + \sqrt[4]{M_n/c^2}) - (M_n - U)a - (M_{n+1} - M_n)a \\ &\le -d_1 - (M_{n+1} - M_n)a \end{aligned}$$

by the definition of L.

In any case we have $V(t_{n+1}) - V(t_n) \le -d - (M_{n+1} - M_n)a$ for some $d > 0$. To prove the inductional assumption for t_{n+1} we need to consider two cases again.

Case A: If $M_{n+1} < M_n$ then we redefine t_n to be $t_{n+1} - \sqrt{M_n/c}$. Now, using this new definition of t_n we have $|x_{t_{n+1}}(s)| \le \sqrt{g(s)}$ and $|x_{t_{n+1}}(u) - x_{t_{n+1}}(v)| \le L(\sqrt{g(\min\{u,v\})})|u - v|$ for $s, u, v \le -(t_{n+1} - t_n)$. We also have $\|x_t\|_g \le M_{n+1}$ for $t \in [t_n, t_{n+1}]$ and hence x satisfies a Lipschitz condition with $L(M_{n+1})$ on that interval.

Case B: If $M_{n+1} \ge M_n$ then we leave t_n as it is, and because $g(s) \ge M_n^2$ for $s \le t_{n+1} - t_n$ we have $|x_{t_n}(s)| \le \sqrt{g(s)}$ and $|x_{t_n}(u) - x_{t_n}(v)| \le L(\sqrt{g(\min\{u,v\})})|u - v|$ for $s, u, v \le -(t_n - t_{n-1})$. Obviously, x satisfies a Lipschitz condition with $L(M_{n+1})$ on the interval $[t_n, t_{n+1}]$.

This finishes our induction step.

If we ever go in Case I during this induction, then the proof is finished. If we always get Case II, then we have

$$V(t_n) - V(t_1) \le -(n-1)d - (M_{n+1} - M_1)a \le -(n-1)d - (r - R)a,$$

which is a contradiction for large n. This proves that Case I must happen at least once and the proof is complete.

Note that we can replace b by a function $b : \mathbf{R} \to \mathbf{R}$ integrable on any finite interval with $\int_0^\infty b(s)ds = \infty$ and we do not have to change much in the proof. In this case we argue that we cannot have Case 2 of Case II infinitely many times, and hence there is an $N > 0$

such that Case 1 holds for $n \geq N$ and so $V(t_n)-V(t_1) \leq \int_{t_1}^{t_N} |b(s)|ds - \int_{t_N}^{t_n} b(s)ds - (r-R)a$, a contradiction for large n.

In order to make the computations in the proof easier we took a stronger condition in Theorem 9, than it is really neccessary. With more careful investigations one could prove that if $g(s)/|s| \to \infty$ as $s \to -\infty$ then the statement of the theorem still holds. For this we must start the proofs from the beginning of the paper by modifying the definition of $\mathcal{S}(R)$ to let the function get closer to g for $s \leq -S$. Then we prove everything the same way as we did modifying the neccessary parts of the proofs.

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