Periodic Solutions of Linear Differential and Integral Equations

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ABSTRACT: In this paper we consider periodic functional differential and integral equations. Massera proved that the existence of a bounded solution implies the existence of a periodic solution for linear (inhomogeneous) ordinary differential equations. Chow proved a similar result for linear functional differential equations with finite delay. We give a new proof for Chow's result that works for linear functional differential equations with infinite delay and also for integral equations. We also give examples that do not satisfy uniform boundedness or uniform ultimate boundedness, the conditions frequently used to prove that there is a periodic solution.

1. Introduction. In this paper we consider functional differential equations with finite or infinite delay with periodic right-hand side of the form

$$x' = F(t, x_t), \tag{1}$$

i.e., $F(t+T,\phi) = F(t,\phi)$ and integral equations with finite or infinite delay

$$x(t) = F(t, x_t). \tag{2}$$

A classical theorem shows that if the solutions of a functional differential equation are uniformly bounded and uniformly ultimately bounded, then it has a T-periodic solution (for references and a proof see [2]). These are very strong conditions. However, for one or two dimensional ordinary differential equations Massera [9] proved that if there is a bounded solution of the differential equation and the solutions can be continued for all future times then there is a T-periodic solution. He also proved that a linear inhomogeneous ordinary differential system has a T-periodic solution if and only if it has a bounded solution. Chow [5] proved a similar result for functional differential equations with finite delay. He considered the functional differential equation in a special form: the right-hand

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side of the equation was a functional linear in ϕ plus a perturbing term depending only on t. What he actually needs is that the solution set of the functional differential equation is convex, i.e. if $x(t)$ and $y(t)$ are solutions and $\alpha \in [0,1]$ then $\alpha x(t) + (1-\alpha)y(t)$ is also a solution. In the following to refer to this condition we say that the equation is convex or the right-hand side of the equation is convex.

In this paper we generalize Chow's Theorem to convex functional differential equations with infinite delay and to convex integral equations. Note that Chow's (and our) theorems require a condition (the existence of a bounded solution) minimally expected for the existence of a periodic solution, since a periodic solution itself is a bounded solution. Thus, these theorems give a necessary and sufficient condition for having a T-periodic solution. In the case of integral equations we need a condition to ensure, that the solutions we consider there are continuous, because in general we have solutions that are not necessarily continuous. From the proof of our theorems it is obvious, that we cannot drop or weaken (by taking a growth condition on F, for example) the convexity condition on the equation, because we use it to prove the convexity of a set. But our proof uses the convexity only at that particular point, one might be able to find some conditions to ensure convexity in another way. Next, we are interested in finding a bounded solution for convex differential and integral equations. In order to do this, we can use several conditions implying that all solutions are bounded: Burton [1] uses Liapunov functions and functionals for functional differential equations, Cantarelli and Risito [4] apply comparison method for holonomic and scleronomic systems, and Pucci and Serrin [10] consider damped quasi-variational systems with estimates on the damping. Also, uniform boundedness or uniform ultimate boundedness of the solutions imply the existence of a bounded solution. In our examples we show some techniques for finding a bounded solution for an ordinary differential equation and an equation with infinite delay when the equation does not satisfy the above conditions.

There are some related results the reader might want to look at. Yoshizawa [11, Theorem 15.10] proved for ordinary differential systems that if a bounded solution exists with a certain stability property, then there is an mT-periodic solution for some $m > 0$ integer. To prove the existence of a T-periodic solution using comparison theorems and Lyapunov functionals for general (non-convex) periodic functional differential equations with finite delay see [6]. For results on general infinite delay equations using continuation method and Lyapunov functionals see [3]. Hatvani and Krisztin [8] used characteristic roots to prove the existence of a periodic solution for linear inhomogeneous infinite delay equations. Their Theorem 1.2 gives a necessary and sufficient condition, which is (considering the results of this paper) equivalent to the existence of a bounded solution, but the author fails to see a direct proof for this equivalence.

2. Functional differential equations.

Let $(\mathcal{C}, \|\cdot\|)$ be the Banach space of continuous functions $\phi : [-h, 0] \to \mathbf{R}^n$ with the supremum norm, $x_t(s) = x(t+s)$ for $-h \leq s \leq 0$, $F : \mathbf{R} \times \mathcal{C} \to \mathbf{R}^n$ be continuous and locally Lipschitz in ϕ and $F(t+T, \phi) = F(t, \phi)$ for some $T > 0$. Then

$$x' = F(t, x_t) \tag{3}$$

is a system of functional differential equations with finite delay and for each $(t_0, \phi) \in \mathbf{R} \times \mathcal{C}$ there is a unique solution $x(t, t_0, \phi)$, that depends continuously on the initial data. Note that the convexity of F we require later implies a global Lipschitz condition on F. Throughout the proof of the following theorem we insist on not using the convexity of F if we do not have to. One feels that if this theorem is true for convex equations it would be true for "almost convex" equations also, so that our work should make these studies easier.

Theorem 1. *Suppose, that $F(t, \phi)$ is a convex functional in ϕ. If there is a bounded solution of (3) defined on an interval $[\beta, \infty)$, then there is a T-periodic solution.*

Proof. Let M be the maximum of h and T and define S as the Banach space of continuous functions mapping $[-M, 0]$ into $\mathbf{R}^n$ with the supremum norm. In the following we will write $x(t, t_0, \psi)$ for the solution of (3) starting at t_0 with the initial function $\psi \in S$ with the understanding, that if $M > h$ then we use only the values of ψ on the interval $[-h, 0]$. Consider the subset

$$\begin{aligned} U := \{\psi \in S : &|\psi(s)| \leq r \text{ for } s \in [-M, 0], \qquad (a) \\ &|\psi(s_1) - \psi(s_2)| \leq K|s_1 - s_2| \text{ for } s_1, s_2 \in [-M, 0], \qquad (b) \\ &|x(t, 0, \psi)| \leq r \text{ for } t \geq 0\}, \qquad (c) \end{aligned}$$

where r is the bound for the existing bounded solution and $|F(t,\phi)| \le K$ for $\|\phi\| \le r$ (from the local Lipschitz condition on F). First of all we need to show that this subset is non-empty. Consider the bounded solution. Since F is a T-periodic function in t, we may assume (by a translation argument), that the bounded solution $x(t)$ is defined on the interval $[-M-h,\infty)$ and it satisfies the equation on $[-M,\infty)$. Define $\psi_0(s) := x(s)$ for $s \in [-M,0]$ and $\overline{\psi}_0(s) := x(s-M)$ for $s \in [-h,0]$. By definition $x(t) = x(t,0,\psi_0)$ and hence ψ_0 satisfies (a) and (c) in the definition of U. For condition (b) we use the local Lipschitz condition of F to get

$$\begin{aligned}
|(\psi_0)(s_1) - (\psi_0)(s_2)| &= |x(s_1,-M,\overline{\psi}_0) - x(s_2,-M,\overline{\psi}_0)| \\
&= |\int_{s_2}^{s_1} x'(s,-M,\overline{\psi}_0)ds| = |\int_{s_2}^{s_1} F(s,x_s(\cdot,-M,\overline{\psi}_0))ds| \\
&\le |\int_{s_2}^{s_1} K ds| \le K|s_1 - s_2|,
\end{aligned}$$

where we used the boundedness of x. This proves that ψ_0 satisfies condition (b) as well as (a) and (c) in the definition of U, so $\psi_0 \in U$, and hence U is not empty. Next, we prove that this set is convex. If $\psi_1,\psi_2 \in U$ then

$$|\alpha\psi_1(s) + (1-\alpha)\psi_2(s)| \le \alpha|\psi_1(s)| + (1-\alpha)|\psi_2(s)| \le r,$$

and

$$\begin{aligned}
&|(\alpha\psi_1(s_1) + (1-\alpha)\psi_2(s_1)) - (\alpha\psi_1(s_2) + (1-\alpha)\psi_2(s_2))| \le \\
&\alpha|\psi_1(s_1) - \psi_1(s_2)| + (1-\alpha)|\psi_2(s_1) - \psi_2(s_2)| \le K|s_1 - s_2|,
\end{aligned}$$

so conditions (a) and (b) are satisfied for the convex linear combination. For condition (c) we note that by the convexity of the equation (this is the only place we really need it), if $\psi_1,\psi_2 \in U$ then

$$\begin{aligned}
|x(t,0,\alpha\psi_1 + (1-\alpha)\psi_2)| &= |\alpha x(t,0,\psi_1) + (1-\alpha)x(t,0,\psi_2)| \\
&\le \alpha|x(t,0,\psi_1)| + (1-\alpha)|x(t,0,\psi_2)| \le r
\end{aligned}$$

using condition (c) for ψ_1 and ψ_2, and hence condition (c) is also satisfied for the convex linear combination. Also, this set is precompact by Ascoli's Theorem. Moreover, it is compact, because it is closed as we now prove. If $\psi_n \to \psi$ in the supremum norm and $\psi_n \in U$, then ψ clearly satisfies (a) and (b) in the definition of U. To prove condition

(c) suppose for contradiction, that $|x(t,0,\psi)| > r$ for some $t > 0$. By the continuous dependence of the solution on the initial function we find an $n > 0$ such that $|x(t,0,\psi_n)| > r$, a contradiction to the fact that $\psi_n \in U$ so ψ_n satisfies (c). This proves that ψ satisfies (c) and hence $\psi \in U$. Therefore, U is closed, so it is compact. Now let us define the map $P : U \to U$ by

$$(P\psi)(s) := x(s+T,0,\psi) \quad \text{for } -M \le s \le 0.$$

First we prove that P maps into U. By (c) for ψ we find that $P\psi$ is bounded by r and hence $P\psi$ satisfies (a). For condition (c) on $P\psi$ we note that by the T-periodicity of F and the uniqueness of the solution we find that

$$|x(t,0,P\psi)| = |x(t,0,x_T(\cdot,0,\psi))| = |x(t+T,T,x_T(\cdot,0,\psi))| = |x(t+T,0,\psi)| \le r.$$

To prove (b) for $P\psi$ we do exactly the same as we did proving that $\psi_0 \in U$. Hence, $P\psi$ satisfies the conditions (a), (b) and (c), so P indeed maps U into U. Moreover, P is continuous from the continuous dependence of the solution on the initial function. By Schauder's fixed point theorem P has a fixed point, i.e. there is a $\psi \in U$ with $P\psi = \psi$. This means that $x(s+T,0,\psi) = \psi(s)$ for $s \in [-h,0]$ and hence by the periodicity of F and the uniqueness of the solution we find, that $x(s+T,0,\psi) = x(s,0,\psi)$ for all $s \ge 0$. Therefore, $x(t,0,\psi)$ is a T-periodic solution of (3) and the proof is complete.

Note that this proof does not require that solutions can be continued for all future times, although the convexity implies that. Also, observe that the periodic solution we find in the proof is bounded by the bound of the bounded solution.

We now consider the functional differential equation

$$x' = F(t,x_t) \tag{4}$$

with infinite delay, i.e. $x_t(s) = x(t+s)$ for $s \le 0$. Define the Banach space $\mathcal{C}$ of bounded, continuous functions $\phi : (-\infty,0] \to \mathbf{R}^n$ with the supremum norm $\|\cdot\|$. We assume that $F(t+T,\phi) = F(t,\phi)$ for some $T > 0$ and $F : \mathbf{R} \times \mathcal{C} \to \mathbf{R}^n$ is a continuous function. We also need a Lipschitz-type condition on F to prove uniqueness and continuous dependence. In the following Lemma we prove this condition using a quite strong-looking condition, but this condition is frequently used in the literature.

Lemma 1. *Suppose that there is a monotone decreasing function* $g : (-\infty, 0] \to [1, \infty)$ *with* $g(s) \to \infty$ *as* $s \to -\infty$ *such that whenever an* $r > 0$ *is given there is a* $K > 0$ *such that*

$$|F(t, \psi_1) - F(t, \psi_2)| \le K \|\psi_1 - \psi_2\|_g$$

for $\|\psi_i\|_g \le r$ $(i = 1, 2)$*, where*

$$\|\phi\|_g := \sup_{s \in (-\infty, 0]} \frac{|\phi(s)|}{g(s)}.$$

Then the solutions are unique and for all $\varepsilon > 0$ *and* $M > 0$ *there is a* $\delta > 0$ *such that if* $\|\phi_1 - \phi_2\|_g < \delta$ *then* $\|x_M(\cdot, 0, \phi_1) - x_M(\cdot, 0, \phi_2)\|_g < \varepsilon$*, where* $x(\cdot, 0, \phi)$ *is the solution of (4) starting at 0 with the initial function* ϕ*.*

Proof. We prove only the last part of the statement (the continuous dependence of the solution on the initial data), the uniqueness easily follows from that. Integrating the equation we have

$$x(t) - x(0) = \int_0^t F(s, x_s) ds.$$

Applying this equality for a solution y and subtracting these we find

$$x(t) - y(t) = x(0) - y(0) + \int_0^t F(s, x_s) - F(s, y_s) ds.$$

Thus, using the Lipschitz condition on F

$$\begin{aligned} |x(t) - y(t)| &\le |x(0) - y(0)| + \int_0^t |F(s, x_s) - F(s, y_s)| ds \\ &\le g(0) \|x_0 - y_0\|_g + \int_0^t K \|x_s - y_s\|_g ds. \end{aligned}$$

Taking the supremum norm on the interval $[0, t]$ on both sides we obtain

$$\sup_{s \in [0,t]} |x(s) - y(s)| \le g(0) \|x_0 - y_0\|_g + \int_0^t K \|x_s - y_s\|_g ds.$$

Because g is monotone decreasing we find that

$$g(0) \sup_{s \in [0,t]} \frac{|x(s) - y(s)|}{g(s - t)} \le \sup_{s \in [0,t]} |x(s) - y(s)|$$

and

$$g(0)\sup_{s\le 0}\frac{|x(s)-y(s)|}{g(s-t)} \le g(0)\sup_{s\le 0}\frac{|x(s)-y(s)|}{g(s)} = g(0)\|x_0-y_0\|_g,$$

therefore, putting together the last three inequalities we arrive at

$$g(0)\|x_t-y_t\|_g \le g(0)\|x_0-y_0\|_g + \int_0^t K\|x_s-y_s\|_g ds.$$

Applying Gronwall's inequality we obtain

$$\|x_t-y_t\|_g \le \|x_0-y_0\|_g e^{\frac{Kt}{g(0)}},$$

which proves the continuous dependence. This completes the proof of the lemma.

Theorem 2. *Suppose, that $F(t,\phi)$ is a convex functional in ϕ. If there is a bounded solution of (4) defined on the interval $(-\infty,\infty)$, then there is a T-periodic solution.*

Proof. We do exactly the same as in Theorem 1, with only slight changes. Define the Banach space S of continuous functions defined on the interval $(-\infty,0]$ with the norm $\|\cdot\|_g$. Define the subset U exactly the same way as we did before. The rest of the proof is very similar to that of Theorem 1 with $M=-\infty$, we emphasize only some points, where more severe modifications are needed. To prove that U is precompact we cannot use Ascoli's Theorem, because we have an infinite interval. Instead, consider a sequence $\{\phi_n\}\subset U$. Using Ascoli's Theorem for the interval $[-1,0]$ we find a subsequence $\{\phi_n^{(1)}\}$ converging in the supremum norm. For this sequence we find a subsequence $\{\phi_n^{(2)}\}$ converging on $[-2,0]$, etc. Let $\psi_n := \phi_n^{(n)}$, then ψ_n converges uniformly on any finite subinterval of $(-\infty,0]$, in particular, it converges pointwise to a function ψ. Since $\psi_n\in U$, ψ_n is bounded by r, and hence ψ is bounded by r, so it is in S. Moreover,

$$\begin{aligned}\|\psi_n-\psi\|_g &\le \max\left\{\sup_{s\le -L}\frac{|\psi_n(s)|+|\psi(s)|}{g(s)}, \sup_{s\in[-L,0]}\frac{|\psi_n(s)-\psi(s)|}{g(s)}\right\}\\ &\le \max\left\{\sup_{s\le -L}\frac{2r}{g(s)}, \sup_{s\in[-L,0]}|\psi_n(s)-\psi(s)|\right\}\le\varepsilon,\end{aligned}$$

where L is chosen large enough, so that $g(s)\ge 2r/\varepsilon$ if $s\le -L$, and n large enough, so that $\sup_{s\in[-L,0]}|\psi_n(s)-\psi(s)|\le\varepsilon$. So this proves, that $\psi_n\to\psi$ in $\|\cdot\|_g$ norm. Therefore, we have found a subsequence of $\{\phi_n\}$ that converges in S, and hence U is precompact. Since

U is closed (can be proved as before), U is compact. Defining P and proving everything about it is the same as before; we can find a periodic solution.

Note that g can be a bit more general than it is stated in Lemma 1. With a little care one can modify the proof of Lemma 1 and Theorem 2, so that g satisfies only the following conditions: $g(s) \to \infty$ as $s \to -\infty$ and $\sup_{0 \le v \le B} \sup_{s \le 0} g(s)/g(s-v) < \infty$ for all $B > 0$. Naturally, if g is monotone decreasing, then $\sup_{0 \le v \le B} \sup_{s \le 0} g(s)/g(s-v) \le 1$ for all $B > 0$.

3. Integral equations. We now consider the integral equation

$$x(t) = F(t, x_t) \tag{5}$$

with infinite delay, i.e., $F : \mathbf{R} \times \mathcal{C} \to \mathbf{R}^n$ is continuous and T-periodic in t, where $\mathcal{C}$ is the Banach space of bounded functions $\phi : (-\infty, 0] \to \mathbf{R}^n$ with the supremum norm. Note, that we do not require ϕ to be continuous; solutions of integral equations can have discontinuity at the initial time (and any time after that) even if the initial function is "nice". Also, this is the reason why we need to ask in our theorems the existing bounded solution to be continuous. The finite delay case is a special case of (5), we emphasize only the differences in the following. We assume that (5) has a unique solution $x(t, t_0, \phi)$ for any $(t_0, \phi) \in \mathbf{R} \times \mathcal{C}$ which depends continuously on the initial data (using the $\|\cdot\|_g$ norm as before, or the $\|\cdot\|$ norm for the finite delay case), see e.g. [7, Chapters 11-13]. We do not restrict our investigations to specific forms of (5) at first, so we need a condition, that replaces the local Lipschitz condition on F in Theorem 1 and 2.

Theorem 3. *Suppose, that for all $r > 0$ there is a continuous function $W : [0, \infty) \to [0, \infty)$ with $W(0) = 0$ such that $|F(t_1, x_{t_1}) - F(t_2, x_{t_2})| \le W(|t_1 - t_2|)$ for all $t_1, t_2 \in \mathbf{R}$ and for any continuous function x with $\sup |x(t)| \le r$ (we may call $F(t, x_t)$ equi-continuous in t uniformly for x with $\sup |x(t)| \le r$). Assume also, that $F(t, \phi)$ is a convex functional in ϕ. If there is a bounded continuous solution of (5), then there is a T-periodic solution.*

Proof. We do exactly the same as in the proof of Theorem 1 and 2 with only small changes. Let S be the set of bounded continuous functions in $\mathcal{C}$, r be the bound for the existing bounded solution and W the function defined above for this r. We define U a little bit differently this time:

$$U := \{\psi \in S : |\psi(s)| \le r \text{ for } s \in [-M, 0], \qquad (a)$$

$$|\psi(s_1) - \psi(s_2)| \le W(|s_1 - s_2|) \text{ for } s_1, s_2 \in [-M, 0], \qquad (b)$$

$$x(t, 0, \psi) \text{ is continuous and } |x(t, 0, \psi)| \le r \text{ for } t \ge 0\}. \qquad (c)$$

For the rest of the proof we need to be careful only when we use the Lipschitz condition on F in the proof of Theorem 1 or 2. Instead, we say (when proving that U is not empty, for example) that

$$|(\psi_0)(s_1) - (\psi_0)(s_2)| = |x(s_1, -M, \overline{\psi}_0) - x(s_2, -M, \overline{\psi}_0)|$$
$$= |F(s_1, x_{s_1}(\cdot, -M, \overline{\psi}_0)) - F(s_2, x_{s_2}(\cdot, -M, \overline{\psi}_0))| \le W(|s_1 - s_2|).$$

Also, Ascoli's theorem can be applied, since condition (b) implies the equi-continuity of the functions in U. The remainder of the proof is exactly the same as before.

Although the equi-continuity of F in t uniformly in x may seem to be a strong condition, we show in the next section that the continuity, periodicity and convexity of F imply this condition, if F (the right-hand side of an *integral* equation) is indeed an integral form. But there are some quite innocent-looking examples (the difference equation $x(t) = a(t) + b(t)x(t - h)$, for example), for which this condition is not satisfied. For these equations we can prove the following

Theorem 4. *Suppose, that $F(t, \phi)$ is a convex functional in ϕ. If there is a bounded, uniformly continuous solution $\overline{x}$ of (5), then there is a T-periodic solution.*

Proof. We need a definition for the proof.

Definition. We say that *the function $y : \mathbf{R} \to \mathbf{R}^n$ is in the equi-continuity class of the uniformly continuous function $x : \mathbf{R} \to \mathbf{R}^n$*, if y is also uniformly continuous and whenever $\delta > 0$ is of the uniform continuity for $\varepsilon > 0$ for x then the same δ works for y in the definition of uniform continuity for ε.

Once again, we define U in a different way. Let r be the bound of the uniformly continuous solution $\overline{x}(t)$, and define

$$U := \{\psi \in S : |\psi(s)| \le r \text{ for } s \in [-M, 0], \qquad (a)$$

$$\text{the solution } x(t, 0, \psi) \text{ is in the equi-continuity class of } \overline{x}, \qquad (b)$$

$$|x(t, 0, \psi)| \le r \text{ for } t \ge 0\}. \qquad (c)$$

Clearly, the functions in U are equi-continuous. The rest of the proof is the same as that of Theorem 3.

4. EXAMPLES

Since either uniform boundedness, or uniform ultimate boundedness, or boundedness of all solutions implies the existence of a bounded solution, we shall give examples not satisfying these properties. Moreover, we demonstrate how our theorem works for functional differential equations with infinite delay.

Now let us consider the system

$$x'(t) = A(t)x(t) + f(t),$$

where A is an $n \times n$ matrix of continuous, T-periodic functions and f is an n dimensional vector of continuous, T-periodic functions. Suppose that all eigenvalues of $A(t) + A^T(t)$ have positive real parts for all $t \in \mathbf{R}$. Then there is a T-periodic solution of this equation.

Proof. As A is continuous and T-periodic we find an $\alpha > 0$ such that $x^T A(t)x \geq \alpha x^T x = \alpha|x|^2$. Also, we can find a $B > 0$ such that $|f(t)| \leq B$. Thus, there is an $M > 0$ such that if $|x| \geq M$ then $x^T(A(t)x + f(t)) \geq \alpha|x|^2 - B|x| > 0$, i.e., if a solution of the differential equation ever gets outside the M-ball it stays outside the M-ball. Let us pick an $x_0 \in \mathbf{R}^n$ on the surface of the M-ball, and consider the solution $x : (-\infty, \infty) \to \mathbf{R}^n$ with $x(0) = x_0$. Thus, as $x(0)$ is on the boundary of the M-ball, $x(s)$ must be in the M-ball for $s \leq 0$. The compactness of the M-ball gives that the set $\{x(-mT) : m \geq 0\}$ has a limit point x_1 in the M-ball. Consider the solution $x(t, 0, x_1)$. We claim that this solution remains in the M-ball. Suppose for contradiction, that $|x(K, 0, x_1)| > M$ for some $K > 0$. Let $\varepsilon > 0$ be small enough, so that $|x(K, 0, x_1)| > M + \varepsilon$. By the continuous dependence of the solution on the initial data we find a $\delta > 0$ such that $|x_1 - y| < \delta$ implies $|x(K, 0, x_1) - x(K, 0, y)| < \varepsilon$ and hence $|x(K, 0, y)| > M$. As x_1 is the limit point of the set $\{x(-mT) : m \geq 0\}$ we find an $m \geq 0$ such that $mT > K$ and $|x_1 - x(-mT, 0, x_0)| < \delta$. Thus, we must have $|x(K, 0, x(-mT, 0, x_0))| > M$. On the other hand, the periodicity of the equation implies that

$$x_0 = x(0, -mT, x(-mT, 0, x_0)) = x(mT, 0, x(-mT, 0, x_0)),$$

and hence $x(s,0,x(-mT,0,x_0))$ remains in the M-ball for $s \in [0,mT]$. This is a contradiction to the fact that $|x(K,0,x(-mT,0,x_0))| > M$, which proves that $x(t,0,x_1)$ is bounded by M for all $t \geq 0$. Since we have a bounded solution, by Theorem 1 we have a periodic solution as well.

Note that we cannot prove that the solution $x(t,0,x_1)$ itself T-periodic or even periodic of some period. For $n \leq 2$ we do know something about this solution: x_1 is an equilibrium point if $n=1$, $x(t,0,x_1)$ is a periodic solution or an equilibrium point for $n=2$. Even for $n=2$ the period of $x(t,0,x_1)$ is of the form mT for some $m>0$, but not necessarily T. Note also, that if the eigenvalues of $A(t)+A^T(t)$ have negative real parts, then we can prove uniform boundedness and uniform ultimate boundedness, which implies the existence of a T-periodic solution.

As our next example consider the functional differential equation with infinite delay

$$x'(t) = a(t)x(t) + \sum_{i=1}^{n} b_i(t)x(t-h_i(t)) + \int_{-\infty}^{t} c(t,s)x(s)ds + f(t),$$

where a, b_i, h_i, c and f are continuous and T-periodic functions (for c it means $c(t+T,s+T) = c(t,s)$), $h_i(t) \geq 0$ and $\int_{-\infty}^{t} |c(t,s)|ds < \infty$ for all $t \in \mathbf{R}$. Note that these conditions are enough to ensure the existence of a g satisfying the conditions of the lemma, so we have uniqueness and continuous dependence of the solutions in the initial data in the $\|\cdot\|_g$ norm. Suppose that $a(t) - \sum_{i=1}^{n} |b_i(t)| - \int_{-\infty}^{t} |c(t,s)|ds \geq \alpha$ for all $t \in \mathbf{R}$ and some $\alpha > 0$. Then there is a T-periodic solution of this equation.

Proof. From continuity and periodicity we find a $B>0$ such that $|f(t)| \leq B$ for all t. This implies that if $\phi(0) \geq M := 1 + (B/\alpha)$ and $\|\phi\| \leq \phi(0)$, then

$$a(t)\phi(0) + \sum_{i=1}^{n} b_i(t)\phi(-h_i(t)) + \int_{-\infty}^{t} c(t,s)\phi(s-t)ds + f(t) \geq$$
$$\left(a(t) - \sum_{i=1}^{n} |b_i(t)| - \int_{-\infty}^{t} |c(t,s)|ds\right)\phi(0) - B \geq \alpha M - B = \alpha > 0.$$

Similarly, if $\phi(0) \leq -M$ and $\|\phi\| \leq |\phi(0)|$, then

$$a(t)\phi(0) + \sum_{i=1}^{n} b_i(t)\phi(-h_i(t)) + \int_{-\infty}^{t} c(t,s)\phi(s-t)ds + f(t) \leq -\alpha < 0.$$

Consider the set

$$S := \{\phi : (-\infty, 0] \to \mathbf{R} : \phi \text{ is a constant function}\}.$$

Denote by

$$S^+ := \{\phi \in S : x(t,0,\phi) > M \text{ for some } t \geq 0, \text{ and } x(s,0,\phi) > -M \text{ for } s \in [0,t]\}$$

and

$$S^- := \{\phi \in S : x(t,0,\phi) < -M \text{ for some } t \geq 0, \text{ and } x(s,0,\phi) < M \text{ for } s \in [0,t]\}.$$

By the continuous dependence of the solution on the initial condition if $\phi \in S^+$, $\psi \in S$ and $\|\phi - \psi\|_g = |\phi(0) - \psi(0)|/g(0)$ is small enough, then $|x(t,0,\phi) - x(t,0,\psi)|$ is arbitrarily small on a compact t-interval. Since for some $t \geq 0$ we have $x(t,0,\phi) > M$, for small enough $\|\phi - \psi\|_g$ we obtain $x(t,0,\psi) > M$. Similarly, $x(s,0,\psi) > -M$ for $s \in [0,t]$ too, so $\psi \in S^+$. This proves that S^+ is an open set in S. A similar proof gives that S^- is also open in S. Clearly, S^+ and S^- are disjoint and from the above estimations $S^+ \supset \{\phi \in S : \phi \geq M\}$ and $S^- \supset \{\phi \in S : \phi \leq -M\}$, so S^+ and S^- are not empty. This implies, that $S^+ \cup S^-$ does not cover all S and hence there is a $\phi \in S$ with $\|\phi\| = |\phi(0)| < M$ and $\phi \notin S^+ \cup S^-$. From our first note in this proof it can be seen, that if $x(t,0,\phi)$ is ever equal to M ($-M$ respectively) for the first time then $x(t,0,\phi)$ become strictly monotone increasing (decreasing) and hence $\phi \in S^+$ ($\phi \in S^-$). Therefore, we must have $|x(t,0,\phi)| < M$ for all $t \geq 0$, so we have found a bounded solution. Using Theorem 2 we prove the existence of a periodic solution.

Note that as the function $a(t) - \sum_{i=1}^n |b_i(t)| - \int_{-\infty}^t |c(t,s)|ds$ is continuous and T-periodic, if it is greater than zero, then we can find an $\alpha > 0$ bounding it away from zero. Also, if we had $a(t) + \sum_{i=1}^n |b_i(t)| + \int_{-\infty}^t |c(t,s)|ds \leq -\alpha < 0$, we could prove uniform boundedness and uniform ultimate boundedness, which, of course, implies the existence of a T-periodic solution. Observe also, that for a class of continuous functions $\phi : (-\infty, 0] \to \mathbf{R}$ we can use the above proof to find a $\beta \in \mathbf{R}$ such that $x(t,0,\beta\phi)$ is a bounded solution. For example, if $|\phi(0)| \geq \|\phi\|$ then we can do this. So far this last note does not have any practical use, but we are in a new territory searching for just one bounded solution, it may be important later.

We now concentrate on showing that the equi-continuity of F in t uniformly for x (in the case of integral equations) is not a strong condition. Consider

$$x(t) = \int_{-\infty}^{t} c(t,s)x(s)ds + f(t),$$

where c and f are continuous, T-periodic functions, and $\int_{-\infty}^{t} |c(t,s)|ds$ is continuous and finite for $t \in \mathbf{R}$. Let $r > 0$ be given, x be any continuous function with $|x(t)| \leq r$, suppose that $t_1 < t_2$ and consider

$$|F(t_1, x_{t_1}) - F(t_2, x_{t_2})| \leq r \int_{t_1}^{t_2} |c(t_2,s)|ds + r \int_{-\infty}^{t_1} |c(t_1,s) - c(t_2,s)|ds + |f(t_1) - f(t_2)|.$$

Clearly, it is enough to prove the upper bound on $|F(t_1, x_{t_1}) - F(t_2, x_{t_2})|$ for small $|t_1 - t_2|$, since for larger numbers it follows from that. Let us look closely at the terms in the upper bound. Since $c(t,s)$ is T-periodic and continuous, $|c(t,s)| \leq C$ for all $t \in \mathbf{R}$, $s \in [t-1, t]$ and some $C > 0$. Consequently, $\int_{t_1}^{t_2} |c(t_2,s)|ds \leq C|t_1 - t_2|$ for $|t_1 - t_2| \leq 1$. To prove the bound on the second term, we only need to find a $\delta > 0$ for each $\varepsilon > 0$ such that if $|t_1 - t_2| < \delta$ then $\int_{-\infty}^{t_1} |c(t_1,s) - c(t_2,s)|ds < \varepsilon$ (we can then define a W by $W(\delta) := \varepsilon$). Since c is T-periodic, so is $\int_{-\infty}^{t} |c(t,s)|ds$. Also, $\int_{-\infty}^{t} |c(t,s)|ds$ is continuous and finite and hence we can find a $K > 0$ (independent of t) such that $\int_{-\infty}^{t-K} |c(t,s)|ds < \varepsilon/4$ for all t. Then the periodicity and continuity of c imply that for some small enough $\delta > 0$ if $|t_1 - t_2| < \delta$ then we have $|c(t_1,s) - c(t_2,s)| < \varepsilon/2K$ for $s \in [t_1 - K, t_1]$. Putting together these remarks we arrive at

$$\begin{aligned}\int_{-\infty}^{t_1} |c(t_1,s) - c(t_2,s)|ds &\leq \int_{-\infty}^{t_1-K} |c(t_1,s)| + |c(t_2,s)|ds + \int_{t_1-K}^{t_1} |c(t_1,s) - c(t_2,s)|ds \\ &< 2\frac{\varepsilon}{4} + \int_{t_1-K}^{t_1} \frac{\varepsilon}{2K} ds = \varepsilon.\end{aligned}$$

This proves the bound on the second term. Clearly, f is continuous and T-periodic, so it is uniformly continuous and hence the third term has the necessary upper bound too. Thus, we see that we do not need any extra conditions on this equation to show that $F(t, x_t)$ is equi-continuous for $|x(t)| \leq r$.

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