Uniform Boundedness and Uniform Ultimate Boundedness for Functional Differential Equations

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1. Introduction.

Consider the functional differential equation

$$x' = f(t, x_t) \tag{1}$$

with finite delay, i.e. let $\mathcal{C}$ be the Banach space of continuous functions $\phi : [-h, 0] \to \mathbf{R}^r$ with the supremum norm $\|\cdot\|$, $x_t(s) = x(t+s)$ for $s \in [-h, 0]$. Assume that f is continuous and locally Lipschitz, so there is a unique solution $x(t, t_0, \phi)$ with $x_{t_0} = \phi$ of the initial value problem for (1) which depends continuously on the initial data.

In this paper we study some conditions in connection with uniform boundedness (UB) and uniform ultimate boundedness (UUB). We give the definitions of some properties (called UB and UUB at a time t_0) that are weaker than UB and UUB. Then we prove theorems for finite and infinite delay equations using Liapunov functionals that ensure these properties. We also give examples to illustrate the connections and differences between the definitions. The importance of this work can be seen by noting that if solutions are UB and UUB at t_0 and the right-hand side of the equation is T-periodic in t, then there is a T-periodic solution (see e.g. [1] for the finite and [2] for the infinite delay case). Also, there are several spaces that can be considered in the infinite delay case, He, Huang and Wang [3] show some of them and prove theorems using Liapunov functionals to ensure UB and UUB. Hering [4], [5] worked with spaces and methods to prove UB and UUB similar to those we are concerned with here.

2. Main results.

First we introduce our definitions of uniform boundedness and uniform ultimate boundedness. They can be found with $t_0 = 0$ in [1] but we need them for arbitrarily fixed t_0.

Definition. We say that the solutions of (1) are *uniform bounded at* t_0 *(UB(*t_0*))* if for all $B_1 > 0$ there is a $B_2 > 0$ such that $t \geq t_0$ and $\phi \in \mathcal{C}$ with $\|\phi\| < B_1$ imply $|x(t, t_0, \phi)| < B_2$.

Definition. We say that the solutions of (1) are *uniform ultimate bounded at* t_0 *(UUB(*t_0*))* if there is a $B > 0$ such that for all $B_3 > 0$ there is a $K > 0$ such that $t \geq t_0 + K$, $\phi \in \mathcal{C}$ with $\|\phi\| < B_3$ imply $|x(t, t_0, \phi)| < B$.

Note, that the only difference between these definitions and those in [1] is the initial time when we start the solutions. By a simple change of time one could prove that the solutions of $x' = f(t, x_t)$ are UB(t_0) (UUB(t_0), respectively) if and only if the solutions of $x' = f(t - t_0, x_t)$ are UB(0) (UUB(0)). It is also clear, that these conditions are generalizations of the properties known as uniform boundedness and uniform ultimate boundedness, since in those we have the constants B_2, B and K independent of t_0 and the estimations for all t_0. With $h = 0$ these definitions work for ordinary differential equations as well.

Theorem 1: *Suppose that* $f(t + T, \phi) = f(t, \phi)$ *and for all* $B > 0$ *there is an* $L > 0$ *such that* $(t, \phi) \in \mathbf{R} \times \mathcal{C}$ *with* $\|\phi\| \leq B$ *implies* $|f(t, \phi)| \leq L$. *Assume also, that the solutions of (1) are UB and UUB at* t_0. *Then there is a* T*-periodic solution of (1).*

Proof: The proof is the same as that of Theorem 4.2.2 in [1], the only difference is that instead of starting the solutions at 0 we need to start them at t_0. Actually, that theorem is the same as this one with $t_0 = 0$ and does not use the stronger UB or UUB.

The condition on f we have in this theorem is called the boundedness condition. Note that, if f is locally Lipschitz, then we have this condition. We will clarify what kind of local Lipschitz condition we are talking about here, and how we can prove the boundedness condition needed for this theorem.

To see the point in these definitions we now give an example, where the solutions are not UB or UUB but they are UB and UUB at 0. To this end we need the function $g : \mathcal{C} \times [-h, 0] \times [-h, 0] \to \mathbf{R}$ defined by

$$g(\phi, \alpha, \beta) = \sup_{\alpha \le s_0 \le \ldots \le s_n \le \beta} \sum_{k=1}^{n} \max\{0, \phi(s_k) - \phi(s_{k-1}) - 1\}.$$

For this definition to make sense we always assume that $\alpha \le \beta$. Kato [6] defined a function g very similar to our function g and proved that it is continuous and locally Lipschitz in ϕ. It is also clear that g is continuous in α and β too. To clarify matters we note that g is a function, which gives the "large increasing variation" of ϕ on the interval $[\alpha, \beta]$. We also define a function, which is (more or less) a sign function:

$$h(s) := \begin{cases} 1, \text{ if } s \ge 1 \\ s, \text{ if } s \in [-1, 1] \\ -1, \text{ if } s \le -1 \end{cases}$$

Now we are ready to define f. Let $t \in [2n, 2n+2]$ for some integer n, then

$$f(t, \phi) := \begin{cases} \min\{g(\phi, 2n - t, 0) - 1, 4|\phi(0)|^2\} h(\phi(0)), \text{ if } t \in [2n, 2n+1] \\ \min\{g(\phi, -1, 2n + 1 - t) - 1, 4|\phi(0)|^2\} h(\phi(0)), \text{ if } t \in [2n+1, 2n+2] \end{cases}$$

where $h = 1$ and $T = 2$. It is clear, that f is continuous and locally Lipschitz, since g and h are. But, unfortunately, we have two slightly different definitions of the local Lipschitz condition:

Definition. We say that a function $f : \mathbf{R} \times \mathcal{C} \to \mathbf{R}^r$ is *weakly locally Lipschitz in its second variable*, if for all $(t_0, \phi) \in \mathbf{R} \times \mathcal{C}$ there is a $\delta > 0$ and there is an $L > 0$ such that $|t - t_0| < \delta$ and $\|\psi - \phi\| < \delta$ imply $|f(t, \psi) - f(t, \phi)| \le L\|\psi - \phi\|$.

Definition. We say that a function $f : \mathbf{R} \times \mathcal{C} \to \mathbf{R}^r$ is *strongly locally Lipschitz in its second variable*, if for all $(t_0, \phi) \in \mathbf{R} \times \mathcal{C}$ and for all $\delta > 0$ there is an $L > 0$ such that $|t - t_0| < \delta$ and $\|\psi - \phi\| < \delta$ imply $|f(t, \psi) - f(t, \phi)| \le L\|\psi - \phi\|$.

The only difference in the two definitions is the way in which δ is defined. In his paper [6] Kato used the weak local Lipschitz condition when he said that g is locally Lipschitz, but unfortunately we need the strong one for f to prove the boundedness condition in Theorem 1. Note, that a weak local Lipschitz condition for f is enough to prove that solutions are unique and depend continuously on the initial data. We are unable to prove the strong local Lipschitz condition for this particular f, but we have the boundedness condition needed. That is why we put $4|\phi(0)|^2$ into the argument of the minimum in the definition of f. Since $4|\phi(0)|^2$ satisfies the boundedness condition, so does f. But $4|\phi(0)|^2$ is a good enough right-hand side for a differential equation to ensure, that the solution blows up fast, so in our example we can be sure that the big influence of g will be in effect as we describe it in the following.

Let us consider how solutions of our example behave. First note that if $x(t) = x(t, t_0, \phi)$ ever becomes 0, then it remains 0 for ever after. Thus, solutions do not change sign. To be definite, consider a solution $x(t) = x(t, t_0, \phi)$ with $\phi(0) > 0$, so $x(t) \geq 0$ for all $t \geq t_0$. Since $g \geq 0$ and $h \leq 1$, $x'(t) \geq -1$. Define n to be the integer with $t_0 \in (2n-2, 2n]$. From the definition of f we see that $x'(2n) = -h(x(2n)) \leq 0$ and hence $x(t)$ is monotone decreasing for $t \in [2n, \infty)$. Moreover, if $x(t) > 1$ then $x'(t) = -1$ and x is a linear function, if $x(t) \leq 1$ then $x'(t) = -x(t)$ and x is an exponentially decreasing function. A similar proof works if $\phi(0) < 0$. Therefore, if we start our solution at $t_0 = 2n$ then the above consideration proves, that the solutions are UB(t_0) (for every $B_1 > 0$ let $B_2 := B_1$) and UUB(t_0) (let $B := 1$, then for $B_3 > 0$ we may choose $K := B_3 - 1$).

But the solutions are not UB or UUB at any other point (we will prove it for only one), and hence they are not UB or UUB. To see this, we take a function ϕ which is oscillatory with amplitude greater than 1. If we start the solution at $t_0 = 2n + 1$, our function g in the definition of f is applied to a segment containing $[-1/2, 0]$ of ϕ while t is in $[2n + 1, 2n + (3/2)]$ and it gives larger and larger numbers as ϕ has higher and higher frequency. (Note, that although g satisfies a weak local Lipschitz condition, ϕ may be bounded by 1, but g is not bounded for these ϕ's, so g does not satisfy a strong local Lipschitz condition.) The minimum makes everything a little bit complicated, but one can argue as follows. From the above we know that $g(x_t, -1, 2n + 1 - t) \geq g(\phi, -1/2, 0)$ for $t \in [2n + 1, 2n + (3/2)]$, where $x(t) = x(t, t_0, \phi)$ is the solution. We can choose ϕ so that $g(\phi, -1/2, 0) > 1$ and $\phi(0) > 1$. From the definition of f it is clear that $x(t)$ is monotone increasing on the interval $[2n + 1, 2n + (3/2)]$. Thus, in the following, we do not have to worry about the function h, since it is 1 along this solution on the intervals considered here. We claim that the solution satisfies $x'(t) = g(x_t, -1, 2n+1-t) - 1$ for $t \in [2n + (5/4), 2n + (3/2)]$. If this is not true, then since $g(x_t, -1, 2n+1-t)$ is monotone decreasing, but $4|x(t)|^2$ is monotone increasing we must have $x'(t) = 4|x(t)|^2$ on the interval $[2n+1, 2n+(5/4)]$. But the solution of this equation (with $t_0 = 2n+1$ and $\phi(0) > 1$) blows up before $2n+(5/4)$, which is not possible, since $x' \le g(x_t, -1, 2n+1-t)-1 \le g(\phi, -1, 0)-1$ (a fixed constant for every ϕ) for $t \in [2n+1, 2n+(3/2)]$. This contradiction shows, that $x'(t) = g(x_t, -1, 2n+1-t) - 1 \ge g(\phi, -1/2, 0) - 1$ for $t \in [2n + (5/4), 2n + (3/2)]$. Hence, the solution can be as large as one wants at $t = 2n$, because one can have any large number of the form $g(\phi, -1/2, 0)$ with bounded ϕ's. Therefore, the solutions are not UB(t_0). Also, since the solution is very large at $t = 2n$ and cannot decrease faster than a linear function with slope 1, we cannot find a suitable B in the definition of UUB(t_0).

The above example shows, that there are differential equations with solutions that are not UB or UUB but UB(t_0) and UUB(t_0) for some t_0. Since we also have $|x(t, t_0, \phi)| \to 0$ for all $(t_0, \phi) \in \mathbf{R} \times \mathcal{C}$, the example shows that dissipativeness (see e.g. [7]) does not imply UB or UUB. With some slight changes one can even show that dissipativeness does not imply UB or UUB at any t_0. Another interesting connection between these properties is shown in the following

Theorem 2: *If the solutions of (1) are uniformly bounded, if they are uniformly ultimately bounded at $\bar{t}_0$, and if the right-hand side of (1) is T-periodic in t, then the solutions are uniformly ultimately bounded.*

Proof: Let $B > 0$ be that of uniform ultimate boundedness at $\bar{t}_0$. Since (1) is T-periodic, using a translation argument it is easy to prove that the solutions are uniformly ultimately bounded at any $\bar{t}_n = \bar{t}_0 + nT$ with the same bound B. Moreover, K is independent of n in the uniform ultimate boundedness. Let $B_3 > 0$ be given, define B_4 to be the B_2 of uniform boundedness for $B_1 = B_3$. For this B_4 we find a $K > 0$ from the uniform ultimate boundedness at $\bar{t}_n$ (for any n). Now, let $(t_0, \phi) \in \mathbf{R} \times \mathcal{C}$ be given with $\|\phi\| < B_3$. Choose the smallest $\bar{t}_n$ with $t_0 \le \bar{t}_n$. From the uniform boundedness we have $\|x_{\bar{t}_n}(\cdot, t_0, \phi)\| < B_4$ and hence from the uniform ultimate boundedness $|x(\bar{t}_n + t, t_0, \phi)| < B$ for $t \ge K$. This proves that if $(t_0, \phi) \in \mathbf{R} \times \mathcal{C}$ with $\|\phi\| < B_3$ then $|x(t_0 + t, t_0, \phi)| < B$ for $t \ge K + T$, which was to be shown.

The next natural question is, how can we prove that solutions are UB and UUB at t_0. We modify Theorem 4.2.11 in [1] to fit our needs.

Theorem 3: *Suppose there are a continuous functional* $V : \mathbf{R} \times \mathcal{C} \to \mathbf{R}$*, a sequence* $t_n \in \mathbf{R}$ *with* $h \le t_n - t_{n-1} \le k$ *for some* $k > 0$ *and constants* $a, b, M, U > 0$ *such that*

(i) $W_1(|\phi(0)|) \le V(t, \phi)$ *and* $V(t_n, \phi) \le W_2(\|\phi\|)$

(ii) $V'(t, x_t) \le M$

(iii) $V'(t, x_t) \le -a|x'(t)| - b$ *for* $|x(t)| \ge U$*,*

where $W_1(r) \to \infty$ *as* $r \to \infty$*. Then the solutions of (1) are uniformly bounded and uniformly ultimately bounded at* t_n *for any* n*. Also,* B_2 *of the uniform boundedness and* B*,* K *of uniform ultimate boundedness does not depend on* n*.*

Proof: First we prove uniform boundedness. Clearly, the proof is the same for any n, and we will see that B_2 does not depend on n, so we prove everything for t_0. Let $B_1 > 0$ be given; we need to find a $B_2 > 0$ such that $\phi \in \mathcal{C}$ with $\|\phi\| < B_1$ implies $|x(t, t_0, \phi)| < B_2$ for $t \ge t_0$. Define $L \ge B_1$ so that $-aL + Mk < -\alpha$ for some $\alpha \in (0, bh)$. Define B_2 by

$$B_2 := W_1^{-1}\left(W_2(U+L) + Mk\left(2 + \frac{W_2(U+L)}{\alpha}\right)\right)$$

and suppose for contradiction that $|x(t)| = |x(t, t_0, \phi)| \ge B_2$ for some $t \ge t_0$. Using (i) we find that $V(t) := V(t, x_t) \ge W_1(|x(t)|) \ge W_1(B_2)$. Let

$$\beta := \frac{W_1(B_2) - W_2(U+L)}{M} = k\left(2 + \frac{W_2(U+L)}{\alpha}\right)$$

and integrate (ii) from $s \in [t-\beta, t]$ to t to get $V(t) - V(s) \le M(t-s)$ or $V(s) \ge V(t) - M(t-s) \ge W_1(B_2) - M\beta = W_2(U+L)$. Thus, using (i) we obtain $\|x_{t_n}\| \ge W_2^{-1}(V(t_n)) \ge U + L$ for all $t_n \in [t-\beta, t]$. Define N to be the largest positive integer so that $t_N \le t$. Since $\|x_{t_0}\| < B_1 \le U + L$ there is an $\overline{N} > 0$ such that $\|x_{t_n}\| \ge U + L$ for $n = \overline{N}+1, ..., N$, but $\|x_{t_{\overline{N}}}\| < U + L$. Note that from the above observation we know that

$$N - \overline{N} \ge \frac{\beta}{k} - 2 = \frac{W_2(U+L)}{\alpha}. \tag{2}$$

Let $S_n = [t_{n-1}, t_n]$. Since $\|x_{t_n}\| \ge U + L$ for $n = \overline{N}+1, ..., N$ there is a $t'_n \in S_n$ such that $|x(t'_n)| \ge U + L$. There are two cases.

Case 1: If $|x(s)| \geq U$ for $s \in S_n$ then integrating (iii) we obtain

$$V(t_n) - V(t_{n-1}) \leq \int_{t_{n-1}}^{t_n} [-a|x'(s)| - b]ds \leq -bh < -\alpha.$$

Case 2: If $|x(s)| = U$ for some $s \in [t_{n-1}, t_n]$ then without loss of generality we may assume that $|x(u)| \geq U$ for all u between s and t'_n and hence integrating (ii) and (iii) we have

$$V(t_n) - V(t_{n-1}) \leq -|\int_s^{t'_n} [a|x'(v)| + b]dv| + Mk \leq -aL + Mk \leq -\alpha$$

again.

In any case we find that $V(t_n) - V(t_{n-1}) \leq -\alpha$ for $n = \overline{N}+1, ..., N$. By the choice of $\overline{N}$ we have

$$\begin{aligned} V(t_N) = V(t_{\overline{N}}) + \sum_{n=\overline{N}+1}^{N} (V(t_n) - V(t_{n-1})) \leq V(t_{\overline{N}}) - \alpha(N - \overline{N}) \\ \leq W_2(\|x_{t_{\overline{N}}}\|) - \alpha(N - \overline{N}) < W_2(U+L) - \alpha(N - \overline{N}) \leq 0 \end{aligned}$$

by (2), which is a contradiction to the fact that $0 \leq V(t, x_t)$ and hence it proves that the solutions are uniformly bounded at t_0.

Consider now the uniform ultimate boundedness. Once again, we will prove everything for t_0 only. The definition of L is slightly different now, we do not have B_1 to ask for $L \geq B_1$. So, choose $L > 0$ large enough to have $-aL + Mk < -\alpha$ for some $\alpha \in (0, bh)$. Let $B > 0$ be the B_2 of the uniform boundedness (at t_n for any n) for $B_1 := U + L$. Let $B_3 > 0$ be given; we need to find a $K > 0$ such that if $\phi \in \mathcal{C}$, $\|\phi\| \leq B_3$ and $t \geq t_0 + K$ then $|x(t)| = |x(t, t_0, \phi)| < B$. In order to do this we need to prove only that $\|x_{t_n}\| < U + L$ for some $t_n \in [t_0, t_0 + K]$, since then from the uniform boundedness at t_n we obtain $|x(s)| < B$ for all $s \geq t_0 + K \geq t_n$. Define $K := (W_2(B_3)/\alpha + 1)k$ and suppose for contradiction that $\|x_{t_n}\| \geq U + L$ for all $t_n \in [t_0, t_0 + K]$. Let $S_n := [t_{n-1}, t_n]$ for $n = 1, 2, ...$ From here on, we do the same as we did in the proof of the uniform boundedness: find a $t'_n \in S_n$ with $|x(t'_n)| \geq U + L$ and using the two cases we prove that $V(t_n) - V(t_{n-1}) \leq -\alpha$ for $n = 1, 2, ...$ with $S_n \subset [t_0, t_0 + K]$. The last estimate works out with a slight difference:

$$V(t_N) = V(t_0) + \sum_{n=1}^{N} (V(t_n) - V(t_{n-1})) \leq W_2(B_3) - \alpha N < 0$$

if $N > W_2(B_3)/\alpha$, and we have this by our definition of K. This is a contradiction to the fact that $0 \leq V(t_N)$ and hence it proves that the solutions are uniformly ultimately bounded at t_0.

Remark. Note that conditions (ii) and (iii) of the theorem are equivalent with the following: There are constants $U, M > 0$ such that

(ii)' $V'(t, x_t) \leq -W_3(|x(t)|) + M$ and

(iii)' $V'(t, x_t) \leq -W_4(|x'(t)|) + M$ for $|x(t)| \geq U$,

where $W_3(U) > M$ and W_4 is convex downward.

Proof: Assume (ii)' and (iii)' holds. Then for any $\alpha \in [0, 1]$ we have

$$V'(t, x_t) \leq -\alpha W_3(|x(t)|) - (1-\alpha) W_4(|x'(t)|) + M.$$

As W_4 is convex downward there are constants $c, d > 0$ such that $W_4(r) \geq cr - d$. Substituting this into the above estimate, assuming that $|x(t)| \geq U$ and reordering we have

$$\begin{aligned} V'(t, x_t) &\leq (-\alpha W_3(|x(t)|) + M) - (1-\alpha)(c|x'(t)| - d) \\ &= -(1-\alpha)c|x'(t)| + (-\alpha W_3(U) + M + (1-\alpha)d), \end{aligned}$$

so taking α sufficiently close to 1 we have the desired (iii) estimate. Clearly (ii) holds.

To prove the opposite direction note that (iii) implies (iii)'. For (ii)' define W_3 by $W_3(U) = 1$, $W_3(r) = M + b + 1$ for $r \geq U + 1$ and linear everywhere else. Then by (ii)

$$V'(t, x_t) \leq M \leq -W_3(|x(t)|) + (M+1)$$

for $|x(t)| \leq U$. Similarly, by (iii)

$$V'(t, x_t) \leq -b \leq -W_3(|x(t)|) + (M+1)$$

for $|x(t)| > U$. Thus (ii)' is satisfied by replacing $U + 1$ by U and $M + 1$ by M, and the proof is complete.

We now give an example for Theorem 3. Let $a, b, c : \mathbf{R} \to \mathbf{R}$ be continuous functions, $h > 0$, and consider

$$x'(t) = -a(t)x(t) + b(t)x(t-h) + c(t). \tag{3}$$

Suppose, that there are constants $k > 0$, $B > 0$ and $\alpha > 0$ such that for $p := 1 + \alpha$ we have

(i) $-a(t) + p|b(t+h)| \leq \min\{-\alpha, -\alpha a(t)\}$,

(ii) $|c(t)| \leq B$ and

(iii) there is a sequence $t_n \in \mathbf{R}$ with $h \leq t_n - t_{n-1} \leq k$ with $\int_{t_n-h}^{t_n} |b(s+h)|ds \leq B$.

Then the solutions of (3) are uniformly bounded and uniformly ultimately bounded at t_0.

Proof: Define

$$V(t, x_t) = |x(t)| + p\int_{t-h}^{t} |b(s+h)| \cdot |x(s)|ds.$$

Clearly we have $x(t) \leq V(t, x_t)$ and

$$V(t_n, x_{t_n}) \leq \|x_{t_n}\| + p\int_{t_n-h}^{t_n} |b(s+h)|\|x_{t_n}\|ds \leq (1+pB)\|x_{t_n}\|.$$

Also, differentiating V we find

$$\begin{aligned} V'(t, x_t) &\leq -a(t)|x(t)| + |b(t)||x(t-h)| + |c(t)| + p|b(t+h)||x(t)| - p|b(t)||x(t-h)| \\ &\leq [-a(t) + p|b(t+h)|]|x(t)| + (1-p)|b(t)||x(t-h)| + B \\ &\leq -\alpha[a(t)|x(t)| + |b(t)||x(t-h)|] + B. \end{aligned}$$

Using (i) from this last estimate we have $V' \leq -\alpha|x'(t)| + B$ and $V' \leq -\alpha|x(t)| + B$, which is enough for Theorem 3 by the remark. Thus, the solutions of (3) are uniformly bounded and uniformly ultimately bounded at t_0.

This result implies the existence of a T-periodic solution if the functions a, b and c are T-periodic. There are other ways to prove the existence of a periodic solution for this special kind of differential equation. The author proved theorems in [8] that imply the existence of a periodic solution under a different set of conditions.

Consider now the functional differential equation

$$x' = f(t, x_t) \tag{4}$$

with infinite delay, i.e., let $\mathcal{C}$ be the Banach space of continuous functions $\phi : (-\infty, 0] \to \mathbf{R}^r$ with $\|\phi\| := \sup_{s\in(-\infty,0]} |\phi(s)| < \infty$, $f : \mathbf{R} \times \mathcal{C} \to \mathbf{R}^r$ is continuous and locally Lipschitz in ϕ with this topology defined on $\mathcal{C}$. Let the function $g : (-\infty, 0] \to [1, \infty)$ be continuous and monotone decreasing with $g(s) \to \infty$ as $s \to -\infty$ and $g(0) = 1$. We define the norm $\|\phi\|_g := \sup_{s \le 0} |\phi(s)|/g(s)$.

Definition. Solutions of (4) are *uniformly bounded at t_0*, if for all $B_1 > 0$ there is a $B_2 > 0$ such that $\phi \in \mathcal{C}$, $\|\phi\| < B_1$ and $t \ge t_0$ imply $|x(t, t_0, \phi)| < B_2$.

Definition. Solutions of (4) are *g-uniformly bounded at t_0*, if for all $B_1 > 0$ there is a $B_2 > 0$ such that $\phi \in \mathcal{C}$, $\|\phi\|_g < B_1$ and $t \ge t_0$ imply $|x(t, t_0, \phi)| < B_2$.

Definition. Solutions of (4) are *uniformly ultimately bounded with bound B at t_0*, if for all $B_3 > 0$ there is a $K > 0$ such that $\phi \in \mathcal{C}$, $\|\phi\| < B_3$ and $t \ge t_0 + K$ imply $|x(t, t_0, \phi)| < B$.

Definition. Solutions of (4) are *g-uniformly ultimately bounded with bound B at t_0*, if for all $B_3 > 0$ there is a $K > 0$ such that $\phi \in \mathcal{C}$, $\|\phi\|_g < B_3$ and $t \ge t_0 + K$ imply $|x(t, t_0, \phi)| < B$.

Since $\|\phi\|_g \le \|\phi\|$, g-uniform boundedness is a stronger condition than uniform boundedness and the same is true for uniform ultimate boundedness. The next theorem proves uniform boundedness and uniform ultimate boundedness for infinite delay equations. We suggest that the reader draw a picture for the proof of this theorem. Although the main idea is not very complicated, the interpretation is very technical.

Theorem 4: *Suppose there are a continuous functional $V : \mathbf{R} \times \mathcal{C} \to \mathbf{R}$, wedges W_i and constants $M, U, a, b > 0$ such that*

(i) $W_1(|\phi(0)|) \le V(t, \phi) \le W_2(\|\phi\|_g)$

(ii) $V'(t, x_t) \le -a|x'(t)| - b$ *for* $|x(t)| \ge U$

(iii) $V'(t, x_t) \le M$,

where $W_1(r) \to \infty$ as $r \to \infty$. Also assume, that $g(s) \ge -\mu s$ for all $s \le 0$ and some $\mu > 0$. Then the solutions of (4) are g-uniformly bounded and uniformly ultimately bounded.

Proof: First we prove the g-uniform boundedness. Let $B_1 > 0$ be arbitrary and $L \ge B_1$ be large enough to have

$$\mu \left(-a(U + L) + \frac{M}{\mu} \right) \le -2b$$

and

$$\left(-a(U + L) + \frac{M}{\mu} \right) \le -2aU.$$

The last estimation clearly implies $aU + M/\mu \le aL$; we will need this later. Also define B_2 by

$$\frac{b(W_1(B_2) - W_2(U+L))}{M} > W_2(U+L) + aL.$$

While these definitions are complicated, they will simplify later work. Suppose for contradiction that there is such a $(t_0, \phi) \in \mathbf{R} \times \mathcal{C}$ such that $\|\phi\|_g < B_1$ and $|x(t, t_0, \phi)| \ge B_2$ for some $t \ge t_0$. Using the very same idea as we used in Theorem 3, we can prove that if

$$\beta := \frac{W_1(B_2) - W_2(U+L)}{M}$$

then $s \in (t - \beta, t]$ implies $\|x_s\| \ge \|x_s\|_g \ge W_2^{-1}(V(s)) > U + L$. Let $v_0 \in [t_0, t]$ be the largest number such that $\|x_{v_0}\|_g \le U + L$ (since $\|x_{t_0}\|_g < B_1 \le U + L$ we can find such a v_0). It is not difficult to see from the definition of v_0 and the monotonicity of g that $|x(v_0)| = U + L$. It is also clear by the above observation and the definition of B_2 that

$$t - v_0 \ge \beta > \frac{W_2(U+L) + aL}{b}. \tag{5}$$

Let $w_0 = u_0 = v_0$, $\eta_0 = 1$ and $V_0 := V(v_0) + aL\eta_0$. We now define the sequences η_n and $u_n \le v_n \le w_n$ by induction. The sequences will have the following property: either

(a) $v_n = t$ and $V(v_n) - V_0 \le -b(v_n - v_0)$ or

(b) $\eta_n \ge 1$, $\eta_n(U+L)$ is the maximum of $|x(s)|$ on the interval $[u_n, v_n]$, $V(v_n) - V_0 \le -b(v_n - v_0) - aL\eta_n$ and $\sup_{s \le -(w_n - u_n)} |x(w_n + s)|/g(s) \le U + L$.

Clearly, (b) holds for $n = 0$. In our induction step we prove, that if (b) holds for η_n, u_n, v_n and w_n, then we can define $\eta_{n+1}, u_{n+1}, v_{n+1}$ and w_{n+1} so that either (a) or (b) holds. To define $u_{n+1}, v_{n+1}, w_{n+1}$ we consider two cases.

Case 1: If $|x(s)| \ge U$ for all $s \in [v_n, t]$ then define $v_{n+1} = t$. Clearly, (ii) implies $V(v_{n+1}) - V(v_n) \le -b(v_{n+1} - v_n)$ and hence (a) holds for v_{n+1}.

Case 2: If there is a $u_{n+1} \in (v_n, t)$ such that $|x(u_{n+1})| = U$ and $|x(s)| \ge U$ for $s \in [v_n, u_{n+1}]$, then let $\eta(U+L)$ be the maximum of $|x(s)|$ on the interval $[v_n, u_{n+1}]$ and $w_{n+1} := u_{n+1} + (\eta + \eta_n)/\mu$.

Case 2/a: If $w_{n+1} \ge t$ then we define $v_{n+1} = t$.

Case 2/b: If $w_{n+1} < t$ then $\|x_{w_{n+1}}\|_g > U + L$. By the above definitions we obtain the followings:

$$\sup_{s \le -(w_{n+1}-u_n)} \frac{|x(w_{n+1}+s)|}{g(s)} \le \sup_{s \le -(w_n - u_n)} \frac{|x(w_n+s)|}{g(s)} \le U + L$$

$$\sup_{-(w_{n+1}-u_n) < s \le -(w_{n+1}-v_n)} \frac{|x(w_{n+1}+s)|}{g(s)} \le \frac{\eta_n(U+L)}{\eta+\eta_n} \le U + L$$

$$\sup_{-(w_{n+1}-v_n) < s \le -(w_{n+1}-u_{n+1})} \frac{|x(w_{n+1}+s)|}{g(s)} \le \frac{\eta(U+L)}{\eta+\eta_n} \le U + L$$

because $w_{n+1} - v_n \ge w_{n+1} - u_{n+1} = (\eta + \eta_n)/\mu$ and hence if $s \le -(w_{n+1} - v_n) \le -(w_{n+1} - u_{n+1}) = -(\eta + \eta_n)/\mu$ then $g(s) \ge \eta + \eta_n$. Now putting these together we find

$$U + L < \|x_{w_n}\|_g = \sup_{-(w_{n+1}-u_{n+1}) < s \le 0} \frac{|x(w_{n+1}+s)|}{g(s)} \le \sup_{-(w_{n+1}-u_{n+1}) < s \le 0} |x(w_{n+1}+s)|,$$

and hence we can find a $v_{n+1} \in (u_{n+1}, w_{n+1}]$ such that $|x(v_{n+1})| > U + L$.

Now, in both Case 2/a and 2/b we estimate $V(v_{n+1}) - V_0$. In the following we will use several times that $v_{n+1} - u_{n+1} \le w_{n+1} - u_{n+1} = (\eta + \eta_n)/\mu$ without even mentioning it. Define $|x(s_1)| = \eta_{n+1}(U + L)$ to be the maximum of $|x(s)|$ on the interval $[u_{n+1}, v_{n+1}]$, and let $s_2 \in [u_{n+1}, s_1]$ be the largest number with $|x(s_2)| \le U$ (in Case 2/a we do not have to do this, in this case we must drop the integral from s_2 to s_1 and $-aL\eta_{n+1}$ from the following estimations). Integrating (ii) and (iii) we have

$$\begin{aligned}
V(v_{n+1}) - V_0 &\le V(v_{n+1}) - V(v_n) + V(v_n) - V_0 \\
&\le \int_{v_n}^{u_{n+1}} -a|x'(s)| - b ds + \int_{s_2}^{s_1} -a|x'(s)| - b ds + \int_{u_{n+1}}^{v_{n+1}} M ds \\
&\quad - b(v_n - v_0) - aL\eta_n \\
&\le -a(\eta(U+L) - U) - b(u_{n+1} - v_0) + \frac{M(\eta+\eta_n)}{\mu} - aL\eta_n - aL\eta_{n+1} \\
&\le aU - b(u_{n+1} - v_0) + \eta\left(-a(U+L) + \frac{M}{\mu}\right) + \left(\frac{M}{\mu} - aL\right)\eta_n - aL\eta_{n+1} \\
&\le aU + \frac{1}{2}\left(-a(U+L) + \frac{M}{\mu}\right) + \left(\frac{M}{\mu} - aL\right)\eta_n \\
&\quad - b(u_{n+1} - v_0) + \frac{(v_{n+1} - u_{n+1})\mu - \eta_n}{2}\left(-a(U+L) + \frac{M}{\mu}\right) - aL\eta_{n+1} \\
&\le -b(v_{n+1} - v_0) - aL\eta_{n+1} + \frac{1}{2}\left(\frac{M}{\mu} + aU - aL\right)\eta_n \\
&\le -b(v_{n+1} - v_0) - aL\eta_{n+1}
\end{aligned}$$

by our definition of L. Therefore, we proved that in Case 2/a v_{n+1} satisfies (a), and in Case 2/b $\eta_{n+1}, u_{n+1}, v_{n+1}, w_{n+1}$ satisfy (b).

This finishes our induction step. Continuing this process we either end up in Case 1 or Case 2/a, or we have sequences u_n, v_n and w_n for all $n \geq 0$. Also, by (i) $V_0 \leq W_2(\|x_{v_0}\|_g) + aL\eta_0 \leq W_2(U+L) + aL$.

Case A: If we end up in Case 1 or in Case 2/a when we started our procedure from v_n, then v_{n+1} satisfies (a) and we have $v_{n+1} = t$ and $V(v_{n+1}) - V_0 \leq -b(v_{n+1} - v_0)$. Using these and (5) we obtain

$$V(t) \leq V_0 - b(t - v_0) \leq W_2(U+L) + aL - b\beta < 0,$$

which is a contradiction.

Case B: If we have the sequences u_n, v_n and w_n for all $n \geq 0$, then $v_n \to v \leq t$ as $n \to \infty$ for some v. Then integrating (ii) and (iii) from v_0 to v_n we have

$$\begin{aligned} V(v_n) - V(v_0) &\leq M(v_n - v_0) - \sum_{i=0}^{n-1} \int_{v_i}^{u_{i+1}} [a|x'(s)| + b] ds \\ &\leq M(t - v_0) - \sum_{i=0}^{n-1} aL = M(t - v_0) - anL \end{aligned}$$

a contradiction for large n to $0 \leq V(v_n)$.

This finishes the proof of the g-uniform boundedness.

For the uniform ultimate boundedness we define $L > 0$ as we did in the proof of g-uniform boundedness. Let $B := B_2$ for $B_1 := U + L + 1$ in the g-uniform boundedness. In order to prove uniform ultimate boundedness it is sufficient to prove that for all $B_3 > 0$ there is a $K > 0$ such that for every (t_0, ϕ) with $\|\phi\| < B_3$ there is an $s \in [t_0, t_0 + K]$ with $\|x_s(\cdot, t_0, \phi)\|_g \leq U + L$; then g-uniform boundedness implies the necessary inequality. Let $B_3 > 0$ be arbitrary, $D := B_3/(\mu(U+L))$, $w_0 := t_0 + D$ and $\eta_0(U+L)$ to be the maximum of $|x(s)|$ on the interval $(-\infty, w_0]$. As x is bounded by B_3 on $(-\infty, t_0]$ and $V(s) \leq V(t_0) + M(s - t_0) \leq W_2(B_3) + MD$, we have $|x(s)| \leq \max\{B_3, W_1^{-1}(W_2(B_3) + MD)\} =: \overline{\eta}(U+L)$ for $s \leq w_0$ and hence $\eta_0 \leq \overline{\eta}$, a bound independent of t_0 and ϕ. Define $K := D + (W_2(B_3) + DM + aL\overline{\eta})/b + 1$ and $t := t_0 + K$. Suppose for contradiction, that there is a (t_0, ϕ) such that $\|x_s\|_g = \|x_s(\cdot, t_0, \phi)\|_g > U + L$ for all $s \in [t_0, t_0 + K]$. Then using the below estimation on g we find

$$\begin{aligned} U + L < \|x_{w_0}\|_g &= \sup_{s \le 0} \frac{|x(w_0 + s)|}{g(s)} \\ &\le \max\left\{ \sup_{s \le -D} \frac{|x(w_0+s)|}{g(s)}, \sup_{-D \le s \le 0} \frac{|x(w_0+s)|}{g(s)} \right\} \\ &\le \max\left\{ \frac{B_3}{\mu D}, \sup_{-D \le s \le 0} \frac{|x(w_0+s)|}{g(s)} \right\} \\ &\le \max\left\{ U + L, \sup_{-D \le s \le 0} |x(w_0+s)| \right\}, \end{aligned}$$

and hence we find a $v_0 \in [t_0, w_0]$ with $|x(v_0)| > U + L$. Let $u_0 = -\infty$. From here on the proof is exactly the same as we did before. Note, that $V(t_0) \le W_2(\|\phi\|_g) \le W_2(\|\phi\|) \le W_2(B_3)$ and hence $V(v_0) \le W_2(B_3) + (v_0 - t_0)M \le W_2(B_3) + DM$. We have Cases A and B again:

Case A: If $v_{n+1} = t$, then from $v_0 \le t_0 + D$ we have $t - v_0 \ge K - D > (W_2(B_3) + DM + aL\overline{\eta})/b$ and hence $V(v_{n+1}) - V_0 \le -b(v_{n+1} - v_0)$ implies

$$V(t) \le V_0 - b(t - v_0) < W_2(B_3) + DM + aL\eta_0 - b\frac{W_2(B_3) + DM + aL\overline{\eta}}{b} \le 0$$

a contradiction.

Case B: We get a contradiction as before.

This proves the uniform ultimate boundedness and the proof is complete.

Note that with a little more care one can prove the exact counterpart of Theorem 3 for infinite delay equations too. The theorem would be the following.

Theorem 5: *Suppose there are a continuous functional $V : \mathbf{R} \times \mathcal{C} \to \mathbf{R}$, wedges W_i, constants $M, U, a, b, k > 0$ and a sequence $t_n \in \mathbf{R}$ with $1 \le t_n - t_{n-1} \le k$ such that*

(i) $W_1(|\phi(0)|) \le V(t, \phi)$ *and* $V(t_n, \phi) \le W_2(\|\phi\|_g)$

(ii) $V'(t, x_t) \le -a|x'(t)| - b$ *for* $|x(t)| \ge U$

(iii) $V'(t, x_t) \le M$,

where $W_1(r) \to \infty$ as $r \to \infty$. Also assume, that $g(s) \ge -\mu s$ for all $s \le 0$ and some $\mu > 0$. Then the solutions of (4) are g-uniformly bounded and uniformly ultimately bounded at t_n.

Also, B_2 of the uniform boundedness and B, K of uniform ultimate boundedness does not depend on n.

Proof: We only mention the changes one needs to make on the proof of Theorem 4. We need to modify the definition of w_{n+1} (starting with w_0) so that w_{n+1} is one of the t_i's in the interval of length k with the left-end point being the previous definition of w_{n+1}, and hence we have $V(w_{n+1}) \leq W_2(\|x_{w_{n+1}}\|_g)$. Also, because we have $w_{n+1} \leq u_{n+1} + (\eta + \eta_n)/\mu + k$ only, we need to do the estimation on $V(v_{n+1}) - V_0$ more carefully, we will have some terms containing k too. To get the same estimation as before, instead of cutting the term $\eta(-a(U+L)+M/\mu)$ into two equal pieces we need to use an $\alpha \in (0,1)$ to cut it. According to this α we need to modify the definition of L too.

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