ON THE ASYMPTOTIC STABILITY IN TERMS OF TWO MEASURES FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

G. Makay

Student in Mathematics, University of Szeged, Szeged, H-6726, Hungary

Key words and phrases: Lyapunov functional, Lyapunov's direct method, process, nonautonomous differential systems, partial stability.

1. INTRODUCTION

Denote by $\mathcal{C}$ the space of the continuous functions $\phi : [-r, 0] \to \mathbf{R}^n$, where $r > 0$ is a constant. In the function space $\mathcal{C}$ we use the norms

$$||\phi|| = \max\{|\phi(s)| : s \in [-r, 0]\}, \ |||\phi||| = \sqrt{\int_{-r}^{0} |\phi(s)|^2 ds} \ ,$$

where $|\cdot|$ denotes an arbitrary norm in $\mathbf{R}^n$. If $x : [t_0 - r, T) \to \mathbf{R}^n$ $(0 \leq t_0 < T \leq \infty)$ is continuous and $t \in [t_0, T)$, we define $x_t \in \mathcal{C}$ by $x_t(s) := x(t + s)$ for $s \in [-r, 0]$.

Consider the functional differential equation

$$\dot{x} = f(t, x_t), \tag{1.1}$$

where $f : \mathbf{R}_+ \times \mathcal{C} \to \mathbf{R}^n$ is continuous and $f(t, 0) = 0$ for all $t \geq 0$. We suppose, that for each $t_0 \in \mathbf{R}_+$ and each $\phi \in \mathcal{C}$ there is a unique solution $x(\cdot; t_0, \phi)$ defined on an interval $[t_0, t_0 + \alpha)$, $\alpha > 0$. We assume, that the function x continuously depends on the initial data t_0, ϕ.

Denote by $\mathcal{K}$ the set of strictly increasing continuous functions $w : \mathbf{R}_+ \to \mathbf{R}_+$ with $w(0) = 0$. If V is a continuous functional from $\mathbf{R}_+ \times \mathcal{C}$ into $\mathbf{R}$, then $\dot{V}$ denotes the derivative of functional V with respect to equation (1.1) defined by

$$\dot{V}(t, \phi) = \dot{V}_{(1.1)}(t, \phi) = \limsup_{\delta \to 0+} \frac{V(t + \delta, x_{t+\delta}(\cdot; t, \phi)) - V(t, \phi)}{\delta}.$$

In this paper we study the asymptotic stability of the zero solution of equation (1.1) using Lyapunov functionals. In this topic the following two theorems are basic :

THEOREM A (see [5, Theorem 4.1]). Suppose, that for every $M > 0$ there is an $L(M) > 0$ such that

$$[t \in \mathbf{R}_+,\ \phi \in \mathcal{C}_M := \{\phi \in \mathcal{C}:\ \|\phi\| \le M\}] \Longrightarrow |f(t,\phi)| \le L(M). \tag{1.2}$$

If there are $M > 0$, a continuous functional $V: \mathbf{R}_+ \times \mathcal{C}_M \to \mathbf{R}$ and functions $w_1, w_2, w_3 \in \mathcal{K}$ such that

$$w_1(|\phi(0)|) \le V(t,\phi) \le w_2(\|\phi\|)\ , \tag{1.3}$$

$$\dot V(t,\phi) \le -w_3(|||\phi|||) \tag{1.4}$$

for all $t \in \mathbf{R}_+$, $\phi \in \mathcal{C}_M$, then the zero solution of equation (1.1) is uniformly asymptotically stable.

THEOREM B ([3, Theorem 5.2.1]). Suppose that conditions (1.2) and (1.3) of Theorem A hold, and, instead of condition (1.4), the inequality

$$\dot V(t,\phi) \le -w_3(|\phi(0)|) \tag{1.5}$$

holds. Then the zero solution of equation (1.1) is uniformly asymptotically stable.

It is an old problem (see [1, p. 252]), whether the boundedness condition can be dropped from Theorems A and B. Note, that - as is usual in the stability theory - the proofs of these theorems need conditions (1.4) or (1.5) only along the solutions of (1.1). We will show by examples that if conditions (1.4) and (1.5) are required only along the solutions of (1.1), then the boundedness condition cannot be dropped.

At first we replace the norms $|\cdot|$, $\|\cdot\|$ and $|||\cdot|||$ in Theorems A and B by abstract "measures" (see [6],[7] and [8]). It will be pointed out that, under these general circumstances, to the boundedness condition there corresponds an estimate between the measures. Our examples will show that these estimates are essential.

2. THEOREMS AND COROLLARIES

Let X be a Banach-space. A continuous function $h: \mathbf{R} \times X \to \mathbf{R}_+$ is called a *measure in* X, if $h(t,0) = 0$.

The continuous function $u: \mathbf{R} \times X \times \mathbf{R}_+ \to X$ is said to be a *process* (see e.g. [3, Chapter 4.1]), if $u(t_0, x, 0) = x$ and $u(t_0 + t_1, u(t_0, x, t_1), t) = u(t_0, x, t_1 + t)$ for all $t_0 \in \mathbf{R}$, $t_1, t \in \mathbf{R}_+$. For example, let $U(t_0, \phi, t - t_0) := x_t(\cdot; t_0, \phi)$, where $x(t; t_0, \phi)$ $(t \ge t_0)$ is the solution of equation (1.1) with $x(t_0; t_0, \phi) = \phi$. It is easy to see, that U is a process.

Let h_0 and h be measures. If u is a process and $t_0 \in R$, $x_0 \in X$, then the function $u(t_0, x_0, \cdot)$ is said to be a *motion.* Let x_* denote the state of this motion at $t_* \geq t_0$, i.e. $x_* := u(t_0, x, t_* - t_0)$. The motion is (h_0, h)-*stable* or *stable in measures* (h_0, h), if for each $t_* \geq t_0$ and $\epsilon > 0$ there is a $\delta(x_0, t_0, t_*, \epsilon) > 0$ such that $h_0(t_*, x - x_*) < \delta$ implies $h(t_* + t, u(t_*, x, t) - u(t_*, x_*, t)) < \epsilon$ for all $t \geq 0$. The stability is *uniform*, if δ can be chosen independently of t_*. The motion $u(t_0, x_0, \cdot)$ is said to be (h_0, h)-*attractive*, if there is a $\gamma(x_0, t_0, t_*) > 0$ such that, if $h_0(t_*, x - x_*) < \gamma$, then $h(t_* + t, u(t_*, x, t) - u(t_*, x_*, t)) \to 0$ $(t \to \infty)$. The attractivity is *uniform*, if γ is independent of t_* and the convergence is uniform in t_*. The motion is said to be *(uniformly) asymptotically stable*, if it is (uniformly) stable and (uniformly) attractive.

For a $V : \mathbf{R} \times X \to \mathbf{R}$ we define the derivative $\dot{V}$ with respect to the process u by

$$\dot{V}(t, x) := \limsup_{\delta \to 0+} \frac{V(t + \delta, u(t, x, \delta)) - V(t, x)}{\delta}.$$

Examples: The zero solution of equation (1.1) is stable in the usual sense (see e.g. [3, Chapter 5.1]), if and only if the zero motion $(U(t_0, 0, \cdot))$ is stable in measures $h_0(t, \phi) = \|\phi\|$ and $h(t, \phi) = |\phi(0)|$. The solution $x(\cdot; t_0, \phi_0)$ of equation (1.1) is stable if and only if the motion $U(t_0, \phi_0, \cdot)$ is stable in the previous measures. The zero solution of the ordinary differential equation $\dot{x} = F(t, x)$ $(x \in \mathbf{R}^n)$ is *partially stable* (see [4]), if the zero motion of this equation is stable in the measures $h_0(t, x) = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$ and $h(t, x) = \sqrt{x_1^2 + ... + x_s^2}$, where $0 < s \leq n$ and $x = (x_1, x_2, ..., x_n)$. The stability of an invariant set A with respect to this equation is equivalent to the stability in measures $h_0(t, x) = h(t, x) = d(x, A)$, where $d(x, A)$ means the distance between x and A in $\mathbf{R}^n$. (For further examples for processes, among them partial differential equations, see e.g. [9].)

Proposition 1. Let the measures h_0, h be given. Suppose that there are a continuous functional $V : \mathbf{R} \times X \to \mathbf{R}$, functions $w_1, w_2, w_3, w_4 \in \mathcal{K}$ and a measure h_1 satisfying the following conditions:

$$w_1(h(t, x)) \leq V(t, x) \leq w_2(h_0(t, x)) \tag{i}$$

$$\dot{V}(t, x) \leq -w_3(h_1(t, x)) \tag{ii}$$

$$h_0(t, x) \leq w_4(h_1(t, x)) \tag{iii}$$

for all $t \in \mathbf{R}$ and $x \in X$.

Then the zero motion of the process u is uniformly asymptotically (h_0, h)-stable.

Proof: At first we prove the uniform stability. Let an $\epsilon > 0$ be given and define $\delta(\epsilon) := w_2^{-1}(w_1(\epsilon))$. Now if $h_0(t_*, x) < \delta$, then $V(t_*, x) \leq w_1(\epsilon)$ by (i). Condition (ii) and the continuity of function $V(t_* +$

$t, u(t_*, x, t))$ implies, that this function is nonincreasing in t, so

$$\begin{aligned} w_1(h(t_* + t, u(t_*, x, t))) &\leq V(t_* + t, u(t_*, x, t)) \leq \\ &\leq V(t_*, u(t_*, x, t_* - t_*)) = V(t_*, x) \leq \\ &\leq w_1(\epsilon). \end{aligned}$$

Consequently, $h(t_* + t, u(t_*, x, t)) \leq \epsilon$ which proves the uniform stability.

The conditions imply

$$\begin{aligned} \dot{V}(t_* + t, u(t_*, x, t)) &\leq -w_3(h_1(t_* + t, u(t_*, x, t))) \leq \\ &\leq -w_3(w_4(h_0(t_* + t, u(t_*, x, t)))) \leq \\ &\leq -w_3(w_4(w_2^{-1}(V(t_* + t, u(t_*, x, t))))), \end{aligned}$$

which is a differential inequality for $V(t_* + t, u(t_*, x, t))$. By [6, Theorem 3.1.1] we get

$$\lim_{t \to \infty} V(t_* + t, u(t_*, x, t)) = 0$$

uniformly in t_*. Obviously, inequality (i) proves the uniform asymptotic stability.

Note, that inequalities (ii) and (iii) could be replaced by

$$\dot{V}(t, x) \leq -w_5(h_0(t, x)),$$

with an appropriate $w_5 \in \mathcal{K}$. In spite of this fact we separated them because it is inequality (iii) that corresponds to the boundedness condition in Theorem A.

In order to formulate the problem corresponding to that of omitting the boundedness condition from Theorem A, we weaken some conditions of Proposition 1.

It can be seen that conditions (ii) and (iii) may be asked only along the motions, even we can assume on the motions in condition (iii) that $h(t_* + t, u(t_*, x, t)) \leq B$ for all $t \geq 0$, where B is a constant independent of t_*, x.

Conditions (ii) and (iii) can be further weakened if we need only (nonuniform) asymptotic stability.

The process u is said to be h_0*-continuous (with respect to* x*)* if for every $\epsilon > 0$, $t_0 \in \mathbf{R}$, $t_1 \geq 0$ and $x_0 \in X$ there is $\delta = \delta(t_0, t_1, x_0, \epsilon) > 0$ such that $h_0(t_0, x - x_0) < \delta$ implies $h_0(t_0 + t_1, u(t_0, x, t_1) - u(t_0, x_0, t_1)) < \epsilon$.

If the process u is h_0-continuous, then conditions (ii) and (iii) can be asked only for sufficiently large values of t.

This modifications result in the following.

Theorem 1. Let measures h_0, h be given. Assume that the process u is h_0-continuous with respect to x. Suppose that there are continuous functional $V : \mathbf{R} \times X \to \mathbf{R}$, functions $w_1, w_2, w_3, w_4 \in \mathcal{K}$, measure h_1, and constants $T, B > 0$ satisfying the following conditions:

$$w_1(h(t,x)) \le V(t,x) \le w_2(h_0(t,x)) \tag{i}$$

for all $t \ge t_0$ and $x \in X$,

$$\dot{V}(t_* + t, u(t_*, x, t)) \le -w_3(h_1(t_* + t, u(t_*, x, t))) \tag{ii}$$

for all $t_* \ge t_0$, $t \ge T$ and $x \in X$, and

$$h_0(t_* + t, u(t_*, x, t)) \le w_4(h_1(t_* + t, u(t_*, x, t))) \tag{iii}$$

for each $t_* \ge t_0$, $t \ge T$ and $x \in X$ such that $h(t_* + s, u(t_*, x, s)) < B$ for all $s \ge T$.

Then the zero motion of the process u is asymptotically (h_0, h)-stable.

Corollary 1 (Theorem A revisited). Suppose that all but condition (1.4) of Theorem A are satisfied. Suppose, in addition, that there is a $T = T(t_0) > 0$ such that the inequality

$$\dot{V}(t, x_t) \le -w_3(|||x_t|||) \tag{1.4$'$}$$

holds for all $t \ge t_0 + T$ and for every solution $x : [t_0 - r, \infty) \to \mathbf{R}^n$ of (1.1).

Then the zero solution of equation (1.1) is asymptotically stable.

Proof: Let $h_0(t, \phi) := ||\phi||$, $h(t, \phi) := |\phi(0)|$ and $h_1(t, \phi) := |||\phi|||$ in Theorem 1. It is enough to prove, that condition (iii) in Theorem 1 follows from the boundedness condition of Theorem A. Inequalities (i) and (ii) imply the stability, so we can assume, that the solutions are bounded above in the measure h with an arbitrary B. Suppose that $T \ge r$. We have from the boundedness (see condition (1.2)), that the absolute value of the derivative of function $x(t; t_0, \phi)$ less or equal than $L(B)$. For every $t \ge t_0$ there exists $t_1 \in [t - r, t]$ with $|x(t_1)| = ||x_t||$. So if $t \ge t_0 + r$, then $s \in [t_1 - ||x_t||/2L(B), t_1 + ||x_t||/2L(B)]$ implies $|x(s)| \ge ||x_t||/2$ and we have the inequality

$$|||x_t||| \ge \sqrt{\frac{||x_t||^3}{8L(B)}}.$$

This means that condition (iii) is satisfied and the proof is complete.

Theorem B cannot be deduced from Theorem 1, because one cannot estimate below the measure $|\phi(0)|$ by the measure $||\phi||$. So we must replace inequality (iii) with a more general condition.

THEOREM 2. Assume that all but condition (iii) of Theorem 1 are satisfied. Suppose, in addition that if $w_2(h_0(t_* + t, u(t_*, x, t))) > \epsilon > 0$ for all $t \geq T$, then

$$\int_T^\infty w_3(h_1(t_* + t, u(t_*, x, t)))dt = \infty.$$

Then the zero motion of the process u is asymptotically (h_0, h)-stable.

Proof: By h_0-continuity of the process u, for every $\alpha > 0$ there exists a $\delta(t_0, x_0, t_*, T, \alpha) > 0$ such that $h_0(t_*, x - x_*) < \delta$ implies $h_0(t_* + T, u(t_*, x, T) - u(t_*, x_*, T)) < \alpha$. From the proof of Proposition 1, for each $\epsilon > 0$ we get an $\alpha(t_0, x_0, t_*, T, \epsilon) > 0$ such that if $h_0(t_* + T, u(t_*, x, T) - u(t_*, x_*, T)) < \alpha$, then $h(t_* + t, u(t_*, x, t) - u(t_*, x_*, t)) < \epsilon$ for all $t \geq T$. This proves the stability of the zero motion.

To complete the proof it is enough to show that

$$\lim_{t\to\infty} V(t_* + t, u(t_*, x, t)) =: v_0 = 0.$$

Suppose $v_0 > 0$. Then $v_0 \leq w_2(h_0(t_* + t, u(t_*, x, t)))$ for all $t \geq T$ and the last condition of the theorem gives

$$\int_T^\infty w_3(h_1(t_* + t, u(t_*, x, t)))dt = \infty.$$

By condition (ii) we have

$$\begin{aligned} 0 &\leq V(t_* + t, u(t_*, x, t)) \leq \\ &\leq V(t_* + T, u(t_*, x, T)) - \int_T^t w_3(h_1(t_* + t, u(t_*, x, t)))dt \to -\infty \end{aligned}$$

$(t \to \infty)$, which is a contradiction.

COROLLARY 2 (Theorem B revisited). Suppose that all but condition (1.5) of Theorem B are satisfied. Suppose, in addition, that there is a $T = T(t_0) > 0$ such that the inequality

$$\dot V(t, x_t) \leq -w_3(|x(t)|) \tag{1.5'}$$

holds for all $t \geq t_0 + T$ and for every solution $x : [t_0 - r, \infty) \to \mathbf{R}^n$ of (1.1).

Then the zero solution of equation (1.1) is asymptotically stable.

Proof: Let $h_0(t, \phi) := \|\phi\|$ and $h(t, \phi) = h_1(t, \phi) := |\phi(0)|$ in Theorem 2. We use the boundedness condition to prove the new condition in Theorem 2. As we saw in the proof of Corollary 1, for every $t \geq t_0 + r$ we have $t_1 \in [t - r, t]$ such that $|x(s)| \geq \|x_t\|/2$ for all $s \in [t_1 - \|x_t\|/2L(B), t_1 + \|x_t\|/2L(B)]$. If $\|x_t\| \geq \epsilon$ for all $t \geq t_0$, then it follows from the last property that

$$\int_{t_0}^\infty w_3(|x(t)|)dt = \infty,$$

which proves the new condition.

3. EXAMPLES

The following two examples show that the boundedness condition cannot be dropped from the Corollary 1 and 2.

Consider the ordinary scalar differential equation

$$\dot{x} = \frac{\dot{\psi}(t)}{\psi(t)} x, \qquad (x \in \mathbf{R}) \tag{3.1}$$

where $\psi : [-1, \infty) \to (0, \infty)$ is continuously differentiable $(r = 1, t_0 = 0)$. Obviously, the functions $x(t) = c\psi(t)$ $(c \in \mathbf{R})$ are the solutions of equation (3.1). In both examples we choose the functional $V(t, \phi) = \|\phi\| + |||\phi|||$. We have to construct a function ψ which is bounded on $[-1, \infty)$ and satisfies an inequality

$$\dot{V}(t, c\psi_t) \leq -w_3(|||c\psi_t|||) \qquad \text{(Cor. 1)} \tag{3.2}$$

respectively

$$\dot{V}(t, c\psi_t) \leq -w_3(|c\psi(t)|) \qquad \text{(Cor. 2)} \tag{3.3}$$

for all $t \geq 1, c \in \mathbf{R}$ and, at the same time, $\psi(t) \not\to 0$ as $t \to \infty$.

Let a function $\psi : [-1, \infty) \to (0, \infty)$ be such that

(a) $\psi(n) = 1 + 1/2^n$,

(b) $\psi(t-1) - \psi(t) \geq 1/2^{n+1}$ for all $t \in [n, n+1)$,

(c) $|||\psi||| \leq 1/2^{n-1}$ for all $t \in [n, n+1)$

for $n = -1, 0, 1, \ldots$. (Continuously differentiable functions with these properties can be constructed from pieces of lines and parabolas.)

Condition (b) implies that $\|\psi_t\|$ is monotone nonincreasing. So

$$\dot{V}(t, c\psi_t) \leq |||c\psi_t|||^{\cdot} \leq -|c| w(\frac{1}{2^n}) \leq -w_3(|||c\psi_t|||)$$

for all $t \geq 0, c \in \mathbf{R}$, i.e. inequality (3.2) holds. Condition (a) implies, that $\psi(t) \not\to 0$ as $t \to \infty$, so the zero solution of equation (3.1) is not asymptotically stable.

We now construct a function ψ satisfying (3.3). Let a sequence of intervals $\{[a_n, b_n]\}_{n=0}^{\infty}$ be given such that $[a_{n+1}-1, b_{n+1}-1] \subset [a_n, b_n]$, $a_0 = 0$, $b_0 = 1/4$. Consider a function ψ having the following properties:

(a) $\psi(t) = 1/2^{n+1}$ for $t \in [b_n, a_{n+1}]$,

(b) $\psi(t-1) - \psi(t) \geq 1/2^{n+1}$ for all $t \in [a_n, a_{n+1})$,

(c) $\dot{\psi}(t) \leq -1$ and $\psi(t) \geq 1 + 1/2^{n+2}$ for $t \in [a_{n+1} - 1, b_{n+1} - 1]$,

(d) $\psi(t)$ monotone decreasing on interval $[b_{n+1}-1, b_n]$,
(e) $\max\{\psi(t) : t \in [a_n, a_{n+1}-1]\} = 1 + 1/2^{n+1}$ and
(f) $\psi(t) \leq 4$ for all $t \geq -1$.

Since $\|\psi_t\|$ is nonincreasing, properties (a) and (b) implies

$$\dot{V}(t, c\psi_t) \leq |c| \cdot |||\psi_t|||^{\cdot} \leq -|c| w(\frac{1}{2^n}) \leq -w_3(|c\psi(t)|)$$

for all $t \in [b_n, a_{n+1}]$ $(n = 0, 1, ...)$ and $c \in \mathbf{R}$ with appropriate functions $w, w_3 \in \mathcal{K}$.

If $t \in [a_n, b_n]$, then

$$\dot{V}(t, c\psi_t) \leq |c| \cdot \|\psi_t\|^{\cdot} \leq -|c| \leq -|c| \frac{\psi(t)}{4} = -\frac{1}{4}|c\psi(t)|.$$

Consequently, (3.3) is satisfied for all $t \geq 1$. On the other hand, condition (e) guarantees $\psi(t) \not\to 0$ $(t \to \infty)$, i.e. the zero solution of equation (3.1) is not asymptotically stable.

REFERENCES

1. Burton T. A., *Volterra Integral and Differential Equations,* Academic Press (1983).
2. Burton T. & Hatvani L., Stability theorems for nonautonomous functional differential equations by Lyapunov functionals, *Tohoku Math. J.* **41,** 65-104 (1989).
3. Hale J., *Theory of functional differential equations,* Springer-Verlag New York-Heidelberg-Berlin (1977).
4. Hatvani L., On partial asymptotic stability and instability, *Acta Sci. Math.* **45,** 219-231 (1983).
5. Hatvani L., On the asymptotic stability of the solutions of functional differential equations, *Coll. Math. Soc. J. Bolyai* **53,** 227-238 (1988).
6. Lakshmikantham V., Leela S. & Martynyuk A. A., *Stability analysis of nonlinear systems,* Marcel Dekker, Inc. New York and Basel (1989).
7. Lakshmikantham V. & Xin Zhi Liu, Perturbing families of Lyapunov functions and stability in terms of two measures, *J. Math. Anal. Appl.* **140,** 107-114 (1989).
8. Movchan A. A., Stability of processes with respect to two metrics, *J. Appl. Math. Mech.* **24,** 1506-1524 (1961).
9. Stephen H. Saperstone, *Semidynamical Systems in Infinite Dimensional Spaces,* Springer-Verlag New York-Heidelberg-Berlin (1981).