An Example On The Asymptotic Stability For Functional Differential Equations

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Denote by $\mathcal{C}$ the space of continuous functions $\phi : [-r, 0] \to \mathbf{R}^n$, where $r > 0$ is a constant. In the function space $\mathcal{C}$ we use the norms

$$\|\phi\| = \max\{|\phi(s)| : s \in [-r, 0]\}, \ |||\phi||| = \sqrt{\int_{-r}^{0} |\phi(s)|^2 ds}\ ,$$

where $|\cdot|$ denotes an arbitrary norm in $\mathbf{R}^n$. If $x : [t_0 - r, T) \to \mathbf{R}^n$ $(0 \le t_0 < T \le \infty)$ is continuous and $t \in [t_0, T)$, we define $x_t \in \mathcal{C}$ by $x_t(s) := x(t + s)$ for $s \in [-r, 0]$.

Consider the functional differential equation

$$\dot{x} = f(t, x_t), \tag{1}$$

where $f : \mathbf{R}^+ \times \mathcal{C} \to \mathbf{R}^n$ is continuous and $f(t, 0) = 0$ for all $t \ge 0$. We suppose that for each $t_0 \in \mathbf{R}^+$ and each $\phi \in \mathcal{C}$ there is a unique solution $x(\cdot; t_0, \phi)$ defined on an interval $[t_0, t_0 + \alpha)$, $\alpha > 0$. We assume that the function x continuously depends on the initial data t_0, ϕ.

Denote by $\mathcal{K}$ the set of strictly increasing continuous functions $w : \mathbf{R}^+ \to \mathbf{R}^+$ with $w(0) = 0$. If V is a continuous functional from $\mathbf{R}^+ \times \mathcal{C}$ into $\mathbf{R}$, then $\dot{V}$ denotes the derivative of the functional V with respect to equation (1) defined by

$$\dot{V}(t, \phi) = \dot{V}_{(1)}(t, \phi) = \limsup_{\delta \to 0+} \frac{V(t + \delta, x_{t+\delta}(\cdot; t, \phi)) - V(t, \phi)}{\delta}.$$

In this paper we study the asymptotic stability of the zero solution of equation (1) using Lyapunov functionals. In this topic the following two theorems are basic :

Theorem A (see [4, Theorem 4.1]). Suppose, that for every $M > 0$ there is an $L(M) > 0$ such that

$$[t \in \mathbf{R}^+,\ \phi \in \mathcal{C}_M := \{\phi \in \mathcal{C}:\ \|\phi\| \le M\}] \Longrightarrow |f(t,\phi)| \le L(M). \tag{2}$$

If there are $M > 0$, a continuous functional $V : \mathbf{R}^+ \times \mathcal{C}_M \to \mathbf{R}$ and functions $w_1, w_2, w_3 \in \mathcal{K}$ such that

$$w_1(|\phi(0)|) \le V(t,\phi) \le w_2(\|\phi\|)\ , \tag{3}$$

$$\dot{V}(t,\phi) \le -w_3(|||\phi|||) \tag{4}$$

for all $t \in \mathbf{R}^+$, $\phi \in \mathcal{C}_M$, then the zero solution of equation (1) is uniformly asymptotically stable.

Theorem B ([3, Theorem 5.2.1]). Suppose that conditions (2) and (3) of Theorem A hold, and, instead of condition (4), the inequality

$$\dot{V}(t,\phi) \le -w_3(|\phi(0)|) \tag{5}$$

holds. Then the zero solution of equation (1) is uniformly asymptotically stable.

A classical question (see [1, p. 252]) asks whether the boundedness condition can be dropped from Theorems A and B. The first attempt to answer this question is by the author [6]. Junji Kato answered the question, as it is stated, in one of his talks [5]. The answer is no; he gave an example satisfying conditions (3), (4) and (5), but the zero solution of his equation is not asymptotically stable. In this paper we radically simplify his equation.

Let us define the right hand side of equation (1) in the following way: Let $r = 1$ and $g : [0,\infty) \to (0,\infty)$ be a continuously differentiable function such that

1. $g(n) = 1 + \frac{1}{2^n}$ and $\dot{g}(n) = 0$;
2. g is monotone decreasing on the interval $[n, n+\frac{1}{2}]$ and monotone increasing on the interval $[n+\frac{1}{2}, n+1]$;
3. $\int_n^{n+1} g(s)ds \le \frac{1}{2^n}$;
4. $g(s) \le 2$ for $s \in [0,\infty)$,

where $n = 0, 1, 2, \ldots$. Clearly there is such a g function. Now let $\alpha_n = g(n+1)/g(n)$ and $p_n = \alpha_n/(1-\alpha_n)$ and define f on the interval $[n - \frac{1}{2}, n + \frac{1}{2}]$ by

$$f(t,\phi) := \begin{cases} \dfrac{\dot{g}(t)}{g(t)}\phi(0), \text{ if } 0 \le \phi(0) \le \alpha_n\|\phi\| \text{ or } t \in [n, n+\dfrac{1}{2}] \\ -p_n\dfrac{\dot{g}(t)}{g(t)}(\phi(0) - \|\phi\|), \text{ if } \phi(0) \ge \alpha_n\|\phi\| \text{ and } t \in [n - \dfrac{1}{2}, n] \\ -f(t, -\phi), \text{ if } \phi(0) < 0 \end{cases}$$

(I) First we need to verify that f is a continuous functional; in particular we need to verify the continuity when we change definitions in the definition of f. If $\phi(0) = \alpha_n\|\phi\|$, then by the first and second part of the definition

$$f(t,\phi) = \frac{\dot{g}(t)}{g(t)}\phi(0) = \alpha_n\frac{\dot{g}(t)}{g(t)}\|\phi\| = -p_n\frac{\dot{g}(t)}{g(t)}(\alpha_n - 1)\|\phi\| = -p_n\frac{\dot{g}(t)}{g(t)}(\phi(0) - \|\phi\|),$$

so they give the same result. When $t = n$ then $\dot{g}(t) = 0$ so by either the first or second part of the definition $f(t,\phi) = 0$. Similarly, if $\phi(0) = 0$, then the first or third part gives $f(t,\phi) = 0$. Thus, f is continuous.

(II) It is clear, that f satisfies a local Lipshitz condition in its second variable. This and (I) imply that there is a unique solution of the initial value problem for (1) and it depends continuously on the initial data.

(III) Let us examine the behaviour of the solutions. We consider the case when $\phi(0) \ge 0$; the case for $\phi(0) < 0$ can be handled similarly. First, note that the supremum norm of a solution is monotone non-increasing because

(a) if $t \in [n, n+\frac{1}{2}]$ then $f(t,\phi) = (\dot{g}(t)/g(t))\phi(0) \le 0$ because g is monotone decreasing on the interval $[n, n+\frac{1}{2}]$, or

(b) if $\phi(0)$ is near $\|\phi\|$ (the second case in the definition of f), then f becomes smaller and smaller as $\phi(0)$ goes to $\|\phi\|$ and $f(t,\phi) = 0$ when $\phi(0) = \|\phi\|$.

Next, note that if we start the solution at time t_0 with an initial function ϕ, for a time t with $t_0 \le t < t_0 + 1$ and t an integer, the solution obeys the first part of the definition of f and (by the definition of α_n) $x(s)$ will never go over $\alpha_n\|x_s\|$ as $s \in [t, \infty)$. Therefore the solution of (1) obeys the first part of the definition of f on the interval $[t, \infty)$, so $x(s) = cg(s)$, where c is a constant ($c = x(t)/g(t)$).

Now let us define the Lyapunov functional

$$V(t,\phi) = \|\phi\| + \int_t^\infty |x(s)|\, ds + \int_t^\infty \int_{s-1}^s x^2(u)duds,$$

where $x(u) = x(u; t, \phi)$ is a solution of (1) with $x_t(\cdot; t, \phi) = \phi$. We need to check that the integrals exist in the definition of V. First we use the facts that $\|x_s\|$ is monotone non-increasing and $x_t = \phi$ to obtain

$$\int_t^\infty |x(s)|\, ds \le \int_t^{t+1} |x(s)|\, ds + \int_{t+1}^\infty |x(s)|\, ds \le \|\phi\| + \int_{t+1}^\infty |x(s)|\, ds.$$

On the other hand, on the interval $[t+1, \infty)$ the solution is of the form $x(s) = cg(s)$, where $c < \|\phi\|$, because $\|g_s\| > 1$ for all $s \ge 0$ and $\|x_s\|$ is monotone non-increasing. Since $\int_{t+1}^\infty g(s) ds \le \int_1^\infty g(s) ds = \sum_{i=1}^\infty \int_i^{i+1} g(s) ds \le \sum_{i=1}^\infty \frac{1}{2^i} = 1$ we have

$$\int_t^\infty |x(s)|\, ds \le \|\phi\| + \int_{t+1}^\infty |x(s)|\, ds = \|\phi\| + \int_{t+1}^\infty cg(s) ds \le \|\phi\| + c \le 2\|\phi\|.$$

Changing the order of integration in the double integral and using the boundedness of g, a similar argument shows that

$$\int_t^\infty \int_{s-1}^s x^2(u) du ds \le 4\|\phi\|^2.$$

Hence we have proved the existence of the integrals in V and that

$$\|\phi\| \le V(t, \phi) \le 3\|\phi\| + 4\|\phi\|^2,$$

so condition (3) of theorems A and B is satisfied. A similar argument shows that V is continuous and satisfies a local Lipshitz condition in its second variable.

To prove conditions (4) and (5) compute

$$\dot V(t, x_t) \le \frac{d}{dt}(\|x_t\|) - |x(t)| - \int_{t-1}^t x^2(u) du \le -|x(t)| - |||x_t|||^2,$$

because $\|x_t\|$ is monotone non-increasing. Hence all the needed conditions are satisfied.

Now since V satisfies all the conditions of theorems A and B, all that is left to prove is that the zero solution is not asymptotically stable. For this we note that $\varepsilon g(s)$ is a solution of (1) starting at $t_0 = 1$ with initial function $\varepsilon g_1(s) = \varepsilon g(1+s)$. This solution satisfies $\|\varepsilon g_1\| = 2\varepsilon$ and it does not go to 0 because $\varepsilon g(n) = \varepsilon(1 + \frac{1}{2^n}) \ge \varepsilon$.

This completes the counterexample proving that the boundedness condition (condition (2)) cannot be dropped from theorems A and B even if we want to prove only asymptotic stability.

We note that this example solves another interesting problem too. Using a similar result of the theory of ordinary differential equations one can observe the following

Conjecture. Suppose that there are a continuous functional $V : \mathbf{R} \times \mathcal{C} \to \mathbf{R}$ which is locally Lipshitz in ϕ, functions $w_i \in \mathcal{K}$ and constant $U \geq 0$ such that

(i) $w_1(|x(t)|) \leq V(t, x_t) \leq w_2(\|x_t\|)$,

(ii) $V'_{(1)}(t, x_t) \leq -w_3(|x(t)|)$ for $|x(t)| \geq U$ and

(iii) $w_1(r) \to \infty$ as $r \to \infty$.

Then the solutions of (1) are uniformly bounded and uniformly ultimately bounded.

Many authors are investigating how one can get uniform boundedness and uniform ultimate boundedness using Liapunov functionals, a good reference list can be found in [2]. Our example shows that this conjecture is false, $Mg(s)$ is a solution of (1) as before and $Mg(n) \geq M$ for all $n = 0, 1, 2, \ldots$, where M can be any large number. This proves that the solutions of (1) are not uniformly ultimately bounded even with $U = 0$ in the conjecture.

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