Periodic Solutions of Dissipative Functional Differential Equations

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Abstract. We consider periodic, finite delay differential equations. The first two theorems prove dissipativeness for these equations generalizing a result of Pliss. We use dissipativeness to prove the existence of a periodic solution by a result of Hale and Lopes. In our special case we give short, elementary proof for their theorem. We also present a theorem using Liapunov functionals to show dissipativeness.

1. Introduction. Let $f : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ be continuous and locally Lipschitz in x with $f(t+T, x) = f(t, x)$ for all (t, x) and some $T > 0$. Under these conditions, Pliss [10, Theorem 2.1] showed that solutions of

$$x' = f(t, x) \tag{1}$$

are dissipative if and only if there is an $r > 0$ such that for each (t_0, x_0) there is a $\tau > t_0$ with $|x(\tau, t_0, x_0)| < r$.

An exact counterpart of this beautiful result would have great application in functional differential equations. However, we show that an exact counterpart is false, but that a variation of it is true. Hale and Lopes [4] proved that dissipativeness implies the existence of a periodic solution for functional differential equations. We will give a simpler proof of their more general result in this special case.

Definition. The equation (1) is said to be *dissipative*, if there is an $r > 0$ such that

$$\limsup_{t \to \infty} |x(t, t_0, x_0)| < r$$

for all t_0 and x_0.

Definition. A *wedge* is a continuous function $W : [0, \infty) \to [0, \infty)$ with $W(0) = 0$ and W is strictly increasing.

The classical result on dissipative behavior for (1) may be stated as follows:

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THEOREM A. Let $V : [0, \infty) \times \mathbf{R}^n \to [0, \infty)$ be continuous and locally Lipschitz in x and suppose that there are wedges W_i and a constant $U > 0$ such that

(i) $W_1(|x|) \leq V(t, x) \leq W_2(|x|)$,

(ii) $V'_{(1)}(t, x) \leq -W_3(|x|)$ if $|x| \geq U$,

(iii) $W_1(s) \to \infty$ as $s \to \infty$.

Then solutions of (1) are dissipative.

This result is true even when $f(t, x)$ is not periodic in t. Moreover, one can prove uniform boundedness and uniform ultimate boundedness, which are much stronger properties than the dissipativeness. For an example, which is dissipative, but neither uniform bounded nor uniform ultimate bounded, we refer to Kato's paper [7].

We now introduce a functional differential equation with finite delay. Let $(\mathcal{C}, \|\cdot\|)$ be the Banach space of continuous functions $\phi : [-h, 0] \to \mathbf{R}^n$ with the supremum norm. Denote $x_t(s) = x(t + s)$ for $-h \leq s \leq 0$ and let $F : \mathbf{R} \times \mathcal{C} \to \mathbf{R}^n$ be continuous with $F(t + T, \phi) = F(t, \phi)$ for some $T > 0$. We also assume that F is locally Lipschitz in ϕ, i.e. for all $M > 0$ there is an $L > 0$ such that $|t| \leq M$, $\|\phi_1\| \leq M$ and $\|\phi_2\| \leq M$ imply $|F(t, \phi_1) - F(t, \phi_2)| \leq L\|\phi_1 - \phi_2\|$. Then

$$x' = F(t, x_t) \tag{2}$$

is a system of functional differential equations and for each $(t_0, \phi) \in \mathbf{R} \times \mathcal{C}$ there is a unique solution $x(t, t_0, \phi)$. From the local Lipshitz condition of F we know that F takes bounded sets of (t, ϕ) into bounded sets of $\mathbf{R}^n$. Note that from this property the periodicity of F implies that F takes bounded sets of ϕ into bounded sets.

DEFINITION. The equation (2) is said to be *dissipative* or *point dissipative*, if there is an $r > 0$ such that

$$\limsup_{t \to \infty} |x(t, t_0, \phi)| < r$$

for all t_0 and $\phi \in \mathcal{C}$.

The classical conjecture parallel to Theorem A may be stated as follows:

CONJECTURE. Suppose that there is a continuous functional $V : \mathbf{R} \times \mathcal{C} \to [0, \infty)$ which is locally Lipschitz in ϕ, together with wedges W_i and a constant $U > 0$ such that

(i) $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(\|\phi\|)$,

(ii) $V'_{(2)}(t, x_t) \leq -W_3(|x(t)|)$ if $|x(t)| \geq U$,

(iii) $W_1(s) \to \infty$ as $s \to \infty$.

Then solutions of (2) are dissipative.

If we do not assume that F is T-periodic, this conjecture is false (as is seen from the example in [8]), but some severe modifications of it are true (cf. [3]).

However, such a V has been constructed for many important systems and it is clear from (ii) that for each (t_0, ϕ) there is a $\tau > t_0$ such that $|x(\tau, t_0, \phi)| < U$. Our example shows that this is not enough to yield dissipativeness. We show that we must ask that for each (t_0, ϕ) there is a $\tau > t_0$ such that $\|x(t_0, \phi)\|^{[\tau-2h,\tau]} < U$, where $\|x\|^{[a,b]} = \sup\{|x(s)| : a \le s \le b\}$.

2. Main results.

THEOREM 1. *The solutions of equation (2) are dissipative if and only if there is an* $r > 0$ *such that for all* $(t_0, \phi) \in \mathbf{R} \times \mathcal{C}$ *there is a* $\tau \ge t_0 + h$ *such that* $\|x(t_0, \phi)\|^{[\tau-2h,\tau]} < r$.

Proof. This proof is very similar to that of Theorem 2.1 in [10]. The major difference is the way we ensure compactness for the functions in the proof. That is why we need the solution to be "small" on an interval of length twice the delay.

The necessity is clear. We prove the sufficiency by way of contradiction. Suppose that there are functions $\overline{\phi}_1, \overline{\phi}_2, ... \in \mathcal{C}$, constants $t_0^{(1)}, t_0^{(2)}, ... \in \mathbf{R}$ and $R_1, R_2, ... \to \infty$ such that

$$\limsup_{t\to\infty} |x(t, t_0^{(k)}, \overline{\phi}_k)| > R_k.$$

We may assume $R_k > r$. By the given condition there is a $\tau_k \ge t_0^{(k)} + h$ such that $\|x(t_0^{(k)}, \overline{\phi}_k)\|^{[\tau_k-2h,\tau_k]} < r$. From the above assumption there is a $\overline{t}_k > \tau_k$ such that $|x(\overline{t}_k, t_0^{(k)}, \overline{\phi}_k)| > R_k$ and hence there exists a $\overline{\Theta}_k \in [\tau_k, \overline{t}_k]$ such that

$$\|x(t_0^{(k)}, \overline{\phi}_k)\|^{[\overline{\Theta}_k-2h,\overline{\Theta}_k]} = r \text{ and } \|x(t_0^{(k)}, \overline{\phi}_k)\|^{[t-2h,t]} > r \text{ for all } t \in (\overline{\Theta}_k, \overline{t}_k).$$

From the periodicity of F in t there are Θ_k, t_k, ϕ_k such that $0 \le \Theta_k < T$, $t_k > \Theta_K$, $\phi_k = x_{\overline{\Theta}_k}(t_0^{(k)}, \overline{\phi}_k)$ with $\|\phi_k\|^{[-2h,0]} = r$, $|x(t_k, \Theta_k, \phi_k)| > R_k$ and $\|x(\Theta_k, \phi_k)\|^{[t-2h,t]} > r$ for all $t \in (\Theta_k, t_k)$. Since $\|x(t_0^{(k)}, \overline{\phi}_k)\|^{[\overline{\Theta}_k-2h,\overline{\Theta}_k]} = r$ and F is bounded for ϕ bounded we have $|\phi_k'| \le M$, where M depends only on r and F. Hence $\phi_k \in \{\phi \in \mathcal{C} : \|\phi\| = r, |\phi(u) - \phi(v)| \le M|u - v|\}$. Thus, by compactness (using Ascoli's Theorem) without loss of generality we may assume that $\Theta_k \to \Theta$ and $\phi_k \to \phi$. Now for this Θ and ϕ there is a $\tau \geq \Theta + h$ such that $\|x(\Theta, \phi)\|^{[\tau-2h,\tau]} < r$. Let $R > r$ be such that $|x(t, \Theta, \phi)| < R$ for all $t \in [\Theta, \tau]$. By the continuous dependence of the solutions on the initial conditions there is a k_0 such that for all $k \geq k_0$ we have $|x(t, \Theta_k, \phi_k)| < R$ for all $t \in [\Theta_k, \tau]$ and $\|x(\Theta_k, \phi_k)\|^{[\tau-2h,\tau]} < r$. Let $k \geq k_0$ such that $R_k > R$. Then $|x(t_k, \Theta_k, \phi_k)| > R_k$ implies $t_k > \tau$. Therefore, $\|x(\Theta_k, \phi_k)\|^{[t-2h,t]} > r$ for all $t \in (\Theta_k, t_k]$ and this contradicts $\|x(\Theta_k, \phi_k)\|^{[\tau-2h,\tau]} < r$. The proof is complete.

Remark. Note, that the same proof works for any set of solutions of (2). In particular, we can prove the following:

Proposition. *The solution $x : [t_0 - h, \infty) \to \mathbf{R}^n$ of (2) is bounded if and only if there is an $r > 0$ such that for all $K \geq t_0 + h$ there is a $\tau \geq K$ such that $\|x\|^{[\tau-2h,\tau]} < r$.*

We do not use the proposition in the following, but this statement is interesting for its own sake. It says that either the solution is bounded or $\lim_{t\to\infty} \|x\|^{[t-2h,t]} = \infty$. A very similar proof shows in the case of ordinary differential equations that the solution is either bounded or $\lim_{t\to\infty} |x(t)| = \infty$.

Example 1. This example shows that the solutions of a general differential equation can be zero on an interval of length less than the delay, but they are not dissipative.

Let $\varepsilon > 0$ be given and $h = 1$. Consider a smooth, 1-periodic function $\phi : [0, \infty) \to \mathbf{R}$ such that $\phi(0) = 1$ and $\phi(t) = 0$ on the interval $[\varepsilon/2, 1 - \varepsilon/2]$. Consider the differential equation

$$x'(t) = x([t])\phi'(t),$$

where $[\cdot]$ is the greatest integer function. Clearly, if $x(t)$ is a solution, then $x(t + 1)$ is a solution too ($T = 1$). It is also clear, that the solutions of this equation are not dissipative; to see this, if we start the solution at $t_0 = 1$ with the initial function $M\phi(t)$ ($t \in [0, 1]$) for an arbitrary $M > 0$ we get the solution in the form $M\phi$ on the interval $[0, \infty)$. Hence we are unable to find a number R independent of the solutions, such that every solution tends to the R-ball. It is also obvious that for every solution there is an interval of length $1 - \varepsilon$ (say the interval $[[t_0] + 1 + \varepsilon/2, [t_0] + 2 - \varepsilon/2]$) on which the solution is zero. This finishes our example.

One can construct an example so that every solution becomes bounded by 2 on an interval of length less than twice the delay, but the equation is not dissipative. This example is much too complicated to be presented in this paper. It involves a function g defined by Kato in [6].

We know then that in the very general case we cannot expect to relax the condition in Theorem 1. However, if we have a growth condition on F we can prove dissipativeness from a weaker condition. To do this we need the following:

GENERALIZED BELLMAN'S LEMMA. Let $Y(t)$, $H(t)$ be positive continuous functions on the interval $[a,b]$, $k \geq 0$ and $\omega : \mathbf{R} \to \mathbf{R}$ a positive continuous, non-decreasing function. Then the inequality

$$Y(t) \leq k + \int_a^t H(s)\omega(Y(s))ds \qquad (a \leq t \leq b)$$

implies the inequality

$$Y(t) \leq \Omega^{-1}\left(\Omega(k) + \int_a^t H(s)ds\right) \qquad (a \leq t \leq b' \leq b),$$

where

$$\Omega(u) = \int_{u_0}^u \frac{ds}{\omega(s)} \qquad (u_0 > 0, u \geq 0)$$

and Ω^{-1} means the inverse function of $\Omega(u)$. $\Omega^{-1}(u)$ certainly exists owing to the monotonicity of $\Omega(u)$. Of course, t must be in a subinterval (a,b') of (a,b) so that the argument $\Omega(k) + \int_a^t H(s)ds$ will be within the domain of definition of Ω^{-1}.

The proof of this lemma can be found in [1].

THEOREM 2. *Consider again the equation (2) and suppose that there are non-negative, T-periodic functions H and G which are integrable over one period, together with a positive, continuous and non-decreasing function ω such that $\int^\infty (1/\omega(s))ds = \infty$ and $|F(t,\phi)| \leq H(t)\omega(\|\phi\|) + G(t)$. Then the solutions of the equation (2) are dissipative if and only if there is an $r > 0$ such that for all $(t_0, \phi) \in \mathbf{R} \times C$ there is a $\tau > t_0$ such that $\|x_\tau(t_0,\phi)\| < r$.*

Proof: The necessity is clear. Let us integrate the equation (2) from τ to t $(t \geq \tau)$:

$$x(t) - x(\tau) = \int_\tau^t F(s, x_s)ds,$$

and hence

$$|x(t)| \leq |x(\tau)| + \int_\tau^t |F(s,x_s)|ds \leq r + \int_\tau^t H(s)\omega(\|x_s\|)ds + \int_\tau^t G(s)ds.$$

Taking the supremum of both sides in t on the interval $[\tau, p]$ and using the property that $\|x_\tau\| < r$, we have

$$\|x_p\| \leq r + \int_\tau^p H(s)\omega(\|x_s\|)ds + \int_\tau^p G(s)ds \leq \left[r + \int_\tau^{\tau+h} G(s)ds\right] + \int_\tau^p H(s)\omega(\|x_s\|)ds,$$

if $p \in [\tau, \tau + h]$. Now using the generalized version of Bellman's Lemma we get

$$\|x_p\| \leq \Omega^{-1}\left(\Omega\left(r + \int_\tau^{\tau+h} G(s)ds\right) + \int_\tau^p H(s)ds\right).$$

Using this inequality for $p = \tau + h$ we find

$$\|x_{\tau+h}\| \leq \Omega^{-1}\left(\Omega\left(r + \int_\tau^{\tau+h} G(s)ds\right) + \int_\tau^{\tau+h} H(s)ds\right).$$

By the periodicity of H and G and the fact that they are integrable on one period we can find an $M > 0$ such that $\int_t^{t+h} G(s)ds \leq M$ and $\int_t^{t+h} H(s)ds \leq M$ for all t. Taking into account the inequality $\|x_\tau\| < r$ we arrive at

$$\|x\|^{[\tau-h,\tau+h]} \leq \Omega^{-1}(\Omega(r + M) + M),$$

and the conditions of Theorem 1 are satisfied. This completes the proof.

REMARK. Once again we note that the same proof works to establish the following:

PROPOSITION. *Suppose that there are non-negative, T-periodic functions H and G which are integrable over one period, together with a positive, continuous and non-decreasing function ω such that $\int^\infty(1/\omega(s))ds = \infty$ and $|F(t,\phi)| \leq H(t)\omega(\|\phi\|) + G(t)$. The solution $x : [t_0 - h, \infty) \to \mathbf{R}^n$ of (2) is bounded if and only if there is an $r > 0$ such that for all $K \geq t_0$ there is a $\tau \geq K$ such that $\|x\| < r$.*

Also observe that Example 1 shows that we cannot expect a better condition in the theorem's statement.

We now show how to use the dissipativeness of the solutions in proving the existence of a periodic solution. Note that uniform boundedness and uniform ultimate boundedness is enough to prove that there is a T-periodic solution of (2) (see, e.g., [2]). It is clear that dissipativeness is a weaker condition than uniform ultimate boundedness. In the following, we restate some results of Hale and Lopes [4]. Their results are more general than necessary for our purposes. The author was surprised to discover (again) that dissipativeness itself is enough to prove the existence of a T-periodic solution. We introduce the following properties:

DEFINITION. The equation (2) is said to be *point dissipative (at $t = 0$)*, if there is an $r > 0$ such that

$$\limsup_{t\to\infty} |x(t, 0, \phi)| < r$$

for all $\phi \in \mathcal{C}$.

DEFINITION. The equation (2) is *compact dissipative (at $t = 0$)*, if there is an $r > 0$ such that for each compact subset K of $\mathcal{C}$ there is a $P > 0$ such that $|x(t, 0, \phi)| < r$ for all $\phi \in K$ and $t \geq P$.

DEFINITION. The equation (2) is *compact uniform bounded (at $t = 0$)*, if for each compact subset K of $\mathcal{C}$ there is a $B > 0$ such that $|x(t, 0, \phi)| < B$ for all $\phi \in K$ and $t \geq 0$.

The next two theorems were also proved by Kato [7], in which he considers many different kinds of boundedness and shows the connections among them. Here we give a short direct proof in our special cases. In the following we assume that all solutions can be continued for all future times. The property that (2) is point dissipative implicitly implies this condition, but we want to be clear on this point.

LEMMA. *If (2) is point dissipative at $t = 0$, then $\limsup_{t\to\infty} |x(t, t_0, \phi)| < r$ for all $(t_0, \phi) \in \mathbf{R} \times \mathcal{C}$, so (2) is dissipative in the sense of the first definition of dissipativeness.*

Proof. For a given (t_0, ϕ), let $mT \in [t_0, t_0 + T)$ and $\psi := x_{mT}(\cdot, t_0, \phi)$. Then $x(t, t_0, \phi) = x(t, mT, \psi) = x(t - mT, 0, \psi)$ and hence

$$\limsup_{t\to\infty} |x(t, t_0, \phi)| = \limsup_{t\to\infty} |x(t - mT, 0, \psi)| < r$$

from point dissipativeness at $t = 0$, and the proof is complete.

THEOREM 3. *If (2) is dissipative (point dissipative), then it is compact uniform bounded.*

Proof. Suppose this is not the case. Then we find a compact set K in $\mathcal{C}$ and sequences $\phi_n \in K$ and $t_n \geq 0$ such that $|x(t_n, 0, \phi_n)| \to \infty$. Let $r > 0$ be the number in the definition of point dissipativeness and let $L > 0$ be such that $|F(t, \phi)| \leq L$ for $\|\phi\| \leq r$. Without loss of generality, we may adjoin to K the compact set $\{\phi \in \mathcal{C} : \|\phi\| \leq r, |\phi(u) - \phi(v)| \leq L|u - v|\}$, and still call it K. We can also assume that n is so large that $x_{t_n}(\cdot, 0, \phi_n)$ is not in K, because K is bounded and $|x(t_n, 0, \phi_n)| \to \infty$. Since $\phi_n = x_0(\cdot, 0, \phi_n) \in K$, we can define τ_n so that $\psi_n := x_{\tau_n}(\cdot, 0, \phi_n) \in K$, but $x_t(\cdot, 0, \phi_n) \notin K$ for $t \in (\tau_n, t_n]$. Using a translation argument we find a $\bar{t}_n \in [0, T]$ such that $x(t, \bar{t}_n, \psi_n) = x(t + (\tau_n - \bar{t}_n), \tau_n, \psi_n) = x(t + (\tau_n - \bar{t}_n), 0, \phi_n)$. Since K and $[0, T]$ are compact, there are subsequences, say ψ_n and $\bar{t}_n$ again, such that $\psi_n \to \psi$ and $\bar{t}_n \to \bar{t}$. Using the lemma for this $\bar{t}$ and ψ we find a $P \geq \bar{t}$ such that $|x(t, \bar{t}, \psi)| < r$ for $t \geq P$. Let $M > 0$ be a number with $|x(t, \bar{t}, \psi)| < M$ on the interval $[\bar{t}, P]$. Take any n large enough to have $|x(t, \bar{t}_n, \psi_n)| < M$ for $t \in [\bar{t}_n, P]$, $|x(t_n, 0, \phi_n)| = |x(t_n - (\tau_n - \bar{t}_n), \bar{t}_n, \psi_n)| \geq M$ and $|x(t, \bar{t}_n, \psi_n)| < r$ for $t \in [P, P + 2h]$. Then we must have $t_n - (\tau_n - \bar{t}_n) > P + 2h$ and also $x_{P+2h}(\cdot, \bar{t}_n, \psi_n) \in \{\phi \in \mathcal{C} : \|\phi\| \leq r, |\phi(u) - \phi(v)| \leq L|u - v|\} \subset K$. But this is a contradiction to the choice of τ_n, because we must have $x_t(\cdot, \bar{t}_n, \psi_n) \notin K$ for $t \in [\bar{t}_n, t_n - (\tau_n - \bar{t}_n)]$. This contradiction shows the required compact uniform boundedness.

Theorem 4. *If (2) is dissipative (point dissipative), then it is compact dissipative.*

Proof. Let $r > 0$ be the number from point dissipativeness. From the previous theorem we find an $R > 0$ such that $\phi \in \{\psi \in \mathcal{C} : \|\psi\| \leq r : |\psi(u) - \psi(v)| \leq L|u - v|\}$ implies $|x(t, 0, \phi)| < R$, where L is from the proof of the previous theorem. We claim that (2) is compact dissipative with R. Suppose for contradiction that there is a compact set K and sequences $\phi_n \in K$ and $t_n \to \infty$ such that $|x(t_n, 0, \phi_n)| \geq R$. As K is compact, there is a subsequence of ϕ_n, say ϕ_n again, such that $\phi_n \to \phi \in K$. For this ϕ we find an $m > 0$ such that $|x(t, 0, \phi)| < r$ for $t \in [mT - 2h, mT]$. Then, using the continuous dependence of the solutions on the initial data, take n large enough to have $|x(t, 0, \phi_n)| < r$ for $t \in [mT - 2h, mT]$ and $t_n > mT$. Using the compact uniform boundedness, we find that $|x(t, 0, \phi_n)| = |x(t, mT, x_{mT}(\cdot, 0, \phi_n))| = |x(t - mT, 0, x_{mT}(\cdot, 0, \phi_n))| < R$ for all $t \geq mT$, which is a contradiction to $t_n > mT$ and $|x(t_n, 0, \phi_n)| \geq R$. This contradiction shows the compact dissipativeness.

Theorem 5. *If (2) is point dissipative, then it has a T-periodic solution.*

Proof. From Theorems 3 and 4 we know that (2) is compact uniform bounded and compact dissipative. Since the proof of this theorem is very similar to the usual proof of the existence of a T-periodic solution assuming uniform boundedness and uniform ultimate boundedness (see [2, Theorem 4.2.2]), we will give only a sketch of the proof. Let $r > 0$ be the number from compact dissipativeness and $|F(t,\phi)| \le L$ for $\|\phi\| \le r$. Let

$$S_0 := \{\phi \in \mathcal{C} : \|\phi\| \le r, |\phi(u) - \phi(v)| \le L|u-v|\}$$

and define $P : \mathcal{C} \to \mathcal{C}$ by $P\phi := x_T(\cdot, 0, \phi)$. From the compact uniform boundedness we find $B_1 > 0$ such that $\|P^n(S_0)\| < B_1$ for $n \ge 0$. Define $L_1 > 0$ such that $\|\phi\| \le B_1$ implies $|F(t,\phi)| \le L_1$ and let

$$S_1 := \{\phi \in \mathcal{C} : \|\phi\| < B_1, |\phi(u) - \phi(v)| \le L_1|u-v|\}.$$

Once again using the compact uniform boundedness we define B_2 and L_2 such that if

$$S_2 := \{\phi \in \mathcal{C} : \|\phi\| \le B_2, |\phi(u) - \phi(v)| \le L_2|u-v|\}$$

then $P^n(S_1) \subset S_2$ for all $n \ge 0$. Also, from compact dissipativeness we find an $m > 0$ such that $P^n(S_1) \subset S_0$ for $n \ge m$. Now all the conditions of Horn's fixed-point theorem (see [5] or [2, Section 3.4]) are satisfied, and hence there is a fixed point of P, which is (of course) a T-periodic solution of (2). The proof is complete.

To satisfy the condition of this theorem we do not need the dissipativeness stated at the beginning of this paper, but we need only point dissipativeness at $t = 0$. In the very same way as one proves the propositions stated after Theorems 1 and 2, one can prove:

THEOREM 6. *The solutions of the equation (2) are point dissipative at $t = 0$ if and only if there is an $r > 0$ such that for all $\phi \in \mathcal{C}$ and $K \ge h$ there is a $\tau \ge K$ such that $\|x(0,\phi)\|^{[\tau-2h,\tau]} < r$.*

Also, if we have the growth condition on F used in Theorem 2, we can prove a similar result with $\|x_\tau(0,\phi)\| < r$ in it.

We now prove theorems using Liapunov functionals to prove dissipativeness.

THEOREM 7. *Suppose there are a functional $V : \mathbf{R} \times \mathcal{C} \to \mathbf{R}$ and constants $a, b, M, U > 0$ such that*

(i) $0 \le V(t,\phi)$,

(ii) $V'(t,x_t) \le M$ *and*

(iii) $V'(t,x_t) \le -a|x'(t)| - b$ *for* $|x(t)| \ge U$.

Then the solutions of (2) are dissipative.

Proof. By our previous theorem we need only to prove that there is an $r > 0$ such that for every $(t_0, \phi) \in \mathbf{R} \times \mathcal{C}$ there is a $\tau \geq t_0 + h$ such that $\|x(\cdot, t_0, \phi)\|^{[\tau-2h,\tau]} < r$. Define $L > 0$ so that $aL - 2Mh > 0$, and let $r := U + L$. Suppose for contradiction that there is $(t_0, \phi) \in \mathbf{R} \times \mathcal{C}$ such that $\|x(\cdot, t_0, \phi)\|^{[t-2h,t]} \geq r$ for all $t \geq t_0 + h$. Define $t_n = t_0 + 2nh$ and $S_n := [t_{n-1}, t_n]$. From our assumption we can find a $t'_n \in S_n$ such that $|x(t'_n)| = |x(t'_n, t_0, \phi)| \geq r = U + L$. We have two cases.

Case 1: If $|x(t)| \geq U$ for all $t \in S_n$, then using (iii)

$$V(t_n) - V(t_{n-1}) \leq \int_{t_{n-1}}^{t_n} V'(s)ds \leq -b(t_n - t_{n-1}) = -2bh.$$

Case 2: If there is a $t \in S_n$ such that $|x(t)| < U$, then by (ii) and (iii)

$$V(t_n) - V(t_{n-1}) \leq M(t_n - t_{n-1}) - |\int_t^{t_n} a|x'(s)|ds| \leq 2Mh - aL < 0.$$

In both cases we have proved that $V(t_n) - V(t_{n-1}) \leq -\alpha < 0$ for some $\alpha > 0$. Therefore, $V(t_n) \leq V(t_0) - n\alpha < 0$ for a large enough n, which is a contradiction to (i). Hence there is an $n > 0$ with $\|x(\cdot, t_0, \phi)\|^{[t_n-2h,t_n]} < r$. The proof is complete.

Note that we can replace b by a function $b : \mathbf{R} \to \mathbf{R}$ integrable on any finite interval with $\int_0^\infty b(s)ds = \infty$ and we do not have to change much in the proof. In this case we argue that we cannot have Case 2 infinitely many times, and hence there is an $N > 0$ such that Case 1 holds for $n \geq N$ and so $V(t_n) - V(t_N) \leq -\int_{t_N}^{t_n} b(s)ds$, a contradiction for large n.

If we consider this theorem purely as a tool to prove the existence of a T-periodic solution, then this theorem is better than the theorem in [9] and better than Theorem 4.2.11 in [2]. We also mention that for ordinary differential equations one can prove a theorem similar to Theorem 4.1.16 in [2].

Theorem 8. *Suppose there is a functional $V : \mathbf{R} \times \mathcal{C} \to \mathbf{R}$ such that*

(i) $0 \leq V(t, \phi)$,

(ii) $V'(t, x_t) \leq -W(|x(t)|)$ *for* $|x(t)| \geq U$.

Then the solutions of the ordinary differential equation are dissipative.

This theorem (considered again as a tool for proving the existence of a T-periodic solution) is better than Theorem 4.1.16 in [2].

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