# A Graph to the Pairing strategies of the 9-in-a-row Game*


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## ABSTRACT

In Maker-Breaker positional games two players, Maker and Breaker, are playing on a finite or infinite board with the goal of claiming or preventing to reach a finite winning set, respectively. For different games there are several winning strategies either for Maker or Breaker. One class of winning strategies are the so-called pairing strategies. Generally, a pairing strategy means that the possible moves of a game are paired up; if one player plays one, the other player plays its pair. In this study we describe all possible pairing strategies for the 9-in-a-row game. Furthermore, as a concept, we define a graph of these pairings in order to find a structure for them. The characterization of that graph will be also given.



## Categories and Subject Descriptors

F.2 [**Analysis of algorithms and problem complexity**]: Nonnumerical Algorithms and Problems; G.2 [**Discrete mathematics**]: Graph Theory, Combinatorics

## Keywords

Positional games, pairing strategies, Hales-Jewett pairing


## 1. INTRODUCTION

In this work, we study the pairing strategies of the 9-in-a-row Maker-Breaker game. Hales and Jewett [7] gave the first pairing strategy to this game showing Breaker's win. However, the uniqueness of the Hales-Jewett pairing or other examples had not been provided since then, until Győrffy et al. [6] showed the following. There exist only 8- and 16-toric pairings (i.e. they are simply the repetitions of a pairing on the $8 \times 8$ and $16 \times 16$ square grids, respectively) where all 16-toric ones can be derived from some 8-toric ones.

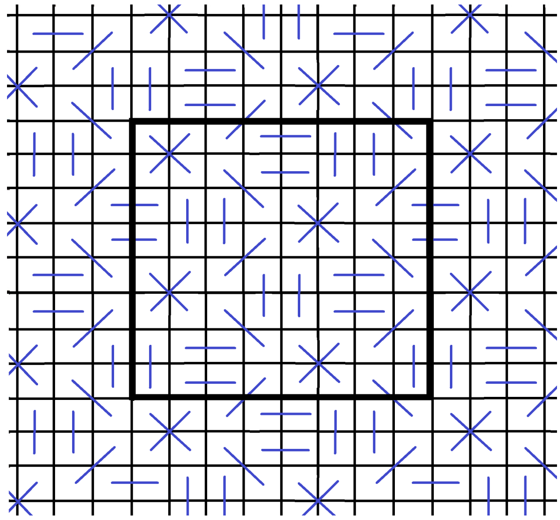

**Figure 1: Hales-Jewett pairing blocks the 9-in-a-row**

After recalling positional games and pairing strategies in general, we focus on the 9-in-a-row game and its pairings. We provide a computer program which generates and distinguish all (194543) different 8-toric pairings. Finally, we create and analyze a graph of these pairings to have a structure of them.

### 1.1 Positional games

A positional game can be defined as a game on a hypergraph $\mathcal{H} = (V, E)$, where $V = V(\mathcal{H})$ and $E = E(\mathcal{H}) \subseteq \mathcal{P}(\mathcal{H}) = \{S : S \subseteq V\}$ are the set of vertices and edges, respectively. Usually, $V$ can be finite or infinite, but an $A \in E$ edge is always finite. The first and second players take elements of $V$ in turns. In the **Maker-Maker** (M-M) version of the game, the player who first takes all elements of some edge $A \in E$ wins the game. In contrary, in the **Maker-Breaker** (M-B) version, Maker wins by taking every element of some $A \in E$, while the other (usually the second) player, Breaker, wins by taking at least one vertex of every edge in $E$. Clearly, there is no draw in this game. The M-M and M-B games are closely related, since if Breaker wins as a second player, then the M-M game is a draw. On the other hand, if the first player has a winning strategy for the M-M game, then Maker also wins the M-B version. For more on these, see Berlekamp, Conway and Guy [3] or Beck [2].


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In this work we deal with the hypergraph of the $k$-in-a-row game, which is defined as follows.

*Definiton 1.* The vertices of the **k-in-a-row hypergraph** $\mathcal{H}_k$ are the squares of the infinite (chess)board, i.e. the infinite square grid. The edges of the hypergraph $\mathcal{H}_k$ are the $k$-element sets of consecutive squares in a row horizontally, vertically or diagonally. We refer to the whole infinite rows as *lines*.

For $k$-in-a-row M-B games Maker wins if $k \leq 5$, see Allis et al. [1] and Breaker wins if $k \geq 8$, see Zetters [5]. There are a Breaker winning pairing strategy only if $k \geq 9$, see Csernenszky et al. [4]. For the case of $k = 9$ the first pairing strategy found by Hales and Jewett can be seen on Fig. 1. For $k = 6, 7$ the problem is open.

### 1.2 Pairing strategies

Given a hypergraph $\mathcal{H} = (V, E)$ and a bijection $\rho : X \to Y$, where $X, Y \subset V(\mathcal{H})$ and $X \cap Y = \emptyset$, is a **pairing** on the hypergraph $\mathcal{H}$. An $(x, \rho(x))$ pair **blocks** an $A \in E(\mathcal{H})$ edge, if $A$ contains both elements of the pair. If the pairs of $\rho$ block all edges, we say that $\rho$ is a **good pairing** of $\mathcal{H}$.

Pairings are one way to show that Breaker has a winning strategy in positional games. A good pairing $\rho$ for a hypergraph $\mathcal{H}$ can be turned to a winning strategy for Breaker in the M-B game on $\mathcal{H}$. Following $\rho$ on $\mathcal{H}$ in a M-B game, for every $x \in X$ chosen by Maker, Breaker chooses $\rho(x)$ or vica versa in case of $x \in Y$ (if $x \notin X \cup Y$ then Breaker can choose an arbitrary vertex). Hence Breaker can block all edges and wins the game. Hereafter we focus on the 9-in-a-row game and its pairings.

## 2. PAIRINGS FOR 9-IN-A-ROW

*Definiton 2.* A pairing is a **domino pairing** on the grid, if all pairs consist of only neighboring cells (horizontally, vertically or diagonally).

Note that the pairing on Fig. 1 is a domino pairing. From Győrffy et al. [6] it follows that if there is a good pairing for $\mathcal{H}_9$ then this pairing is a domino pairing in which the dominoes are following each other by 8-periodicity in each line and all squares are covered by a pair. To handle the periodicity we define the concept of $k$-toric pairings.

*Definiton 3.* A pairing of the infinite board is **k-toric** if it is an extension of a $k \times k$ square, where $k$ is the smallest possible.

In [6] it was proved that a good pairing of $\mathcal{H}_9$ is either 8-toric or 16-toric. Furthermore, all 16-toric pairings can derive from two (or more) 8-toric pairings. Fig. 2 shows a 16-toric (but not 8-toric) good pairing. The four $8 \times 8$ squares differs from each other only in the colored squares, where the bold pairs show the actual pairs and the thin ones show the difference. From now we only deal with the 8-toric pairings of $\mathcal{H}_9$. A good 8-toric pairing is uniquely determined by an $8 \times 8$ section of it, by definition. Furthermore, that $8 \times 8$ section contains exactly one pair in each 32 (eight vertical, eight horizontal and 16 diagonal) torus lines. Three examples, other then the Hales-Jewett pairing, can be seen on Fig. 3. A diagonal torus line is colored on the middle one.

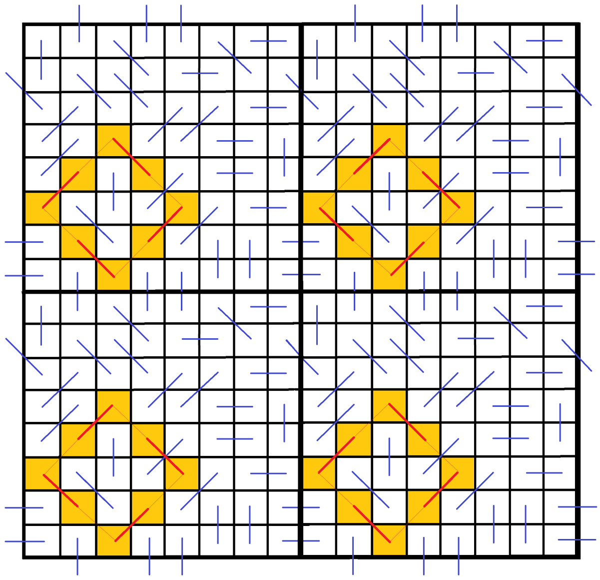

**Figure 2: 16-toric pairing for the 9-in-a-row game**

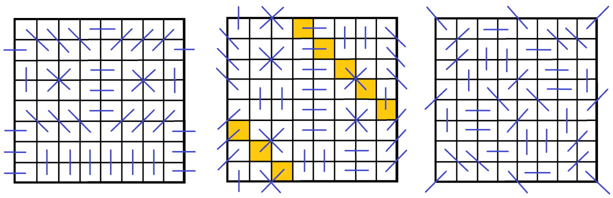

**Figure 3: Other 8-toric examples**

### 2.1 Generate pairings

To find all possible 8-toric pairing strategies of $\mathcal{H}_9$ on the infinite board we wrote a computer program that will be introduced in this section. The main challenge here is not only finding all pairings, but deciding whether two pairings are the same.

We store a pairing in the $8 \times 8$ table such that each cell represents the actual pair of the cell according to the 8 possible pairs: 0 means East, 1 South-East, and so on, 7 North-East. Naturally, if a cell's pair is on the East, then its pair has its own pair on the West, i.e. we fill the table two cells at a time. The algorithm itself is the usual backtracking algorithm: we find possible pairs for the next cell in the table having no pair so far, try all those by recursively calling the table filling function. While checking whether a pair is possible, we also make sure that there can be no overblocking, so we keep track of the blocked edges. A detailed example can be found in webpage [8].

From previous experiences we know, that the running time is crucial, since there are too many such pairings. We try to reduce the number of cases had to be considered. We consider two pairing strategies on the infinite board to be the same, if they can be transformed into each other by translation, mirroring and rotation. Thus, in order not to find the same pairing several times, we apply all transformations for any pairing found on the $8 \times 8$ table. From these transformed pairings we select the smallest one with respect to the lexicographical order. That also means that such a pairing must start with 0 and 4 in the first row of the $8 \times 8$ table, so we can also reduce the number of searched cases by starting fill the table with these two numbers. Naturally, we keep in mind that the $8 \times 8$ table is expanded in (say) an 8-toric way to the whole infinite board while applying these transformations. More precisely:

1. We either mirror or not (2 possible cases) the table to the vertical line between columns 4 and 5.
2. We rotate the table by 0, 90, 180, and 270 degrees (4 possible cases).
3. We try all toric (that is, modulo 8) translation that results in a table starting with 0 and 4.

We select the lexicographically smallest table as a representative for the actual pairing. This method reduce the number of all pairing checked to 6210560, and the program found the 194543 different pairings in about 4 minutes on a desktop computer with a 3.2 GHz Core i7 processor using 12 Mb of memory. The pairings themselves can be downloaded at the page [8]. Interestingly, the number of the different pairings turns out to be a prime number.

Since we have such many different pairings, an obvious way to find a structure can be to store the pairings in a graph. In the next section we will show a natural method to find connections between pairings.

## 2.2 Graph of pairings

While trying to find pairings by hand one can observe, that we can move a pair along the blocked edge by one step to create a new pairing using the following method.

1. Move the first pair on the table. This move creates a cell (say $A$) without a pair, and another cell (say $B$) with two pairs.
2. Move the pair containing cell $B$ which was not the just moved pair so that cell $B$ has one pair after the move. But then another cell may have two pairs.
3. Repeat step 2 as long as it creates a cell with two pairs.
4. This method will end when the last move creates a new pair for cell $A$, which had no pair before the move.

Naturally, we should keep in mind that we are on an 8-toric pairing and move the pairs accordingly. Since we are on a finite table, this method will either end at step 4, or create a repeating cycle. But the later one is not possible. Note that cell $A$ cannot be part of the cycle, as it has no pair, and it would break the repetition. The first move that entered

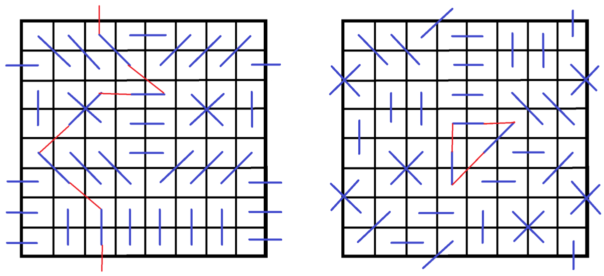

**Figure 4: Two examples of connections between pairings**

the cycle creates a hole "behind" (outside the cycle), and when the cycle comes to the same cell, the pair will move backwards, and the cycle is not entered again. Also, it is easy to see that we get an optimal pairing by this method. Since the original pairing was optimal, moving a pair (an 8-toric way) along the blocked edge keeps that direction (i.e. 8 edges) blocked. Since the method ends in step 4, there are no cells without a pair. We also move the pairs on a torus, so no overblocking is possible. We say that two pairings are connected, if one can obtain the second pairing from the first one by the method described above (of course, we consider only different pairings as it was defined in the previous section). This relation is symmetric: moving back the last pair of the above method gives back the first pairing from the second one. This creates a graph, where the vertices are the pairings and the edges are defined by the moving transition. Fig. 4 shows two examples for this moving transition. In both cases, the first pairing contains only the blue pairs, and the red dominoes show the transition to the other pairing. After computing all possible different pairings our program can easily find this graph. It tries to move all pairings (by trying to free up each cell in the $8 \times 8$ board), and use the method described in the previous section to find the lexicographically smallest representative for the new pairing. It takes about 1 minute to finish this task on the same hardware as in the previous section.

In the next section we will investigate the properties of the obtained graph.

## 2.3 Analyzing the graph

The basic parameters of the obtained graph can be seen in Tab. 1. The graph is not connected, which means that repeating the moving transition described in the previous section we cannot reach an arbitrary pairing from another. One of the 14 components of the graph is a giant component containing almost all (194333) vertices. The diameter of this component is 34, which shows us that even this giant component does not seem to be a "small-world" network. There are 5-5 smaller components of 10 and 16 vertices and 1-1 components of size 6, 26, 48. Note that every graph component containing 16 vertices is the net of a 4-dimensional cube. Fig. 5 shows some small components.

The graph is triangle-free, moreover, the length of all induced cycles is four. The degree distribution of the graph can be seen in Tab. 2.

**Table 1: Basic parameters of the constructed graph**

| vertices | edges | #components | max degree | min degree | avg. degree |
|---|---|---|---|---|---|
| 194543 | 532107 | 14 | 11 | 1 | 5.47 |

**Table 2: Degree distribution of the graph**

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|
| 17 | 392 | 395 | 39811 | 66185 | 53222 | 25309 | 7547 | 1472 | 183 | 10 |

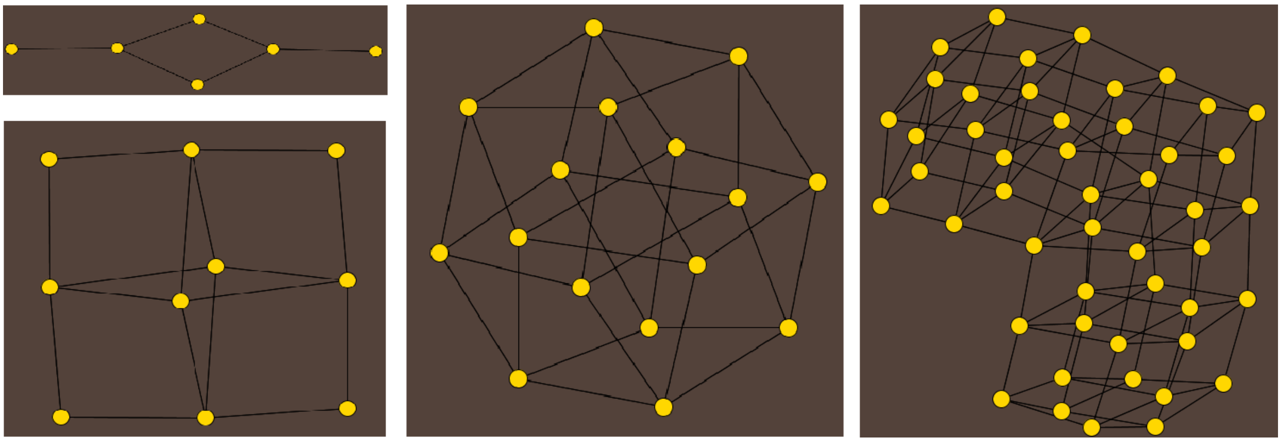

**Figure 5: Some components of the obtained graph**

# 3. CONCLUSIONS

In this study, we investigated the 9-in-a-row Maker-Breaker positional game focusing on its pairing strategies which guarantee Breaker's win. We found all different 8-toric pairing strategies using a computer program. The main concepts of the program were described in detail. In order to find a structure of the 194543 pairings, we arranged them into a graph where the vertices are the pairings itself and the edges are some moving transitions of pairs. Analyzing the graph and calculating standard parameters may help in a better understanding of pairing strategies in general.

# 4. REFERENCES


[1] L. V. Allis, H. J. van den Herik and M. P. Huntjens. Go-Moku solved by new search techniques. *Proc. 1993 AAAI Fall Symp. on Games: Planning and Learning, AAAI Press Tech. Report FS93-02*, pp. 1-9, Menlo Park, CA.

[2] J. Beck. *Combinatorial Games, Tic-Tac-Toe Theory.* Cambridge University Press, 2008.

[3] E. R. Berlekamp, J. H. Conway and R. K. Guy. *Winning Ways for your mathematical plays*, Volume **2**. Academic Press, New York, 1982.

[4] A. Csernenszky, R. Martin and A. Pluhár. On the Complexity of Chooser-Picker Positional Games. *Integers* 11, 2011.

[5] R. K. Guy and J. L. Selfridge, Problem S.10, *Amer. Math. Monthly* **86** (1979); solution T.G.L. Zetters **87** (1980) 575–576.

[6] L. Győrffy, G. Makay and A. Pluhár. Pairing strategies for the 9-in-a-row game. Submitted, 2016. `http://www.math.u-szeged.hu/~lgyorffy/predok/9_pairings.pdf` downloaded: 28. 08. 2016.

[7] A. W. Hales and R. I. Jewett. Regularity and positional games. *Trans. Amer. Math. Soc.* **106** (1963) 222–229; M.R. # 1265.

[8] G. Makay. Personal homepage `http://www.math.u-szeged.hu/~makay/amoba/` downloaded: 06. 04. 2016.