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A Simple Proof for Sturm's Separation Theorem

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Consider the second order linear differential equation

$$y'' + f(x)y' + g(x)y = 0, \tag{1}$$

where $f, g : \mathbf{R} \to \mathbf{R}$ are continuous. Sturm's theorem says that the zeros of two linearly independent solutions of (1) separate one another. The standard proof of this theorem is based upon some properties of the *Wronsky determinant* (see e.g. [**1**, p. 124]). In this note we present a proof using only the elementary calculus.

Theorem A. *Let y_1, y_2 be linearly independent solutions of equation* (1). *If ξ, η ($\xi < \eta$) are succesive zeros of y_1, then y_2 has one and only one zero in the interval (ξ, η).*

Proof. Suppose the contrary. Then we can assume, without loss of generality, that $y_1(x) > 0$ and $y_2(x) > 0$ for all $x \in (\xi, \eta)$ (if y is a solution, then $-y$ is also a solution). Since y_1, y_2 are linearly independent solutions, $y_2(x) > 0$ for all $x \in [\xi, \eta]$, too. Define the set

$$\mathcal{C} = \{c \in \mathbf{R} : y_1(x) \le c y_2(x) \text{ for every } x \in [\xi, \eta]\}$$

Since y_1 is bounded above and y_2 is bounded away from zero, $\mathcal{C}$ is not empty. Let $c_0 = \inf \mathcal{C}$.It is easy to see that $y_1(x) \le c_0 y_2(x)$ on the interval $[\xi, \eta]$ and there is an $x_0 \in (\xi, \eta)$ with $y_1(x_0) = c_0 y_2(x_0)$. Obviously, $y_1'(x_0) = c_0 y_2'(x_0)$; therefore, solutions $y_1, c_0 y_2$ satisfy the same initial conditions at x_0. By uniqueness we obtain $y_1 = c_0 y_2$, a contradiction to the linear independence of y_1, y_2.

Since the role of y_1 and y_2 can be changed, the zero of y_2 on (ξ, η) is unique.

This proof works also for some nonlinear equations.

Theorem B. *Suppose that the second order equation*

$$F(y'', y', y, x) = 0 \qquad (F : \mathbf{R}^4 \to \mathbf{R} \textit{ continuous}) \tag{2}$$

satisfies the following conditions:

(a) *if y is a solution of* (2), *then cy is also a solution for all $c \in \mathbf{R}$;*

(b) *the solutions of initial value problems for* (2) *are unique.*

Then Theorem A is true for the equation (2).

Example: Consider the second order nonlinear equation

$$y''(y')^2 + y^3 = 0. \tag{3}$$

Let the function S_0 be defined by

$$x = \int_0^{S_0(x)} \frac{ds}{\sqrt[4]{1-s^4}}$$

for $x \in [0, \hat{\pi}/2]$, where

$$\hat{\pi} := \frac{\pi/2}{\sin(\pi/4)}.$$

It can be seen **[3]** that the $2\hat{\pi}$-periodic function

$$S(x) := \begin{cases} S_0(x), \text{ if } 0 \le x < \hat{\pi}/2 \\ S_0(\hat{\pi} - x), \text{ if } \hat{\pi}/2 \le x < \hat{\pi} \\ -S_0(x - \hat{\pi}), \text{ if } \hat{\pi} \le x < 3\hat{\pi}/2 \\ -S_0(2\hat{\pi} - x), \text{ if } 3\hat{\pi}/2 \le x < 2\hat{\pi} \end{cases}$$

is the solution of equation (3) satisfying the initial conditions $S(0) = 0,\ S'(0) = 1$. The function S can be considered as a generalization of function sin. Since the equation is autonomous and it satisfies conditions (a) and (b) in Theorem B (see **[2]**), every solution is of the form

$$y(x) = cS(x + \alpha) \qquad (c, \alpha \in \mathbf{R}).$$

According to Theorem B, the zeros of two linearly independent solutions of equation (3) separate one another.

REFERENCES

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3. Á. Elbert, A half-linear second order differential equation, *Coll. Math. Soc. J. Bolyai,* 30 (1981) 153-180.