ON THE ASYMPTOTIC STABILITY OF THE SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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1. Introduction

Denote by $\mathcal{C}$ the space of continuous functions $\phi : (-\infty, 0] \to \mathbf{R}^n$ and $\mathcal{BC} \subseteq \mathcal{C}$ the space of bounded continuous functions. In the function space $\mathcal{BC}$ we use the measure

$$\|\phi\|_g := \int_{-\infty}^{0} |\phi(s)| g(s) ds,$$

where $g : \mathbf{R}_- \to \mathbf{R}_+$ with $\int_{-\infty}^{0} g(s)ds = 1$ and $|\cdot|$ an arbitrary norm in $\mathbf{R}^n$. If $x : (-\infty, T) \to \mathbf{R}^n$ is continuous and $t \in (-\infty, T)$, we define $x_t \in \mathcal{C}$ by $x_t(s) := x(t+s)$ for $s \leq 0$.

Consider the functional differential equation

$$\dot{x} = f(t, x_t), \tag{1.1}$$

where $f : \mathbf{R} \times \mathcal{C} \to \mathbf{R}^n$ is continuous and $f(t, 0) = 0$ for all $t \in \mathbf{R}$. We suppose, that for each $t_0 \in \mathbf{R}$ and each $\phi \in \mathcal{C}$ there is a unique solution $x(\cdot; t_0, \phi)$ defined on an interval $(-\infty, t_0 + \alpha)$, $\alpha > 0$, and satisfying (1.1) on (t_0, α). We assume, that the function x continuously depends on the initial data t_0, ϕ. Sufficient conditions implying existence, uniqueness and continuous dependence can be found in [**11**].

Denote by $\mathcal{K}$ the set of strictly increasing continuous functions $W : \mathbf{R}_+ \to \mathbf{R}_+$ with $W(0) = 0$. If V is a continuous functional from $\mathbf{R} \times \mathcal{C}$ into $\mathbf{R}$, then $\dot{V}$ denotes the derivative of functional V with respect to equation (1.1) defined by

$$\dot{V}(t, \phi) = \dot{V}_{(1.1)}(t, \phi) = \limsup_{\delta \to 0+} \frac{V(t+\delta, x_{t+\delta}(\cdot; t, \phi)) - V(t, \phi)}{\delta}.$$

In this paper we study the asymptotic stability of the zero solution of equation (1.1) using Lyapunov functionals. For the historical overview consider the following functional differential equations with finite delay

$$\dot{x} = f(t, x_t), \tag{1.2}$$

where $x_t(s) = x(t+s)$ for $s \in [-r, 0]$. The first Lyapunov-type theorem for this equation can be found in the book of Krasovskii [**8**].

Theorem A: *Suppose that there are a functional V and functions $W_1, W_2, W_3 \in \mathcal{K}$ such that*

$$W_1(|\phi(0)|) \le V(t, \phi) \le W_2(\|\phi\|),$$

$$\dot{V}_{(1.2)}(t, \phi) \le -W_3(\|\phi\|),$$

where $t \in \mathbf{R}$, $\phi : [-r, 0] \to \mathbf{R}^n$ is continuous and $\|\phi\| = sup\{|\phi(s)| : s \in [-r, 0]\}$. Then the zero solution of equation (1.2) is uniformly asymptotically stable.

This theorem cannot be generalized to the case of equations with infinite delay, because if we have the supremum norm in the estimate of $\dot{V}$, then $V(t, x_t)$ has to tend to $-\infty$ for every nonzero solution. Driver [**5**] proved the following theorem for equations with infinite delay using a pseudo norm $|\phi(0)|$.

Theorem B: *Suppose that for some $M > 0$ there exists $H > 0$ such that $|f(t, \phi)| \le H$ whenever $t \in \mathbf{R}$ and $\phi \in \mathcal{C}_M := \{\psi \in \mathcal{C} : \|\psi\| \le M\}$. If, in addition, there are a continuous functional $V : \mathbf{R} \times \mathcal{BC} \to \mathbf{R}$ which is locally Lipschitz with respect to the second variable, and functions $W_1, W_2 \in \mathcal{K}$ such that $V(t, 0) = 0$, $W_1(|\phi(0)|) \le V(t, \phi)$ and $\dot{V}_{(1.1)}(t, \phi) \le -W_2(|\phi(0)|)$, then the zero solution of equation (1.1) is asymptotically stable.*

The first condition of this theorem is commonly called "the boundedness condition". For equations with finite delay the following well known theorems are useful.

Theorem C ([**6**, Theorem 5.2.1]): *Suppose that the boundedness condition holds for the function f and there are a continuous functional V and functions $W_1, W_2, W_3 \in \mathcal{K}$ such that*

$$W_1(|\phi(0)|) \le V(t, \phi) \le W_2(\|\phi\|), \tag{1.3}$$

$$\dot{V}_{(1.2)}(t,\phi) \le -W_3(|\phi(0)|). \tag{1.4}$$

Then the zero solution of equation (1.2) is uniformly asymptotically stable.

THEOREM D ([**7**, Theorem 4.1]): *Suppose that all but condition* (1.4) *of Theorem C hold, and*

$$\dot{V}_{(1.2)}(t,\phi) \le -W_3(|||\phi|||), \tag{1.5}$$

where $|||\phi||| = \sqrt{\int_{-r}^{0} |\phi(s)|ds}$. *Then the zero solution of equation (1.2) is uniformly asymptotically stable.*

The main problem with these theorem is that the boundedness condition either holds or not for a given differential equation. It is an old problem (see [**2**, p. 252]), whether the boundedness condition can be dropped from Theorems C and D ([**7**],[**10**]). Burton and Hatvani ([**3**]) proved the following Theorem E and Burton [**1**] proved Theorem F which do not use this condition.

THEOREM E: *Suppose that there are a functional* V *and functions* $W_1, W_2, W_3, W_4 \in \mathcal{K}$ *such that*

$$W_1(|\phi(0)|) \le V(t,\phi) \le W_2(|\phi(0)|) + W_3(|||\phi|||), \tag{1.6}$$

$$\dot{V}_{(1.2)}(t,\phi) \le -W_4(|||\phi|||). \tag{1.7}$$

Then the zero solution of equation (1.2) is uniformly asymptotically stable.

THEOREM F: *Suppose that all but condition* (1.7) *of Theorem E hold and*

$$\dot{V}_{(1.2)}(t,\phi) \le -W_4(|\phi(0)|) \tag{1.8}$$

holds. Then the zero solution of equation (1.2) is uniformly asymptotically stable.

In the following we generalize these theorems to the case of equations with infinite delay. The trouble with these theorems is the use of norm $|||\cdot|||$ which has no meaning in the space $\mathcal{BC}$; therefore we will use the measure $\|\cdot\|_g$.

The other problem we investigate is how conditions in our theorems corresponding to (1.7) and (1.8) in Theorems E and F can be weakened. In practice we have a coefficient $\eta(t)$ in the right-hand side of the estimate of $\dot{V}$. We prove theorems based on such estimate for uniform asymptotic stability of functional differential equations with infinite delay.

2. Main results

First of all we prove a proposition needed by the proof our first theorem. This proposition states that if the $\|\cdot\|_g$ measure of a solution is large at one moment, then it is large in a small interval as well.

Proposition 1: *Suppose, that $x \in \mathcal{C}_M$ and $g : \mathbf{R}_- \to \mathbf{R}_+$ is continuous and bounded function. Then there exists $W \in \mathcal{K}$ such that for all $\varepsilon > 0$ there exists a $\delta_0 > 0$ such that $\|x_{t_1}\|_g \geq \varepsilon$ implies $\|x_t\|_g \geq W(\varepsilon)$ for all $t \in (t_1 - \delta_0, t_1 + \delta_0)$.*

Proof: Suppose that $g(s) \leq K$ and $\varepsilon \leq \|x_{t_1}\|_g$. There exists a constant $S > 0$ such that $\int_{-\infty}^{-S} g(s)ds \leq \varepsilon/3M$ (we can assume, that S as a function of ε is monotone decreasing). Then $\int_{-\infty}^{0} |x_{t_1}(s)|g(s)ds \leq \varepsilon/3 + \int_{-S}^{0} |x_{t_1}(s)|g(s)ds$, so $2\varepsilon/3 \leq \int_{-S}^{0} |x_{t_1}(s)|g(s)ds$. Define the following constants and set:

$$r := \varepsilon/3S \qquad Q_r := \{s \in [-S,0] : |x_{t_1}(s)|g(s) \geq r\},$$

$$u := r/2M \qquad R_u := \{s \in [-S,0] : g(s) > u\},$$

$$v := r/2K \qquad S_v(t) := \{s \in [-S,0] : |x_t(s)| > v\}.$$

Then $2\varepsilon/3 \leq \int_{-S}^{0} |x_{t_1}(s)|g(s)ds = \int_{Q_r} + \int_{Q_r^c} \leq \mu(Q_r)MK + \mu(Q_r^c)r$, so $\mu(Q_r) \geq (2\varepsilon/3 - Sr)/(MK - r) \geq \varepsilon/3MK$. Moreover, it can be proved easily that $Q_r \subseteq R_u \cap S_v(t_1)$ and $\mu(R_u \cap S_v(t_1)) \geq \mu(Q_r) \geq \varepsilon/3MK$. Since R_u is an open set, there are $n > 0$, $a_1, b_1, ..., a_n, b_n$ such that $(a_i, b_i) \subseteq R_u$ $(a_i < b_i \leq a_{i+1})$ and $\mu((\cup(a_i,b_i)) \cap S_v(t_1)) \geq \varepsilon/6MK$. Let $\delta_0 := \varepsilon/(12MKn)$ and $-\delta_0 < \delta < \delta_0$ (δ_0 depends only on ε, M, K and g). Obviously

$$\begin{aligned} S_v(t_1) - \delta &= \{s - \delta : s \in [-S,0], |x_{t_1}(s)| > v\} \\ &= \{w \in [-S-\delta, -\delta] : |x_{t_1+\delta}(w)| > v\}, \end{aligned}$$

so $(S_v(t_1) - \delta) \cap [-S,0] \subseteq S_v(t_1+\delta)$. $(a_i,b_i) \subseteq [-S,0]$ implies, that

$$\begin{aligned} \mu((a_i,b_i) \cap S_v(t_1+\delta)) &\geq \mu((a_i,b_i) \cap ((S_v(t_1) - \delta) \cap [-S,0])) \\ &= \mu((a_i,b_i) \cap (S_v(t_1) - \delta)) \\ &\geq \mu((a_i,b_i) \cap S_v(t_1)) - \delta_0, \end{aligned}$$

so $\sum \mu((a_i,b_i) \cap S_v(t_1+\delta)) \geq \sum \mu((a_i,b_i) \cap S_v(t_1)) - n\delta_0 \geq \varepsilon/12MK$. If $t = t_1 + \delta$ $(t \in (t_1 - \delta_0, t_1 + \delta_0))$, then

$$\begin{aligned} \|x_t\|_g = \int_{-\infty}^{0} |x_t(s)|g(s)ds &\geq \int_{(\cup(a_i,b_i)) \cap S_v(t)} |x_t(s)|g(s)ds \\ &\geq vu\mu((\cup(a_i,b_i)) \cap S_v(t)) \geq \varepsilon^3/(432M^2S^2K^2) =: W(\varepsilon) \end{aligned}$$

and the proof is complete.

DEFINITION: Let $S_0 := \inf\{S \geq 0 : \int_{-S}^{0} g(s)ds > 0\}$.

LEMMA 1: *Let $S > S_0$ and $\nu_0 := \int_{-S}^{0} g(s)ds > 0$. If for some $t' \in \mathbf{R}$ we have $|x(t)| + \|x_t\|_g \geq \varepsilon$ for all $t \in [t' - S, t']$, then there exists a $t'' \in [t' - S, t']$ such that $\|x_{t''}\|_g \geq \varepsilon\nu_0/(1 + \nu_0)$.*

Proof: Suppose that $\|x_t\|_g < \varepsilon\nu_0/(1 + \nu_0)$ for all $t \in [t' - S, t']$. Then from $|x(t)| + \|x_t\|_g \geq \varepsilon$ we have $|x(t)| \geq \varepsilon/(1 + \nu_0)$, so

$$\begin{aligned}\|x_{t'}\|_g = \int_{-\infty}^{0} |x_{t'}(s)|g(s)ds &\geq \int_{-S}^{0} |x_{t'}(s)|g(s)ds \\ &\geq \int_{-S}^{0} \varepsilon/(1 + \nu_0)g(s)ds = \varepsilon\nu_0/(1 + \nu_0),\end{aligned}$$

which is a contradiction, and the proof is complete.

Now we can prove our first theorem.

THEOREM 1: *Assume that $g : \mathbf{R}_- \to \mathbf{R}_+$ is bounded and continuous. Suppose that there exists $V : \mathbf{R} \times \mathcal{C} \to \mathbf{R}$ a continuous functional, $W_1, W_2, W_3 \in \mathcal{K}$ such that*

(a) $W_1(|x(t)|) \leq V(t, x_t) \leq W_2(|x(t)| + \|x_t\|_g)$,

(b) $\dot{V}(t, x_t) \leq -W_3(\|x_t\|_g)$.

Then the zero solution of equation (1.1) is uniformly asymptotically stable.

Proof: The uniform stability follows immediately. Let γ be the number for 1 in uniform stability, so if $t_0 \in \mathbf{R}$ and $\|x_{t_0}\| < \gamma$ then $|x(t)| \leq 1$ for all $t \geq t_0$. Moreover $V(t_0, x_{t_0}) \leq W_2(|x(t_0)| + \|x_{t_0}\|_g) \leq W_2(2)$. Let $S > S_0$ and $0 < \nu_0 := \int_{-S}^{0} g(s)ds$.
For every $\omega > 0$ we must find a $T > 0$ such that if $t_0 \in \mathbf{R}$, then $\|x_{t_0}\| < \gamma$ implies $|x(t)| < \omega$ for all $t \geq t_0 + T$. Let N be a natural number, $\delta_0 > 0$ be a real number and $T := (S + 2\delta_0)N$. Suppose, for a contradiction, that there exists $t' \geq t_0 + T$ such that $|x(t')| \geq \omega$. Now for $t \in [t_0, t']$ we have

$$\begin{aligned}|x(t)| + \|x_t\|_g \geq W_2^{-1}(V(t, x_t)) &\geq W_2^{-1}(V(t', x_{t'})) \\ &\geq W_2^{-1}(W_1(|x(t')|)) \geq W_2^{-1}(W_1(\omega)) =: \varepsilon.\end{aligned}$$

From Lemma 1 we have $t_i \in [t_0 + \delta_0(2i - 1) + (i - 1)S, t_0 + (2i - 1)\delta_0 + iS]$ $(i = 1, 2, ..., N)$ such that $\|x_{t_i}\|_g \geq \varepsilon\nu_0/(1 + \nu_0)$. From Proposition 1 we have the above δ_0 and $W \in \mathcal{K}$

such that $\|x_t\|_g \geq W(\varepsilon\nu_0/(1+\nu_0))$ for all $t \in (t_i - \delta_0, t_i + \delta_0)$ $(i = 1, 2, ..., N)$. Obviously the intervalls $(t_i - \delta_0, t_i + \delta_0)$ are disjoint. Now

$$\begin{aligned} V(t_0+T, x_{t_0+T}) &\leq V(t_0, x_{t_0}) + \int_{t_0}^{t_0+T} \dot{V}(s, x_s)ds \leq W_2(2) - \sum \int_{t_i-\delta_0}^{t_i+\delta_0} W_3(\|x_s\|_g)ds \\ &\leq W_2(2) - \sum \int_{t_i-\delta_0}^{t_i+\delta_0} W_3(W(\varepsilon\nu_0/(1+\nu_0)))ds \\ &= W_2(2) - 2N\delta_0 W_3(W(\varepsilon\nu_0/(1+\nu_0))). \end{aligned}$$

Now let $N := [W_2(2)/(2\delta_0 W_3(W(\varepsilon\nu_0/(1+\nu_0)))] + 1$, so $V(t_0+T, x_{t_0+T}) < 0$ which is a contradiction, and the proof is complete.

Theorem 1 can be generalized, if we use a function η in condition (b) in the following way: $\dot{V}(t, x_t) \leq -\eta(t) W_3(\|x_t\|_g)$. In this case we have the following inequality:

$$V(t_0+T, x_{t_0+T}) \leq W_2(2) - W_3(W(\varepsilon\nu_0/(1+\nu_0))) \sum \int_{t_i-\delta_0}^{t_i+\delta_0} \eta(s)ds,$$

so we need η to be "large" in the following sense.

Definition: The function $\eta : \mathbf{R} \to \mathbf{R}_+$ is *uniformly weakly integrally positive with parameters* δ *and* S (UWIP(δ, S)) iff for all $M > 0$ there exists a $Q_M > 0$ such that if $Q \geq Q_M$, and t_i, δ_i (i=1,2,...) satisfy $\delta_i \geq \delta$, $t_i + \delta_i < t_{i+1} - \delta_{i+1} < t_i + \delta_i + S$ and $I := \cup(t_i - \delta_i, t_i + \delta_i)$, then $\sum \int_{I \cap [t_1-\delta_1, t_1-\delta_1+Q]} \eta(s)ds > M$.

Theorem 2: *Suppose that all but condition (b) are satisfied in Theorem 1. If we have a function* $\eta : \mathbf{R} \to \mathbf{R}_+$ *which is UWIP*$(\delta, 2S)$ *for every* $\delta > 0$ *and for some* $S > S_0$*, and if*

(b) $\dot{V}(t, x_t) \leq -\eta(t) W_3(\|x_t\|_g)$,

then the zero solution of equation (1.1) is uniformly asymptotically stable.

Proof: The proof is left to the reader.

Conditions of Theorem 2 can be further weakened, if the function g satisfies a stronger condition.

Proposition 2: *Suppose that the functions* $x : \mathbf{R} \to \mathbf{R}_+$ *and* $g : \mathbf{R}_- \to \mathbf{R}_+$ *are bounded, and there exists a positive, monotone increasing function* $g_1 : \mathbf{R}_- \to \mathbf{R}_+$ *such* *that $g_1(s) \le g(s)$. Then for every $\delta_0 > 0$ and $\varepsilon > 0$ there exists $\gamma > 0$ such that if $\|x_{t_1}\|_g \ge \varepsilon$ for some $t_1 \ge t_0$ then $\|x_t\|_g \ge \gamma$ for all $t \in [t_1, t_1 + \delta_0]$.*

Proof: Let $M, K > 0$ be given such that $|x(t)| \le M$ and $g(s) \le K$. Then there exists $S > 0$ such that $\int_{-\infty}^{-S} g(s))ds \le \varepsilon/2M$ so (as in proof of Proposition 1) we have $\int_{-S}^{0} |x_{t_1}(s)|g(s)ds \ge \varepsilon/2$. If $t \in [t_1, t_1+\delta_0]$ then

$$\begin{aligned}
\int_{-\infty}^{0} |x_t(s)|g(s)ds \ge \int_{-S-(t-t_1)}^{-(t-t_1)} |x_t(s)|g(s)ds &= \int_{-S}^{0} |x_{t_1}(s)|g(s-(t-t_1))ds \\
&= \int_{-S}^{0} |x_{t_1}(s)|g(s)g(s-(t-t_1))/g(s)ds \\
&\ge \int_{-S}^{0} |x_{t_1}(s)|g(s)g_1(s-(t-t_1))/Kds \\
&\ge \int_{-S}^{0} |x_{t_1}(s)|g(s)g_1(-S-\delta_0)/Kds \ge \varepsilon g_1(-S-\delta_0)/2K =: \gamma
\end{aligned}$$

and the proof is complete.

This proposition can be used instead of Proposition 1 in the proof of Theorem 2, so we have proved the following

THEOREM 3: *Suppose that all but condition (b) of Theorem 2 are satisfied. Assume, that there exist a positive, monotone increasing function g_1, with $g_1(s) \le g(s)$, and an η, which is UWIP($\delta, 2S$) for some $\delta > 0$ and $S > S_0$ such that*
(b) $\dot V(t, x_t) \le -\eta(t) W_3(\|x_t\|_g)$.
Then the zero solution of equation (1.1) is uniformly asymptotically stable.

In the following we prove a theorem using the measure $|\phi(0)|$ and a function η in the above estimate of $\dot V$. The following proposition corresponds to Proposition 1 in Theorems 1, 2 and 3.

PROPOSITION 3: *Suppose, that $g(s) \le K$ and $|x(t)| \le M$. Let $\varepsilon > 0$ be given and let $S' > 0$ be such that $\int_{-\infty}^{-S'} g(s)ds \le \varepsilon/3M$ and $S > S'$. If there exists $t_1 \ge t_0 + S$ such that $|x(t)| + \|x_t\|_g \ge \varepsilon$ for all $t \in [t_1 - S, t_1]$ then there exists $Q \subseteq [t_1 - S, t_1]$ such that $\mu(Q) \ge \varepsilon/4MK$ and $|x(t)| \ge \varepsilon/4SK$ for all $t \in Q$.*

Proof: If it is not the case, then we have $t_2 \in [t_1 - \varepsilon/4MK, t_1]$ such that $|x(t_2)| < \varepsilon/4SK$. We can suppose, that M and K are large enough that the inequality $S \ge S' +$

$\varepsilon/4MK$ holds. Then from $|x(t_2)| + \|x_{t_2}\|_g \geq \varepsilon$ we have $\|x_{t_2}\|_g \geq \varepsilon - \varepsilon/4SK$. We can assume that $4SK \geq 6$ so $\|x_{t_2}\|_g \geq 5\varepsilon/6$. Then we can easily prove the inequality $\varepsilon/2 \leq \int_{-S'}^{0} |x_{t_2}(s)|g(s)ds$. Let $r := \varepsilon/4S'$ and

$$Q_r := \{s \in [-S', 0] : |x_{t_2}(s)|g(s) \geq r\}.$$

Now (as in Proposition 1) we have $\mu(Q_r) \geq \varepsilon/4MK$. Moreover if $s \in Q_r$ then $|x_{t_2}(s)| \geq r/g(s) \geq \varepsilon/4SK$. Let $Q = Q_r + t_2 = \{s + t_2 : s \in Q_r\}$ and the proof is complete.

DEFINITION: The function $\eta : \mathbf{R} \to \mathbf{R}_+$ is *uniformly positive in measure with parameter S* (UPIM(S)) iff for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $t \geq t_0 + S$, $Q \subseteq [t - S, t]$ and $\mu(Q) \geq \varepsilon$ then $\int_Q \eta(s)ds \geq \delta$.

THEOREM 4: *Assume that $g : \mathbf{R}_- \to \mathbf{R}_+$ is bounded. Suppose that there exists $V : \mathbf{R} \times \mathcal{C} \to \mathbf{R}$ a continuous functional, $W_1, W_2, W_3 \in \mathcal{K}$ such that*

(a) $W_1(|x(t)|) \leq V(t, x_t) \leq W_2(|x(t)| + \|x_t\|_g)$,

(b) $\dot{V}(t, x_t) \leq -\eta(t)W_3(|x(t)|)$,

where η is UPIM(S) for all $S > 0$.

Then the zero solution of equation (1.1) is uniformly asymptotically stable.

Proof: The solution is uniformly stable. Let γ be the number for 1 in uniform stability, so if $t_0 \in \mathbf{R}$ and $\|x_{t_0}\| < \gamma$ then $|x(t)| \leq 1$ for all $t \geq t_0$. Moreover $V(t_0, x_{t_0}) \leq W_2(|x(t_0)| + \|x_{t_0}\|_g) \leq W_2(2)$.

For all $\omega > 0$ we must find a $T > 0$ such that if $t_0 \in \mathbf{R}$, then $\|x_{t_0}\| < \gamma$ implies $|x(t)| < \omega$ for all $t \geq t_0 + T$. Let N be a natural number and $T := SN$, where S is given in Proposition 3. Suppose for a contradiction, that there exists $t' \geq t_0 + T$ such that $|x(t')| \geq \omega$. Now for $t \in [t_0, t']$ we have

$$\begin{aligned} |x(t)| + \|x_t\|_g &\geq W_2^{-1}(V(t, x_t)) \geq W_2^{-1}(V(t', x_{t'})) \\ &\geq W_2^{-1}(W_1(|x(t')|)) \geq W_2^{-1}(W_1(\omega)) =: \varepsilon. \end{aligned}$$

From Proposition 3 we have $Q_i \subseteq [t_0 + (i-1)S, t_0 + iS]$ $(i = 1, 2, ..., N)$ such that $|x(t)| \geq \varepsilon/4SK$ for all $t \in Q_i$ and $\mu(Q_i) \geq \varepsilon/4MK$. Now

$$\begin{aligned} V(t_0 + T, x_{t_0+T}) &\leq V(t_0, x_{t_0}) + \int_{t_0}^{t_0+T} \dot{V}(s, x_s)ds \leq W_2(2) - \sum \int_{Q_i} \eta(s)W_3(|x(s)|)ds \\ &\leq W_2(2) - W_3(\varepsilon/4SK) \sum \int_{Q_i} \eta(s)ds \leq W_2(2) - W_3(\varepsilon/4SK)N\delta, \end{aligned}$$

where δ is in the definition of uniform positivity in measure for $\varepsilon/4MK$. Define

$$N := [W_2(2)/(W_3(\varepsilon/4SK)\delta)] + 1,$$

so $V(t_0 + T, x_{t_0+T}) < 0$, which is a contradiction. Then the proof is complete.

Note that this theorem for $\eta \equiv 1$ can be found in the papers of Burton ([**1**]) and Burton and Zhang ([**4**]).

At the end we state two propositions in order to generalize our theorems. First of all we generalize the measure $\|\cdot\|_g$.

Definition: $\|\phi\|_g^h := \int_{-\infty}^0 h(|\phi(s)|)g(s)ds$, where $h : \mathbf{R}_+ \to \mathbf{R}_+$ and $\int_{-\infty}^0 g(s)ds < \infty$.

Definition: The measures $\|\cdot\|_{g_1}^{h_1}$ and $\|\cdot\|_{g_2}^{h_2}$ are equivalent, iff for all $M > 0$ there exist $W_1, W_2 \in \mathcal{K}$ such that $W_1(\|\phi\|_{g_1}^{h_1}) \leq \|\phi\|_{g_2}^{h_2} \leq W_2(\|\phi\|_{g_1}^{h_1})$ for all $\phi \in \mathcal{C}_M$.

It is obvious, that if two measures are equivalent, then they can be used interchangable in theorems using Lyapunov functionals for local stability. The following theorems prove equivalencies, which generalize our theorems.

Proposition 4: *Let the functions $h_1, h_2 : \mathbf{R}_+ \to \mathbf{R}_+$ with $W_1, W_2 \in \mathcal{K}$ be such that $W_1(h_1(s)) \leq h_2(s) \leq W_2(h_1(s))$ and $g_1, g_2 : \mathbf{R}_- \to \mathbf{R}_+$ with $W_3, W_4 \in \mathcal{K}$ such that $W_3(g_1(s)) \leq g_2(s) \leq W_4(g_1(s))$. Suppose that there exists $K > 0$ such that $g_1(s), g_2(s) \leq K$ and for all $M > 0$ there exists $H(M) > 0$ such that $h_1(s), h_2(s) \leq H$ for all $s \in [0, M]$. Then the measures $\|\cdot\|_{g_1}^{h_1}$ and $\|\cdot\|_{g_2}^{h_2}$ are equivalent.*

Proof: Let $M > 0$ be given, $\phi \in \mathcal{C}_M$, $H = H(M)$ and $\varepsilon := \|\phi\|_{g_2}^{h_2}$. If $\varepsilon = 0$ then we have $\|\phi\|_{g_1}^{h_1} \geq W(\|\phi\|_{g_2}^{h_2})$ for all $W \in \mathcal{K}$, so we can assume $\varepsilon > 0$. There exists $0 < S = S(\varepsilon, M)$ such that $\int_{-\infty}^{-S} g_2(s)ds \leq \varepsilon/3H$. (We can assume, that $S(\varepsilon)$ is a monotone decreasing function.) Then (as in Proposition 1)

$$2\varepsilon/3 \leq \int_{-S}^0 h_2(|\phi(s)|)g_2(s)ds.$$

Let $r := \varepsilon/3S$ and $Q_r := \{s \in [-S, 0] : h_2(|\phi(s)|)g_2(s) \geq r\}$. Then we have the inequality

$$\mu(Q_r) \geq (2\varepsilon/3 - Sr)/(HK - r) \geq \varepsilon/3HK$$

and if $s \in Q_r$, then $h_2(|\phi(s)|) \geq r/g_2(s) \geq r/K$ and $g_2(s) \geq r/h_2(|\phi(s)|) \geq r/H$. So

$$\begin{aligned}
\|\phi\|_{g_1}^{h_1} &\geq \int_{Q_r} h_1(|\phi(s)|)g_1(s)ds \\
&\geq \int_{Q_r} W_2^{-1}(h_2(|\phi(s)|))W_4^{-1}(g_2(s))ds \\
&\geq W_2^{-1}(r/K)W_4^{-1}(r/H)\mu(Q_r) \\
&\geq W_2^{-1}(r/K)W_4^{-1}(r/H)\varepsilon/3HK =: W(\varepsilon) \\
&= W(\|\phi\|_{g_2}^{h_2}),
\end{aligned}$$

and the proof is complete.

Proposition 5: *Measures* $\|x_t\|_g + |x(t)|$ *and* $W_1(\|x_t\|_g) + W_2(|x(t)|)$ *are equivalent.*

Proof:

$$W_1(|x(t)|) + W_2(\|x_t\|_g) \leq W(|x(t)|) + W(\|x_t\|_g) \leq 2W(|x(t)| + \|x_t\|_g),$$

where $W(x) = \max\{W_1(x), W_2(x)\}$. On the other hand,

$$W_1(|x(t)|) + W_2(\|x_t\|_g) \geq W(|x(t)|) + W(\|x_t\|_g) \geq W((|x(t)| + \|x_t\|_g)/2),$$

where $W(x) = \min\{W_1(x), W_2(x)\}$ and the proof is complete.

It is left to the reader to state the generalized version of Theorems 1 through 4 using Propositions 4 and 5.

3. Example

Consider the following functional differential equation with infinite delay

$$\dot{x}(t) = -a(t)f(x(t)) + b(t)\int_{-\infty}^{t} C(t-s)g(x(s))ds, \tag{3.1}$$

where $f, g : \mathbf{R}^n \to \mathbf{R}$, $a, b : \mathbf{R} \to \mathbf{R}$ and $C : \mathbf{R}_+ \to \mathbf{R}$. Define the functional

$$V(t, x_t) = |x(t)| + \int_{-\infty}^{t}\int_{t}^{\infty} \beta(u)|C(u-s)|du|g(x(s))|ds$$

where $\beta : \mathbf{R} \to \mathbf{R}_+$.

First we prove condition (a) of our theorems. It is obvious that $|x(t)| \leq V(t, x_t)$. Suppose, that there exist functions $\beta : \mathbf{R} \to \mathbf{R}_+$ and $W_1, W_3 \in \mathcal{K}$ such that

$$\int_t^\infty \beta(u)|C(u-s)|du \leq W_1(|C(t-s)|), \text{ and} \tag{3.2}$$

$$|g(x)| \leq W_3(|x|). \tag{3.3}$$

From Proposition 4 we have $W_2 \in \mathcal{K}$ such that

$$V(t, x_t) \leq |x(t)| + W_2\left(\int_{-\infty}^t |C(t-s)||x(s)|ds\right),$$

and condition (a) is satisfied if

$$\int_0^\infty |C(s)|ds < \infty. \tag{3.4}$$

For condition (b) we prove an inequality for the derivative of the functional V with respect to equation (3.1).

$$\begin{aligned}\dot{V}_{(3.1)}(t, x_t) = &- a(t)|f(x(t))| + |b(t)| \int_{-\infty}^t |C(t-s)||g(x(s))|ds - \\ &\int_{-\infty}^t \beta(t)|C(t-s)||g(x(s))|ds + \int_t^\infty \beta(u)|C(u-t)|du|g(x(t))|\end{aligned}$$

Suppose, that

$$|g(x)| \leq c|f(x)| \tag{3.5}$$

for some $c \geq 0$, then

$$\begin{aligned}\dot{V}_{(3.1)}(t, x_t) \leq &- \left[a(t) - c\int_t^\infty \beta(u)|C(u-t)|du\right]|f(x(t))| - \\ &[\beta(t) - |b(t)|]\int_{-\infty}^t |C(t-s)||g(x(s))|ds\end{aligned}$$

Now we present an example for Theorem 2. Suppose, that there are functions β, W_1 and W_3, W_4 and a constant c such that conditions (3.2), (3.4) and (3.5) are hold,

$$|g(x)| \geq W_4(|x|) \tag{3.6}$$

and $\left[a(t) - c\int_t^\infty \beta(u)|C(u-t)|du\right] \geq 0$. Then

$$\dot V_{(3.1)}(t, x_t) \leq -\left[\beta(t) - |b(t)|\right] \int_{-\infty}^{t} |C(t-s)||g(x(s))|ds \leq$$
$$-\left[\beta(t) - |b(t)|\right] W_5 \left(\int_{-\infty}^{t} |C(t-s)||x(s)|ds \right)$$

and condition (b) is satisfied if the function $[\beta(t) - |b(t)|]$ is UWIP$(\delta, 2S)$ for every $\delta > 0$ and some $S > S_0$. So we proved the following

Corollary 1: *Suppose, that there are $\beta : \mathbf{R} \to \mathbf{R}_+$, $c \geq 0$ and $W_1, W_3 \in \mathcal{K}$ such that conditions* (3.2), (3.3), (3.4), (3.5) *and* (3.6) *are hold,* $\left[a(t) - c\int_t^\infty \beta(u)|C(u-t)|du\right] \geq 0$, *and $[\beta(t) - |b(t)|]$ is UWIP$(\delta, 2S)$ for some $\delta > 0$ and $S > S_0$. Then the zero solution of equation (3.1) is uniformly asymptotically stable.*

For Theorem 4 we assume that $\beta(t) = |b(t)|$, $|f(x)| \geq W_2(|x|)$ for some $W_2 \in \mathcal{K}$ and the function $\left[a(t) - c\int_t^\infty |b(u)||C(u-t)|du\right]$ is UPIM(S) for all $S > 0$. Then

$$\dot V_{(3.1)}(t, x_t) \leq -\left[a(t) - c\int_t^\infty |b(u)||C(u-t)|du\right] W_2(|x|),$$

and condition (b) of Theorem 4 is satisfied. Therefore

Corollary 2: *If there are $W_1, W_2, W_3 \in \mathcal{K}$ and $c \geq 0$ such that conditions* (3.1), (3.2), (3.3), (3.4) *and* (3.5) *hold, $|f(x)| \geq W_2(|x|)$ for some $W_2 \in \mathcal{K}$ and the function* $\left[a(t) - c\int_t^\infty |b(u)||C(u-t)|du\right]$ *is UPIM(S) for all $S > 0$, then the zero solution of equation (3.1) is uniformly asymptotically stable.*

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