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Some Extremal Values of the Number of Congruences of a Finite Lattice

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Some Extremal Values of the Number of Congruences of a Finite Lattice

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March 19, 2018

Abstract

We investigate the possible values of the numbers of congruences of finite lattices of an arbitrary but fixed cardinality. We first construct finite lattices with small numbers of congruences; then, motivated by a result of Freese and continuing Czédli's recent work, we prove that the third, fourth and fifth largest numbers of congruences of an n–element lattice are: $5 \cdot 2^{n-5}$ if $n \geq 5$, respectively 2^{n-3} and $7 \cdot 2^{n-6}$ if $n \geq 6$. We also determine the structures of the n–element lattices having $5 \cdot 2^{n-5}$, 2^{n-3}, respectively $7 \cdot 2^{n-6}$ congruences, which show the structures of their congruence lattices, as well.

Keywords: (finite) lattice, (principal) congruence, (prime) interval, atom, (ordinal, horizontal) sum.

MSC 2010: primary: 06B10; secondary: 06B05.

1 Introduction

In this paper, we study the smallest, as well as the largest numbers of congruences of lattices of an arbitrary finite cardinality. The problem of the existence of lattices with certain values for the cardinalities of their sets of congruences, filters and ideals was raised in Mureşan [25],[26] and further studied in Czédli and Mureşan [11]. The idea behind our current article is that, in the finite case (in which the number of filters and that of ideals equal the number of elements), the above–mentioned problem can be tackled by identifying the smallest and the largest possible numbers of congruences and then filling the gap between these, for an arbitrary appropriately large number of elements.

Our main result is Theorem 5.6, on the third, fourth and fifth largest possible numbers of congruences of a finite lattice, which thus continues the work of Freese [12] and Czédli [8] on the largest and second largest number of congruences of finite lattices of an arbitrary but fixed cardinality. Our investigation is also motivated by Czédli's independent work on semilattices [9] and further relates to the representation problem for lattices in the form of congruence lattices of lattices. The investigation of this representation problem goes back to R. P. Dilworth and was mile–stoned by Grätzer and Schmidt [23], Wehrung [30], Růžička [28], Grätzer and Knapp [20], and Ploščica [27], and surveyed in Grätzer [15] and Schmidt [29]. A lot of results have been proved on the representation problem of two or more lattices and certain maps among them by (complete) congruences; for example, see Grätzer and Schmidt [24], Grätzer and Lakser [21], Czédli [1],[6]. Even the posets and monotone maps among them have been characterized by principal congruences of lattices; for example, see Grätzer [16],[17],[18],[19], Grätzer and Lakser [22], and Czédli [3],[2],[4],[5],[7]. Finally, the above-mentioned trends, focusing on the sizes of congruence lattices, on the structures formed by congruences, and on maps among these structures, have recently met in Czédli and Mureşan [11], enriching the first two trends and even related to the third one.

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2 Definitions, Notations and Immediate Properties

We shall denote by $\mathbb{N}$ the set of the natural numbers and by $\mathbb{N}^* = \mathbb{N}\setminus\{0\}$. $\amalg$ will be the disjoint union of sets. For any set M, $|M|$ shall be the cardinality of M, $\mathrm{Part}(M)$ and $\mathrm{Eq}(M)$ will be the bounded lattices of the partitions and the equivalences on M, respectively, $\Delta_M = \{(x,x) \mid x \in M\}$, $\nabla_M = M^2$ and $\mathrm{eq} : \mathrm{Part}(M) \to \mathrm{Eq}(M)$ shall be the canonical lattice isomorphism; if $\pi = \{M_1, \ldots, M_n\}$ for some $n \in \mathbb{N}^*$, then $\mathrm{eq}(\pi)$ will simply be denoted by $\mathrm{eq}(M_1, \ldots, M_n)$.

All lattices shall be non–empty and, unless mentioned otherwise, they shall be designated by their underlying sets; their operations and order relation will be denoted in the usual way, and $\prec$ will denote their succession relation. The *trivial lattice* shall be the one–element lattice. $\cong$ shall denote the existence of a lattice isomorphism.

For any lattice L, $\mathrm{Con}(L)$, $\mathrm{Filt}(L)$ and $\mathrm{Id}(L)$ shall be the lattices of the congruences, filters and ideals of L, respectively. Clearly, if L' is the dual of L, then $\mathrm{Con}(L) \cong \mathrm{Con}(L')$. If L is a bounded lattice, then we shall denote by $\mathrm{Con}_0(L) = \{\theta \in \mathrm{Con}(L) \mid 0/\theta = \{0\}\}$ and by $\mathrm{Con}_{01}(L) = \{\theta \in \mathrm{Con}(L) \mid 0/\theta = \{0\}, 1/\theta = \{1\}\}$; note that $\mathrm{Con}_0(L)$ and $\mathrm{Con}_{01}(L)$ are sublattices of $\mathrm{Con}(L)$ and bounded lattices, with smallest element Δ_L and greatest element $\bigvee \mathrm{Con}_0(L) = \mathrm{eq}(\bigvee_{\theta\in\mathrm{Con}_0(L)} L/\theta)$ and $\bigvee \mathrm{Con}_{01}(L) = \mathrm{eq}(\bigvee_{\theta\in\mathrm{Con}_{01}(L)} L/\theta)$, respectively, where the second join is in the lattice $\mathrm{Part}(L)$ [13]. Now let $a, b \in L$, arbitrary. Following [8], we denote by $\mathrm{con}(a, b)$ the principal congruence of L generated by (a, b). For any $\theta \in \mathrm{Con}(L)$, we know from [14] that every class of θ is a convex sublattice of L, thus an interval if L is finite, and, clearly, if S is a sublattice of L, then $\theta \cap S^2 \in \mathrm{Con}(S)$. If L has a 0, then $\mathrm{At}(L)$ shall be the set of the atoms of L. $[a)_L$ and $(a]_L$ shall be the principal filter, respectively ideal of L generated by a, and $[a, b]_L = [a)_L \cap (b]_L$ shall be the interval of L bounded by a and b; recall that $[a, b]_L$ is called a *prime interval* iff $a \prec b$, and it is called a *narrows* iff it is a prime interval such that a is meet–irreducible and b is join–irreducible [8],[14]. In the particular case when $a, b \in \mathbb{N}$ and $\leq$ is the natural order on $\mathbb{N}$, $[a, b]_{(\mathbb{N},\leq)}$ shall simply be denoted by $[a, b]$.

$\dotplus$ shall be the ordinal sum and $\boxplus$ shall be the horizontal sum. Recall that, for any lattice $(L, \leq_L, 1_L)$ with largest element and any lattice $(M, \leq_M, 0_M)$ with smallest element, the *ordinal sum* of L with M is defined by identifying $c = 1_L = 0_M \in L \cap M$ and letting $L \dotplus M = ((L \setminus \{c\}) \amalg \{c\} \amalg (M \setminus \{c\}), \leq_L \cup \leq_M \cup \{(x, y) \mid x \in L, y \in M\})$; for every $\alpha \in \mathrm{Con}(L)$ and every $\beta \in \mathrm{Con}(M)$, we shall denote by $\alpha \dotplus \beta = \mathrm{eq}((L/\alpha \setminus \{c/\alpha\}) \cup \{c/\alpha \cup c/\beta\} \cup (M/\beta \setminus \{c/\beta\}))$; note that $\mathrm{Con}(L \dotplus M) = \{\alpha \dotplus \beta \mid \alpha \in \mathrm{Con}(L), \beta \in \mathrm{Con}(M)\} \cong \mathrm{Con}(L) \times \mathrm{Con}(M)$ [26]. Also, for any bounded lattices $(L, \leq_L, 0_L, 1_L)$ and $(M, \leq_M, 0_M, 1_M)$ with $|L|, |M| > 2$, the *horizontal sum* of L with M is defined by identifying $0 = 0_L = 0_M, 1 = 1_L = 1_M \in L \cap M$ and letting $L \boxplus M = ((L \setminus \{0, 1\}) \amalg \{0, 1\} \amalg (M \setminus \{0, 1\}), \leq_L \cup \leq_M, 0, 1)$; for every $\alpha \in \mathrm{Con}(L)$ and every $\beta \in \mathrm{Con}(M)$, we shall denote by $\alpha \boxplus \beta = \mathrm{eq}((L/\alpha \setminus \{0/\alpha, 1/\alpha\}) \cup \{0/\alpha \cup 0/\beta, 1/\alpha \cup 1/\beta\} \cup (M/\beta \setminus \{0/\beta, 1/\beta\})) \in \mathrm{Eq}(L \boxplus M)$. Clearly, the ordinal sum of bounded lattices is associative, while the horizontal sum is both associative and commutative. For any $n \in \mathbb{N}^*$, we shall denote by $\mathcal{L}_n$ the n–element chain and by $M_n = \boxplus_{i=1}^{n} \mathcal{L}_3$, so that $M_1 = \mathcal{L}_3$, $M_2 = \mathcal{L}_2^2$ and M_3 is the five–element modular non–distributive lattice (the *diamond*); we shall also denote by $N_5 = \mathcal{L}_3 \boxplus \mathcal{L}_4$ the five–element non–modular lattice (the *pentagon*).

3 The Congruence Lattices of Some Constructions of Lattices

Following [11], we call a triple (κ, λ, μ) of nonzero cardinalities *CFI–representable* iff there exists a lattice L such that $\kappa = |\mathrm{Con}(L)|$, $\lambda = |\mathrm{Filt}(L)|$ and $\mu = |\mathrm{Id}(L)|$, case in which we say that L *CFI–represents* the triple (κ, λ, μ). Of course, if L is finite, then all its filters and all its ideals are principal, thus L CFI–represents the triple $(|\mathrm{Con}(L)|, |L|, |L|)$, with $|\mathrm{Con}(L)| \in \mathbb{N}^*$. For instance, given any $n \in \mathbb{N}$, the Boolean algebra $\mathcal{L}_2^n$ CFI–represents $(2^n, 2^n, 2^n)$ and, if $n \geq 2$, then the chain $\mathcal{L}_n = \dotplus_{i=1}^{n-1} \mathcal{L}_2$ CFI–represents $(2^{n-1}, n, n)$ since $\mathrm{Con}(\mathcal{L}_n) \cong \mathrm{Con}(\mathcal{L}_2)^{n-1} \cong \mathcal{L}_2^{n-1}$. For every lattice K with greatest element and every lattice M with smallest element, clearly, $K \dotplus M$ CFI–represents $(|\mathrm{Con}(K)| \cdot |\mathrm{Con}(M)|, |K| + |M| - 1, |K| + |M| - 1)$, hence, for any $n \in \mathbb{N}^*$, $K \dotplus \mathcal{L}_n$ CFI–represents $(2^{n-1} \cdot |\mathrm{Con}(K)|, |K| + n - 1, |K| + n - 1)$.

Most of the times, we shall use the remarks in this paper without referencing them.

Remark 3.1. The only triples which are CFI–represented by lattices of cardinality at most 4 are: $(1, 1, 1)$, $(2, 2, 2)$, $(4, 3, 3)$, $(4, 4, 4)$ and $(8, 4, 4)$.

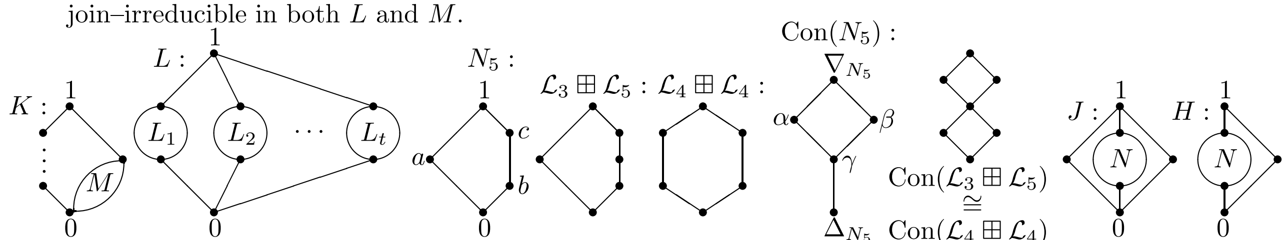
Remark 3.2. [26] If L and M are bounded lattices with $|L|, |M| > 2$, then, clearly, $\mathrm{Con}_{01}(L \boxplus M) = \{\alpha \boxplus \beta \mid \alpha \in \mathrm{Con}_{01}(L), \beta \in \mathrm{Con}_{01}(M)\} \cong \mathrm{Con}_{01}(L) \times \mathrm{Con}_{01}(M)$ and, if we consider the following two conditions:

(0L1M) 0 is meet–irreducible in L and 1 is join–irreducible in M,

(0M1L) 0 is meet–irreducible in M and 1 is join–irreducible in L,

then:

- $\mathrm{Con}(L \boxplus M) = \mathrm{Con}_{01}(L \boxplus M) \cup \{\nabla_{L \boxplus M}\} \cong (\mathrm{Con}_{01}(L) \times \mathrm{Con}_{01}(M)) \dotplus \mathcal{L}_2$, if none of the conditions **(0L1M)** and **(0M1L)** is fulfilled;
- $\mathrm{Con}(L \boxplus M) = \mathrm{Con}_{01}(L \boxplus M) \cup \{\mathrm{eq}(L \setminus \{0\}, M \setminus \{1\}), \nabla_{L \boxplus M}\}$ if **(0L1M)** holds and **(0M1L)** fails, and $\mathrm{Con}(L \boxplus M) = \mathrm{Con}_{01}(L \boxplus M) \cup \{\mathrm{eq}(L \setminus \{1\}, M \setminus \{0\}), \nabla_{L \boxplus M}\}$ if **(0M1L)** holds and **(0L1M)** fails, so that $\mathrm{Con}(L \boxplus M) \cong (\mathrm{Con}_{01}(L) \times \mathrm{Con}_{01}(M)) \dotplus \mathcal{L}_3$ if exactly one of the conditions **(0L1M)** and **(0M1L)** is fulfilled;
- $\mathrm{Con}(L \boxplus M) = \mathrm{Con}_{01}(L \boxplus M) \cup \{\mathrm{eq}(L \setminus \{0\}, M \setminus \{1\}), \mathrm{eq}(L \setminus \{1\}, M \setminus \{0\}), \nabla_{L \boxplus M}\} \cong (\mathrm{Con}_{01}(L) \times \mathrm{Con}_{01}(M)) \dotplus \mathcal{L}_2^2$ if both of the conditions **(0L1M)** and **(0M1L)** are fulfilled, that is if 0 is meet–irreducible and 1 is join–irreducible in both L and M.



By the previous remark, if M is a bounded lattice in which 0 is meet–reducible and $K = \mathcal{L}_k \boxplus (M \dotplus \mathcal{L}_2)$ for some $k \in \mathbb{N} \setminus \{0, 1, 2\}$, as in the picture above, then $\mathrm{Con}(K) \cong (\mathrm{Con}_0(M) \times \mathrm{Con}(\mathcal{L}_{k-2})) \dotplus \mathcal{L}_3 \cong (\mathrm{Con}_0(M) \times \mathcal{L}_2^{k-3}) \dotplus \mathcal{L}_3$.

Let $t \in \mathbb{N}^*$, $L_1, L_2, \ldots, L_t$ be bounded lattices and $L = \boxplus_{i=1}^{t} (\mathcal{L}_2 \dotplus L_i \dotplus \mathcal{L}_2)$, as in the previous diagram. Clearly, for every $i \in [1, t]$, $\mathrm{Con}_{01}(\mathcal{L}_2 \dotplus L_i \dotplus \mathcal{L}_2) \cong \mathrm{Con}(L_i)$ and $\mathcal{L}_2 \dotplus L_i \dotplus \mathcal{L}_2$ has the 0 strictly meet–irreducible and the 1 strictly join–irreducible, while $(\mathcal{L}_2 \dotplus L_i \dotplus \mathcal{L}_2) \boxplus (\mathcal{L}_2 \dotplus L_j \dotplus \mathcal{L}_2)$ has the 0 meet–reducible and the 1 join–reducible for every $j \in [1, t] \setminus \{i\}$, thus, by the previous remark:

- if $t = 2$, then $\mathrm{Con}(L) \cong (\mathrm{Con}(L_1) \times \mathrm{Con}(L_2)) \dotplus \mathcal{L}_2^2$;
- if $t \geq 3$, then $\mathrm{Con}(L) \cong (\prod_{i=1}^{t} \mathrm{Con}(L_t)) \dotplus \mathcal{L}_2$.

For example, $N_5 = (\mathcal{L}_2 \dotplus \mathcal{L}_1 \dotplus \mathcal{L}_2) \boxplus (\mathcal{L}_2 \dotplus \mathcal{L}_2 \dotplus \mathcal{L}_2)$ has $\mathrm{Con}(N_5) = \{\Delta_{N_5}, \alpha, \beta, \gamma, \nabla_{N_5}\} \cong (\mathcal{L}_1 \times \mathcal{L}_2) \dotplus \mathcal{L}_2^2 \cong \mathcal{L}_2 \dotplus \mathcal{L}_2^2$, where $\alpha = \mathrm{eq}(\{0, b, c\}, \{a, 1\})$, $\beta = \mathrm{eq}(\{0, a\}, \{b, c, 1\})$ and $\gamma = (\Delta_{\mathcal{L}_2} \dotplus \nabla_{\mathcal{L}_1} \dotplus \Delta_{\mathcal{L}_2}) \boxplus (\Delta_{\mathcal{L}_2} \dotplus \nabla_{\mathcal{L}_2} \dotplus \Delta_{\mathcal{L}_2}) = \mathrm{eq}(\{0\}, \{a\}, \{b, c\}, \{1\})$, with the elements and congruences of N_5 denoted as in the figure above, while $M_3 = \boxplus_{i=1}^{3} (\mathcal{L}_2 \dotplus \mathcal{L}_1 \dotplus \mathcal{L}_2)$ has $\mathrm{Con}(M_3) \cong (\mathcal{L}_1 \times \mathcal{L}_1) \dotplus \mathcal{L}_2 \cong \mathcal{L}_2$. Similarly, $\mathcal{L}_3 \boxplus \mathcal{L}_5 = (\mathcal{L}_2 \dotplus \mathcal{L}_1 \dotplus \mathcal{L}_2) \boxplus (\mathcal{L}_2 \dotplus \mathcal{L}_3 \dotplus \mathcal{L}_2)$ and $\mathcal{L}_4 \boxplus \mathcal{L}_4 = (\mathcal{L}_2 \dotplus \mathcal{L}_2 \dotplus \mathcal{L}_2) \boxplus (\mathcal{L}_2 \dotplus \mathcal{L}_2 \dotplus \mathcal{L}_2)$, so that $\mathrm{Con}(\mathcal{L}_3 \boxplus \mathcal{L}_5) \cong \mathrm{Con}(\mathcal{L}_4 \boxplus \mathcal{L}_4) \cong \mathcal{L}_2^2 \dotplus \mathcal{L}_2^2$.

Also, in particular, if N is a bounded lattice, $J = \mathcal{L}_3 \boxplus (\mathcal{L}_2 \dotplus N \dotplus \mathcal{L}_2) \boxplus \mathcal{L}_3$ and $H = (\mathcal{L}_2 \dotplus N \dotplus \mathcal{L}_2) \boxplus \mathcal{L}_3$, as depicted in the picture above, then, since $\mathcal{L}_3 \cong \mathcal{L}_2 \dotplus \mathcal{L}_1 \dotplus \mathcal{L}_2$, we have $\mathrm{Con}(J) \cong (\mathcal{L}_1 \times \mathrm{Con}(N) \times \mathcal{L}_1) \dotplus \mathcal{L}_2 \cong \mathrm{Con}(N) \dotplus \mathcal{L}_2$ and $\mathrm{Con}(H) \cong (\mathrm{Con}(N) \times \mathcal{L}_1) \dotplus \mathcal{L}_2^2 \cong \mathrm{Con}(N) \dotplus \mathcal{L}_2^2$, hence:

- J CFI–represents $(|\mathrm{Con}(N)| + 1, |N| + 4, |N| + 4)$;
- H CFI–represents $(|\mathrm{Con}(N)| + 3, |N| + 3, |N| + 3)$.

Thus, for all $k, n \in \mathbb{N}^*$, remembering the notation $M_n = \boxplus_{i=1}^{n} \mathcal{L}_3$ and noting that $\mathcal{L}_4 \cong \mathcal{L}_2 \dotplus \mathcal{L}_2 \dotplus \mathcal{L}_2$:

- if $n \geq 5$, then $(2, n, n)$ is CFI–represented by M_{n-2};
- if $n \geq 6$, then $(3, n, n)$ and $(4, n, n)$ are CFI–represented by $\mathcal{L}_4 \boxplus M_{n-4}$ and $M_{n-3} \dotplus \mathcal{L}_2$, respectively;
- if $n \geq 7$, then $(5, n, n)$ and $(6, n, n)$ are represented by $\mathcal{L}_4 \boxplus \mathcal{L}_4 \boxplus M_{n-6}$ and $(\mathcal{L}_4 \boxplus M_{n-4}) \dotplus \mathcal{L}_2$, respectively;
- if (k, n, n) is CFI–representable, then $(k + 1, n + 4, n + 4)$ and $(k + 3, n + 3, n + 3)$ are CFI–representable.

4 Finite Lattices Whose Numbers of Congruences Are Small or Powers of Two

Proposition 4.1. *Let $n \in \mathbb{N}$ such that $n \geq 7$. Then:*

(i) for any $j \in [1, n-1]$, $(2^j, n, n)$ is CFI–representable;

(ii) if $n \neq 8$, then, for any $k \in [2, n+1]$, (k, n, n) is CFI–representable.

Proof. (i) For all $n \geq 8 \geq 5$, $(2, n, n)$ is CFI–representable and, if $j \in \mathbb{N}^*$ is such that a lattice L CFI–represents $(2^j, n-1, n-1)$, then $L \dotplus \mathcal{L}_2$ CFI–represents $(2^{j+1}, n, n)$. For $n = 7$, $2^{n-1} = 2^6 = 64$, and: $(2,7,7)$, $(4,7,7)$, $(8,7,7)$, $(16,7,7)$, $(32,7,7)$ and $(64,7,7)$ are CFI–represented by M_5, $M_4 \dotplus \mathcal{L}_2$, $M_3 \dotplus \mathcal{L}_3$, $\mathcal{L}_2^2 \dotplus \mathcal{L}_2^2$, $\mathcal{L}_2^2 \dotplus \mathcal{L}_4$ and $\mathcal{L}_7$, respectively. Now an easy induction argument proves (i).
(ii) For any $n \in \mathbb{N}$ with $n \geq 7$, $(2, n, n)$, $(3, n, n)$ and $(4, n, n)$ are CFI–representable, and, if (k, n, n) is CFI–representable for some $k \in \mathbb{N}^*$, then $(k+3, n+3, n+3)$ is CFI–representable, thus, if (k, n, n) is CFI–representable for any $k \in [2, n]$, then $(k, n+3, n+3)$ is CFI–representable for any $k \in [2, n+3]$. Thus it suffices to prove that, for any $n \in \{7, 9, 11\}$ and any $k \in [2, n+1]$, (k, n, n) is CFI–representable; then (ii) follows by induction. Furthermore, for any $n \in \mathbb{N}$ with $n \geq 7$, $(5, n, n)$ and $(6, n, n)$ are CFI–representable, hence it remains to prove that, for any $n \in \{7, 9, 11\}$ and any $k \in [7, n+1]$, (k, n, n) is CFI–representable.

Since $\mathcal{L}_2^2$ CFI–represents $(4,4,4)$, it follows that $\mathcal{L}_3 \boxplus (\mathcal{L}_2 \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_2)$ CFI–represents $(4+3, 4+3, 4+3) = (7,7,7)$. $(8,7,7)$ is CFI–represented by $M_3 \dotplus \mathcal{L}_3$.

Since $M_3 \dotplus \mathcal{L}_2$ CFI–represents $(4,6,6)$, it follows that $\mathcal{L}_3 \boxplus (\mathcal{L}_2 \dotplus M_3 \dotplus \mathcal{L}_2 \dotplus \mathcal{L}_2) = \mathcal{L}_3 \boxplus (\mathcal{L}_2 \dotplus M_3 \dotplus \mathcal{L}_3)$ CFI–represents $(4+3, 6+3, 6+3) = (7,9,9)$. Since $\boxplus_{i=1}^4 \mathcal{L}_3$ CFI–represents $(2,6,6)$ and $\mathcal{L}_2^2$ CFI–represents $(4,4,4)$, it follows that $(\boxplus_{i=1}^4 \mathcal{L}_3) \dotplus \mathcal{L}_2^2$ CFI–represents $(2 \cdot 4, 6+4-1, 6+4-1) = (8,9,9)$. Since $\mathcal{L}_2^2 \dotplus \mathcal{L}_2$ CFI–represents $(8,5,5)$, it follows that $\mathcal{L}_3 \boxplus (\mathcal{L}_2 \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_2 \dotplus \mathcal{L}_2) = \mathcal{L}_3 \boxplus (\mathcal{L}_2 \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_3)$ CFI–represents $(8+1, 5+4, 5+4) = (9,9,9)$. $(7,6,6)$ is CFI–represented by $\mathcal{L}_4 \boxplus \mathcal{L}_4$ (as well as $\mathcal{L}_3 \boxplus \mathcal{L}_5$), hence, for instance, $\mathcal{L}_3 \boxplus (\mathcal{L}_2 \dotplus (\mathcal{L}_4 \boxplus \mathcal{L}_4) \dotplus \mathcal{L}_2)$ CFI–represents $(7+3, 6+3, 6+3) = (10,9,9)$.

Since $(6,7,7)$, $(7,7,7)$ and $(8,7,7)$ are CFI–representable by the above and (i), it follows that $(6+1, 7+4, 7+4) = (7,11,11)$, $(7+1, 7+4, 7+4) = (8,11,11)$ and $(8+1, 7+4, 7+4) = (9,11,11)$ are CFI–representable. Since M_5 CFI–represents $(2,7,7)$ and N_5 CFI–represents $(5,5,5)$, it follows that $M_5 \dotplus N_5$ CFI–represents $(2 \cdot 5, 7+5-1, 7+5-1) = (10,11,11)$. Since N_5 CFI–represents $(5,5,5)$, it follows that $\mathcal{L}_3 \boxplus (\mathcal{L}_2 \dotplus (\mathcal{L}_3 \boxplus (\mathcal{L}_2 \dotplus N_5 \dotplus \mathcal{L}_2)) \dotplus \mathcal{L}_2)$ CFI–represents $(5+3+3, 5+3+3, 5+3+3) = (11,11,11)$. $\mathcal{L}_4 \boxplus M_2 = \mathcal{L}_4 \boxplus \mathcal{L}_2^2$, M_3 and $\mathcal{L}_2$ CFI–represent $(3,6,6)$, $(2,5,5)$ and $(2,2,2)$, respectively, hence $(\mathcal{L}_4 \boxplus \mathcal{L}_2^2) \dotplus M_3 \dotplus \mathcal{L}_2$ CFI–represents $(3 \cdot 2 \cdot 2, 6+5-3+1, 6+5-3+1) = (12,11,11)$. $\square$

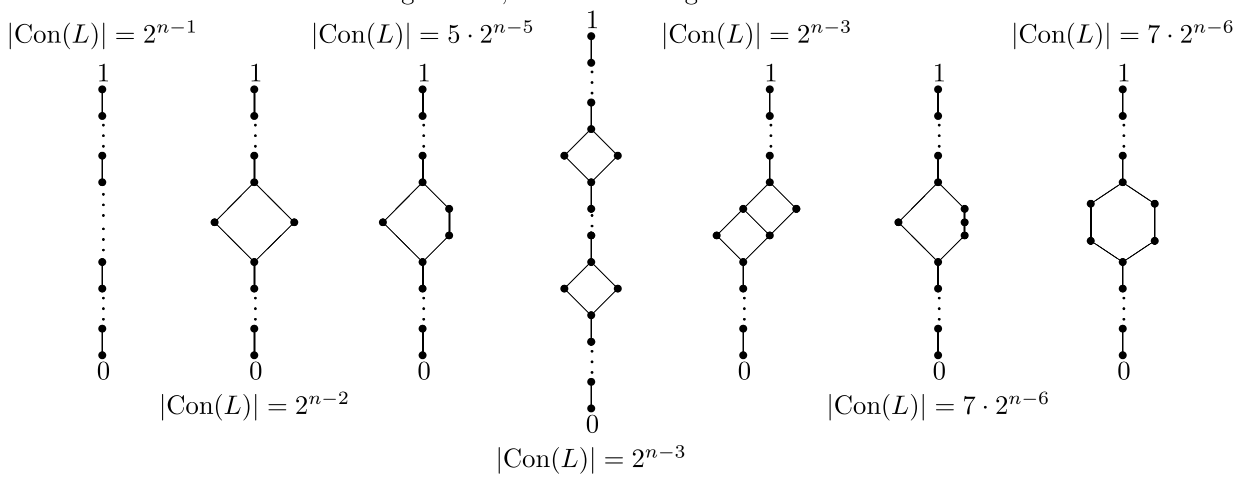
Note 4.2. Many results can be derived from Proposition 4.1. For instance, using ordinal sums, we get that, for any $n, m, l \in \mathbb{N}^*$ such that (l, m, m) is CFI–representable and $n \geq 7$: for any $j \in [1, n-1]$, $(2^j \cdot l, n+m-1, n+m-1)$ is CFI–representable, and, if $n \neq 8$, then, for any $k \in [2, n+1]$, $(k \cdot l, n+m-1, n+m-1)$ is CFI–representable. Thus, for instance, if $n \in \mathbb{N}$ is such that $n \geq 7$ and $n \neq 8$, then, for any $k \in [2, n+1]$, $(k^2, 2n-1, 2n-1)$ is CFI–representable and, more generally, $(k^s, sn-s+1, sn-s+1)$ is CFI–representable for any $s \in \mathbb{N}^*$.

These results suggest that, in order to fill in the gap between the numbers of congruences listed above and the largest possible numbers of congruences of finite lattices, from the next section, it might be useful to represent the numbers of congruences in base 2; this is why, in our main theorem from the next section, we express the numbers of congruences in base 2, apart from the fact that it helps clarify the ordering of those numbers.

5 On the Largest Numbers of Congruences of Finite Lattices

Let $n \in \mathbb{N}^*$ and L be a lattice with $|L| = n$. Since L is finite, its meet–irreducibles are strictly meet–irreducible, its join–irreducibles are strictly join–irreducible, and, for any $u, v \in L$ with $u < v$, $[u, v]_L$ contains at least one successor of u and one predecessor of v. So, for any $a, b \in L$, $[a, b]_L$ is a narrows iff $a \prec b$, b is the unique successor of a and a is the unique predecessor of b in L.

By [9], the first and second largest possible number of congruences of L, along with the structures of the n–element lattices L with these numbers of congruences, are represented in the figure below; we will show that the third, fourth and fifth largest possible number of congruences of L, along with the structures of the n–element lattices L with these numbers of congruences, are as in the figure below:



Lemma 5.1. [8] *If L is non–trivial, then:*

(i) $\emptyset \neq \mathrm{At}(\mathrm{Con}(L)) \subseteq \{\mathrm{con}(a,b) \mid a,b \in L, a \prec b\}$;

(ii) for any $\theta \in \mathrm{At}(\mathrm{Con}(L))$, $|\mathrm{Con}(L/\theta)| \geq |\mathrm{Con}(L)|/2$;

(iii) for any $a,b \in L$ *such that* $a \prec b$: $[a,b]_L$ *is a narrows iff* $L/\mathrm{con}(a,b) = \{\{a,b\}\} \cup \{\{x\} \mid x \in L \setminus \{a,b\}\}$ *iff* $|L/\mathrm{con}(a,b)| = |L| - 1$;

(iv) for any $a,b \in L$ *such that* $a \prec b$ *and* $|L/\mathrm{con}(a,b)| = |L| - 2$, *we have one of the following situations:*

- *a is meet–reducible, case in which* $a \prec c$ *for some* $c \in L \setminus \{b\}$ *such that* $b \prec b \vee c$, $c \prec b \vee c$, $[a, b \vee c]_L = \{a,b,c,b \vee c\} \cong \mathcal{L}_2^2$ *and* $L/\mathrm{con}(a,b) = \{\{a,b\},\{c,b \vee c\}\} \cup \{\{x\} \mid x \in L \setminus \{a,b,c,b \vee c\}\}$;
- *b is join–reducible, case in which, dually,* $c \prec b$ *for some* $c \in L \setminus \{a\}$ *such that* $a \wedge c \prec a$, $a \wedge c \prec c$, $[a \wedge c, b]_L = \{a \wedge c, a, c, b\} \cong \mathcal{L}_2^2$ *and* $L/\mathrm{con}(a,b) = \{\{b \wedge c, c\},\{a,b\}\} \cup \{\{x\} \mid x \in L \setminus \{b \wedge c, c, a, b\}\}$.

Remark 5.2. Let $a,b \in L$ with $a \neq b$. Also, let $\theta \in \mathrm{Con}(L)$. If $a \prec b$ and $a/\theta \neq b/\theta$, then, clearly, $a/\theta \prec b/\theta$. If $a/\theta \prec b/\theta$, then there exists no $u \in [a,b]_L \setminus (a/\theta \cup b/\theta)$, because otherwise we would have $a/\theta < u/\theta < b/\theta$. Let us also note that $a/\theta \leq b/\theta$ iff $a \vee b \in b/\theta$ iff $a \wedge b \in a/\theta$ iff $a \leq x$ for some $x \in b/\theta$ iff $w \leq b$ for some $w \in a/\theta$.

By Lemma 5.1, (iii), if $[a,b]_L$ is a narrows, then $\mathrm{con}(a,b)$ collapses a single pair of elements, thus, clearly, $\mathrm{con}(a,b) \in \mathrm{At}(\mathrm{Con}(L))$. Since $a/\mathrm{con}(a,b) = b/\mathrm{con}(a,b)$, we have $|L/\mathrm{con}(a,b)| \leq |L| - 1$, hence the second equivalence in Lemma 5.1, (iii), is clear.

By Lemma 5.1, (iii), if $|L/\mathrm{con}(a,b)| < |L| - 1$, as in Lemma 5.1, (iv), then $[a,b]_L$ is not a narrows, hence a is meet–reducible, so that a has a successor different from b, or b is join–reducible, so that b has a predecessor different from a. With the notations in Lemma 5.1, (iv), if $|L| - |L/\mathrm{con}(a,b)| = 2$ and, for instance, a is meet–reducible, then, simply, the fact that $(a,b),(c,b \vee c) = (a \vee c, b \vee c) \in \mathrm{con}(a,b)$ implies that $L/\mathrm{con}(a,b) = \{\{a,b\},\{c,b \vee c\}\} \cup \{\{x\} \mid x \in L \setminus \{a,b,c,b \vee c\}\}$, with $a/\mathrm{con}(a,b) \neq x/\mathrm{con}(a,b) \neq c/\mathrm{con}(a,b)$ for all $x \in L \setminus \{a,b,c,b \vee c\}$ and $a/\mathrm{con}(a,b) \neq c/\mathrm{con}(a,b)$, which, along with the fact that $a \prec c$, as above, proves that $a/\mathrm{con}(a,b) \prec c/\mathrm{con}(a,b) = (b \vee c)/\mathrm{con}(a,b)$.

Lemma 5.3. *For any* $a,b \in L$ *such that* $a \prec b$ *and* $|L| - |L/\mathrm{con}(a,b)| = 3$, *we have one of the following situations:*

- *a is meet–reducible, so that* $a \prec c$ *for some* $c \in L \setminus \{b\}$, *case in which one of the following is fulfilled:*

(i) $b \prec b \vee c$, $c \prec b \vee c$, $[a, b \vee c]_L = \{a, b, c, b \vee c\} \cong \mathcal{L}_2^2$ *and* $L/\mathrm{con}(a, b) = \{\{a, b, c, b \vee c\}\} \cup \{\{x\} \mid x \in L \setminus \{a, b, c, b \vee c\}\}$*;*

(ii) $b \prec b \vee c$, $c \prec b \vee c$ *and, for some* $d \in L \setminus \{a, b, c, b \vee c\}$, $d \prec a$, $[d, b \vee c]_L = \{d, a, b, c, b \vee c\} \cong \mathcal{L}_2 \dotplus \mathcal{L}_2^2$ *and* $L/\mathrm{con}(a, b) = \{\{d, a, b\}, \{c, b \vee c\}\} \cup \{\{x\} \mid x \in L \setminus \{a, b, c, b \vee c, d\}\}$*;*

(iii) $c \prec b \vee c$ *and, for some* $d \in L \setminus \{a, b, c, b \vee c\}$, $b \prec d \prec b \vee c$, $[a, b \vee c]_L = \{a, b, c, d, b \vee c\} \cong N_5$ *and* $L/\mathrm{con}(a, b) = \{\{a, b, d\}, \{c, b \vee c\}\} \cup \{\{x\} \mid x \in L \setminus \{a, b, c, b \vee c, d\}\}$*;*

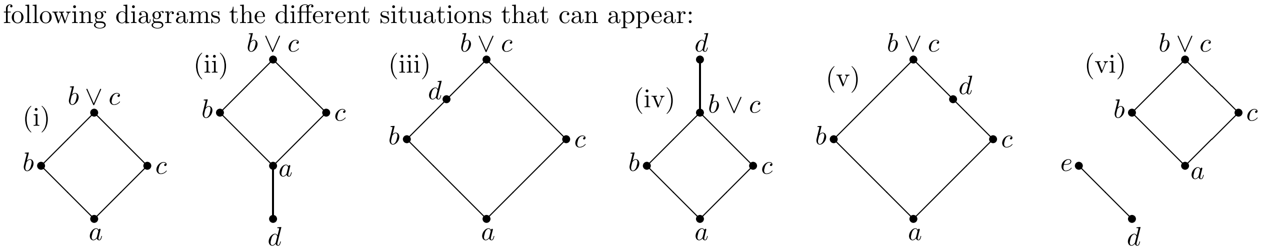
(iv) $b \prec b \vee c$, $c \prec b \vee c$ *and, for some* $d \in L \setminus \{a, b, c, b \vee c\}$, $b \vee c \prec d$, $[a, d]_L = \{a, b, c, b \vee c, d\} \cong \mathcal{L}_2^2 \dotplus \mathcal{L}_2$ *and* $L/\mathrm{con}(a, b) = \{\{a, b\}, \{c, b \vee c, d\}\} \cup \{\{x\} \mid x \in L \setminus \{a, b, c, b \vee c, d\}\}$*;*

(v) $b \prec b \vee c$ *and, for some* $d \in L \setminus \{a, b, c, b \vee c\}$, $c \prec d \prec b \vee c$, $L/\mathrm{con}(a, b) = \{\{a, b\}, \{c, d, b \vee c\}\} \cup \{\{x\} \mid x \in L \setminus \{a, b, c, b \vee c, d\}\}$ *and* $[a, b \vee c]_L = \{a, b, c, d, b \vee c\} \cong N_5$*;*

(vi) $b \prec b \vee c$, $c \prec b \vee c$, $[a, b \vee c]_L = \{a, b, c, b \vee c\} \cong \mathcal{L}_2^2$ *and, for some* $d, e \in L \setminus \{a, b, c, b \vee c\}$ *such that* $d \prec e$, $L/\mathrm{con}(a, b) = \{\{a, b\}, \{c, b \vee c\}, \{d, e\}\} \cup \{\{x\} \mid x \in L \setminus \{a, b, c, b \vee c, d, e\}\}$*;*

- *b is join–reducible, so that the dual of the previous situation is fulfilled, as in Lemma 5.1, (iv).*

Proof. Let $\theta = \mathrm{con}(a, b)$. We have $a \prec b$ and $|L/\theta| = |L| - 3 = n - 3$, hence $[a, b]_L$ is not a narrows, according to Lemma 5.1, (iii), thus a is meet–reducible or b is join–reducible. We analyse the case when a is meet–reducible, so that $a \prec c$ for some $c \in L \setminus \{b\}$; the case when b is join–reducible is dual to this one. We depict in the following diagrams the different situations that can appear:



If $a/\theta = b/\theta = c/\theta$, so that $(b \vee c)/\theta = a/\theta$, then, since a/θ is a convex sublattice of L and $|L| - |L/\theta| = 3$, we have $a/\theta = \{a, b, c, b \vee c\} = [a, b \vee c]_L \cong \mathcal{L}_2^2$, so that $b \prec b \vee c$ and $c \prec b \vee c$, and $L/\theta = \{\{a, b, c, b \vee c\}\} \cup \{\{x\} \mid x \in L \setminus \{a, b, c, b \vee c\}\}$; this is case (i) in the enunciation of the present lemma.

If $a/\theta \neq c/\theta$, then, since $a \prec c$, by Remark 5.2 it follows that $a/\theta \prec c/\theta = (b \vee c)/\theta$. Since $|L| - |L/\theta| = 3 > 2$, we get that there exists $d \in L \setminus \{a, b, c, b \vee c\}$ such that $\{d\} \subsetneq d/\theta$. The fact that $|L| - |L/\theta| = 3$ shows that there are three possible situations:

- $d \in a/\theta$, case in which $a/\theta = \{a, b, d\}$, $c/\theta = \{c, b \vee c\}$ and $x/\theta = \{x\}$ for all $x \in L \setminus \{a, b, c, b \vee c, d\}$;
- $d \in c/\theta$, case in which $a/\theta = \{a, b\}$, $c/\theta = \{c, b \vee c, d\}$ and $x/\theta = \{x\}$ for all $x \in L \setminus \{a, b, c, b \vee c, d\}$;
- $d \notin a/\theta \cup c/\theta$, case in which $a/\theta = \{a, b\}$, $c/\theta = \{c, b \vee c\}$, $d/\theta = \{d, e\}$ for some $e \in L \setminus \{a, b, c, b \vee c, d\}$ and $x/\theta = \{x\}$ for all $x \in L \setminus \{a, b, c, b \vee c, d, e\}$.

If $d \in a/\theta$, then a/θ is a three–element lattice, thus $a/\theta = \{a, b, d\} \cong \mathcal{L}_3$, so that $d < a < b$ or $a < b < d$ since $a \prec b$. The convexity of a/θ ensures us that, if $d < a < b$, then $a/\theta = [d, b]_L$, so $d \prec a$, hence $\{d, a, b, c, b \vee c\} \cong \mathcal{L}_2 \dotplus \mathcal{L}_2^2$; this is case (ii) in the enunciation. If $a < b < d$, then $c \ngeq b < d \leq d \vee c \in (a \vee c)/\theta = c/\theta = \{c, b \vee c\}$, thus $b < d \leq d \vee c = b \vee c \neq d$, so that $b < d < b \vee c$. Therefore $\{a, b, c, d, b \vee c\} \cong N_5$, and, since $d/\theta \prec (b \vee c)/\theta$ and any $x \in L$ with $d < x < b \vee c$ would be such that $x \notin d/\theta \cup (b \vee c)/\theta$, Remark 5.2 shows that $d \prec b \vee c$; this is case (iii).

If $d \in c/\theta$, then $c/\theta = \{c, b \vee c, d\} \cong \mathcal{L}_3$, so that $d < c < b \vee c$ or $c < d < b \vee c$ or $c < b \vee c < d$. If $c < b \vee c < d$, then $\{a, b, c, b \vee c, d\} \cong \mathcal{L}_2^2 \dotplus \mathcal{L}_2$ and $\{c, b \vee c, d\} = c/\theta = [c, d]_L$, so that $b \vee c \prec d$; this is case (iv). If $c < d < b \vee c$, then $\{a, b, c, d, b \vee c\} \cong N_5$ and $\{c, d, b \vee c\} = c/\theta = [c, b \vee c]_L$, so that $c \prec d \prec b \vee c$; this is case (v). Finally, if $d < c < b \vee c$, then $\{a, b\} = a/\theta = (a \wedge c)/\theta = (a \wedge d)/\theta$, hence $b > a \geq a \wedge d \in \{a, b\}$, thus $a \wedge d = a \neq d$, so we obtain $a < d < c$, which contradicts the fact that $a \prec c$.

The remaining possibility is that $d/\theta = e/\theta$ for some $e \in L \setminus \{a, b, c, b \vee c, d\}$, so that $c/\theta = \{c, b \vee c\} \cong \mathcal{L}_2$ and $d/\theta = \{d, e\} \cong \mathcal{L}_2$, thus $c \prec b \vee c$ and either $d \prec e$ or $e \prec d$; this is case (vi). □

Remark 5.4. Remark 3.1 and the fact that $2^{1-1} = 1 = 2^{2-2}$, $2^{2-1} = 2 = 2^{3-2}$ and $2^{3-1} = 4 = 2^{4-2}$ give us: if $|\mathrm{Con}(L)| < 2^{n-1}$, then $n \geq 4$; if $|\mathrm{Con}(L)| < 2^{n-2}$, then $n \geq 5$.

Theorem 5.5. *(i)* [12],[8] $|\mathrm{Con}(L)| \leq 2^{n-1}$ *and:* $|\mathrm{Con}(L)| = 2^{n-1}$ *iff* $L \cong \mathcal{L}_n$*.*

(ii) [8] *if* $|\mathrm{Con}(L)| < 2^{n-1}$*, then* $|\mathrm{Con}(L)| \leq 2^{n-2}$ *and:* $|\mathrm{Con}(L)| = 2^{n-2}$ *iff* $L \cong \mathcal{L}_k \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_{n-k-2}$ *for some* $k \in [1, n-3]$*.*

Following the line of the proof from [8] of Theorem 5.5, now we prove:

Theorem 5.6. *If* $|\mathrm{Con}(L)| < 2^{n-2}$*, then* $n \geq 5$ *and:*

(i) $|\mathrm{Con}(L)| \leq 5 \cdot 2^{n-5} = 2^{n-3} + 2^{n-5}$*, and:* $|\mathrm{Con}(L)| = 5 \cdot 2^{n-5}$ *iff* $L \cong \mathcal{L}_k \dotplus N_5 \dotplus \mathcal{L}_{n-k-3}$ *for some* $k \in [1, n-4]$*;*

(ii) *if* $|\mathrm{Con}(L)| < 5 \cdot 2^{n-5}$*, then* $|\mathrm{Con}(L)| \leq 2^{n-3}$*, and:* $|\mathrm{Con}(L)| = 2^{n-3}$ *iff either* $n \geq 6$ *and* $L \cong \mathcal{L}_k \dotplus (\mathcal{L}_2 \times \mathcal{L}_3) \dotplus \mathcal{L}_{n-k-4}$ *for some* $k \in [1, n-5]$*, or* $n \geq 7$ *and* $L \cong \mathcal{L}_k \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_m \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_{n-k-m-4}$ *for some* $k, m \in \mathbb{N}^*$ *such that* $k + m \leq n - 5$*;*

(iii) *if* $|\mathrm{Con}(L)| < 2^{n-3}$*, then* $|\mathrm{Con}(L)| \leq 7 \cdot 2^{n-6} = 2^{n-4} + 2^{n-5} + 2^{n-6}$*, and:* $|\mathrm{Con}(L)| = 7 \cdot 2^{n-6}$ *iff* $n \geq 6$ *and, for some* $k \in [1, n-7]$*,* $L \cong \mathcal{L}_k \dotplus (\mathcal{L}_3 \boxplus \mathcal{L}_5) \dotplus \mathcal{L}_{n-k-6}$ *or* $L \cong \mathcal{L}_k \dotplus (\mathcal{L}_4 \boxplus \mathcal{L}_4) \dotplus \mathcal{L}_{n-k-6}$*.*

Proof. Assume that $|\mathrm{Con}(L)| < 2^{n-2} < 2^{n-1}$, so that $n \geq 5$ by Remark 5.4. We shall prove the statements in the enunciation by induction on $n \in \mathbb{N}$, $n \geq 5$. We shall identify the lattices up to isomorphism.

The five–element lattices are: M_3, N_5, $\mathcal{L}_2 \dotplus \mathcal{L}_2^2$, $\mathcal{L}_2^2 \dotplus \mathcal{L}_2$ and $\mathcal{L}_5$, whose numbers of congruences are: 2, 5, 8, 8 and $2^4 = 16$, respectively. The five–element lattices with strictly less than $2^{5-2} = 8$ congruences are M_3 and N_5, out of which $N_5 \cong \mathcal{L}_1 \dotplus N_5 \dotplus \mathcal{L}_{5-1-3}$, is of the form in (i) and has $5 = 5 \cdot 2^{5-5}$ congruences, while M_3 has $2 < 4 = 2^{5-3}$ congruences. From this fact and Remark 5.4, it follows that, if $|\mathrm{Con}(L)| = 2^{n-3}$, then $n \geq 6$.

The six–element lattices are: M_4, $\mathcal{L}_4 \boxplus \mathcal{L}_2^2$, $M_3 \dotplus \mathcal{L}_2$, $\mathcal{L}_2 \dotplus M_3$, $(\mathcal{L}_2^2 \dotplus \mathcal{L}_2) \boxplus \mathcal{L}_3$, $(\mathcal{L}_2 \dotplus \mathcal{L}_2^2) \boxplus \mathcal{L}_3$, $\mathcal{L}_3 \boxplus \mathcal{L}_5$, $\mathcal{L}_4 \boxplus \mathcal{L}_4$, $\mathcal{L}_2 \times \mathcal{L}_3$, $N_5 \dotplus \mathcal{L}_2$, $\mathcal{L}_2 \dotplus N_5$, $\mathcal{L}_2^2 \dotplus \mathcal{L}_3$, $\mathcal{L}_3 \dotplus \mathcal{L}_2^2$, $\mathcal{L}_2 \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_2$ and $\mathcal{L}_6$, whose numbers of congruences are: 2, 3, 4, 4, 6, 6, 7, 7, 8, 10, 10, 16, 16, $16 = 2^{6-2}$ and $32 = 2^{6-1}$, respectively. So, the third largest number of congruences of a six–element lattice is $10 = 5 \cdot 2^{6-5}$, the fourth largest is $8 = 2^{6-3}$ and the fifth largest is $7 = 7 \cdot 2^{6-6}$. As above, we notice that $N_5 \dotplus \mathcal{L}_2$ and $\mathcal{L}_2 \dotplus N_5$ are of the form in (i), $\mathcal{L}_2 \times \mathcal{L}_3$ is of the first form in (ii) and $\mathcal{L}_3 \boxplus \mathcal{L}_5$ and $\mathcal{L}_4 \boxplus \mathcal{L}_4$ are of the forms in (iii).

It is easy to construct, as above, the 7–element lattices, and see that the ones with strictly less than $2^{7-2} = 32$ congruences are: the ones having $20 = 5 \cdot 2^{7-5}$ congruences, namely $N_5 \dotplus \mathcal{L}_3$, $\mathcal{L}_3 \dotplus N_5$ and $\mathcal{L}_2 \dotplus N_5 \dotplus \mathcal{L}_2$, all of the form in (i); the ones having $16 = 2^{7-3}$ congruences, namely $(\mathcal{L}_2 \times \mathcal{L}_3) \dotplus \mathcal{L}_2$ and $\mathcal{L}_2 \dotplus (\mathcal{L}_2 \times \mathcal{L}_3)$, which are of the first form in (ii), as well as $\mathcal{L}_2^2 \dotplus \mathcal{L}_2^2$, which is of the second form in (ii); the ones having $14 = 7 \cdot 2^{7-6}$ congruences, namely $(\mathcal{L}_3 \boxplus \mathcal{L}_5) \dotplus \mathcal{L}_2$, $\mathcal{L}_2 \dotplus (\mathcal{L}_3 \boxplus \mathcal{L}_5)$, $(\mathcal{L}_4 \boxplus \mathcal{L}_4) \dotplus \mathcal{L}_2$ and $\mathcal{L}_2 \dotplus (\mathcal{L}_4 \boxplus \mathcal{L}_4)$, all of the forms in (iii); and the ones having strictly less than 14 congruences.

Now assume that $n \geq 8$ and lattices of cardinality at most $n-1$ fulfill the statements in the enunciation. Note that, in the rest of this proof, whenever $|\mathrm{Con}(L)| = 5 \cdot 2^{n-5}$, L is of the form in (i), whenever $|\mathrm{Con}(L)| = 2^{n-3}$, L is of one of the forms in (ii) and, whenever $|\mathrm{Con}(L)| = 7 \cdot 2^{n-6}$, L is of one of the forms in (iii).

Let $\theta \in \mathrm{At}(\mathrm{Con}(L))$. By Lemma 5.1, (i), at least one such θ exists, and $\theta = \mathrm{con}(a, b)$ for some $a, b \in L$ with $a \prec b$. Then $a/\theta = b/\theta$, so that $|L/\theta| \leq n-1$, hence $|\mathrm{Con}(L/\theta)| \leq 2^{n-2}$ by Theorem 5.5, (i). By Lemma 5.1, (i), $|\mathrm{Con}(L/\theta)| \geq |\mathrm{Con}(L)|/2$.

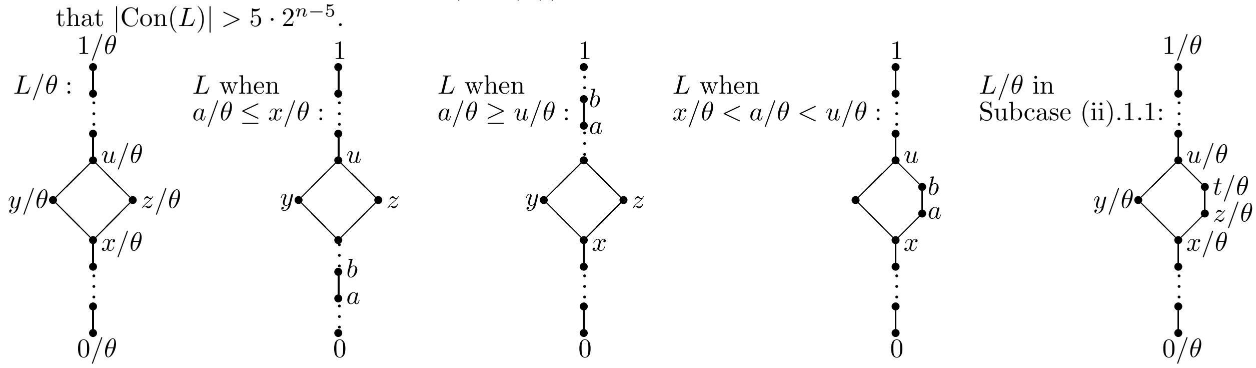
(i) By the hypothesis of the theorem, $|\mathrm{Con}(L)| < 2^{n-2}$. Assume by absurdum that $|\mathrm{Con}(L)| > 5 \cdot 2^{n-5}$, so that $|\mathrm{Con}(L/\theta)| > 5 \cdot 2^{n-6} > 4 \cdot 2^{n-6} = 2^{n-4} = 2^{(n-3)-1}$, thus $|L/\theta| > n-3$ by Theorem 5.5, (i), hence $|L/\theta| \in \{n-1, n-2\}$.

Case (i).1: Assume that $|L/\theta| = n-1$, so that, according to Lemma 5.1, (ii), $L/\theta = \{\{a, b\}\} \cup \{\{x\} \mid x \in L \setminus \{a, b\}\}$ and $[a, b]_L$ is a narrows, thus b is the unique successor of a and a is the unique predecessor of b. Since $|\mathrm{Con}(L/\theta)| > 5 \cdot 2^{n-6} = 5 \cdot 2^{(n-1)-5}$, Theorem 5.5 and the induction hypothesis ensure us that $|\mathrm{Con}(L/\theta)| \in \{2^{n-2}, 2^{n-3}\}$.

Subcase (i).1.1: Assume that $|\mathrm{Con}(L/\theta)| = 2^{n-2} = 2^{(n-1)-1}$, so that $\{\{a, b\}\} \cup \{\{x\} \mid x \in L \setminus \{a, b\}\} = L/\theta \cong \mathcal{L}_{n-1}$ by Theorem 5.5, (i), and thus, for any $x, y \in L \setminus \{a, b\}$, either $x/\theta \leq a/\theta$ or $a/\theta = b/\theta \leq x/\theta$, and either $x/\theta \leq y/\theta$ or $y/\theta \leq x/\theta$, so that, by the form of the classes of θ and Remark 5.2, either $x \leq a$ or $b \leq x$, and either $x \leq y$ or $y \leq x$, therefore $L \cong \mathcal{L}_n$. But then $|\mathrm{Con}(L)| = 2^{n-1}$, which contradicts the hypothesis that $|\mathrm{Con}(L)| < 2^{n-2}$ of the present theorem.

Subcase (i).1.2: Assume that $|\mathrm{Con}(L/\theta)| = 2^{n-3} = 2^{(n-1)-2}$, so that, according to Theorem 5.5, (ii), $L/\theta \cong \mathcal{L}_k \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_{n-k-3} \cong \mathcal{L}_k \dotplus (\mathcal{L}_3 \boxplus \mathcal{L}_3) \dotplus \mathcal{L}_{n-k-3}$ for some $k \in [1, n-4]$. If we denote the elements of L/θ as in the leftmost diagram below, with $x, y, z, u \in L$, and we also consider the facts that $|L| - |L/\theta| = 1$, a has the unique successor b and b has the unique predecessor a, $a/\theta = b/\theta = \{a, b\}$ and $v/\theta = \{v\}$ for all $v \in L \setminus \{a, b\}$, then we notice that L is in one of the following situations, represented in the three diagrams to the right of that of L/θ:

- if $a/\theta = b/\theta \leq x/\theta$, then $b \leq x$ and $L \cong \mathcal{L}_2 \dotplus L/\theta \cong \mathcal{L}_{k+1} \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_{n-k-3}$, while, if $a/\theta = b/\theta \geq u/\theta$, then $a \geq u$ and $L \cong L/\theta \dotplus \mathcal{L}_2 \cong \mathcal{L}_k \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_{n-k-2}$, but in these situations $|\mathrm{Con}(L)| = 2^{n-2}$, which contradicts the hypothesis that $|\mathrm{Con}(L)| < 2^{n-2}$ of the theorem;
- if $x/\theta < a/\theta = b/\theta < u/\theta$, then $x < a < b < u$, hence $\{a, b\} \cap \{y, z\} \neq \emptyset$, so that $L \cong \mathcal{L}_k \dotplus (\mathcal{L}_3 \boxplus \mathcal{L}_4) \dotplus \mathcal{L}_{n-k-3} \cong \mathcal{L}_k \dotplus N_5 \dotplus \mathcal{L}_{n-k-3}$, thus $|\mathrm{Con}(L)| = 2^{k-1} \cdot 5 \cdot 2^{n-k-4} = 5 \cdot 2^{n-5}$, which contradicts the assumption that $|\mathrm{Con}(L)| > 5 \cdot 2^{n-5}$.



Case (i).2: Now assume that $|L/\theta| = n-2$, which means that we are in the situation from Lemma 5.1, (iv), and assume, for instance, that a is meet–reducible, so that $a \prec c$ for some $c \in L \setminus \{b\}$, and we have $b \prec b \vee c$ and $c \prec b \vee c$, so that $a/\theta = \{a, b\} \prec \{c, b \vee c\} = c/\theta$ by Remark 5.2, $[a, b \vee c]_L = \{a, b, c, b \vee c\} \cong \mathcal{L}_2^2$, and $x/\theta = \{x\}$ for all $x \in L \setminus \{a, b, c, b \vee c\}$; the dual case is analogous to this one. Since $|\mathrm{Con}(L/\theta)| > 5 \cdot 2^{n-6} > 4 \cdot 2^{n-6} = 2^{n-4} = 2^{(n-2)-2}$, Theorem 5.5 ensures us that $|\mathrm{Con}(L/\theta)| = 2^{(n-2)-1} = 2^{n-3}$ and $\{\{a, b\}, \{c, b \vee c\}\} \cup \{\{x\} \mid x \in L \setminus \{a, b, c, b \vee c\}\} = L/\theta \cong \mathcal{L}_{n-2}$. So L/θ is a chain, thus, for all $x, y \in L \setminus \{a, b, c, b \vee c\}$, we have either $x/\theta \leq a/\theta \prec c/\theta$ or $a/\theta \prec c/\theta = (b \vee c)/\theta \leq x/\theta$, and either $x/\theta \leq y/\theta$ or $y/\theta \leq x/\theta$, so that, by the form of the classes of θ and Remark 5.2, we have either $x \leq a$ or $b \vee c \leq x$, and either $x \leq y$ or $y \leq x$, so that $L \cong \mathcal{L}_k \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_{n-k-2}$ for some $k \in [1, n-3]$, with $\{a, b, c, b \vee c\}$ being the sublattice of L isomorphic to $\mathcal{L}_2^2$; but then $|\mathrm{Con}(L)| = 2^{n-2}$, which contradicts the hypothesis that $|\mathrm{Con}(L)| < 2^{n-2}$ of the theorem.

Therefore, indeed, $|\mathrm{Con}(L)| \leq 5 \cdot 2^{n-5}$. Now assume that $|\mathrm{Con}(L)| = 5 \cdot 2^{n-5}$, so that $|\mathrm{Con}(L/\theta)| \geq 5 \cdot 2^{n-6} > 4 \cdot 2^{n-6} = 2^{n-4}$, thus, as above, $|L/\theta| \in \{n-1, n-2\}$. By Case (i).1, the equality $|\mathrm{Con}(L)| = 5 \cdot 2^{n-5}$ shows that, if $|L/\theta| = n-1$, then, for some $k \in [1, n-4]$, $L/\theta \cong \mathcal{L}_k \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_{n-k-3}$ and $L \cong \mathcal{L}_k \dotplus N_5 \dotplus \mathcal{L}_{n-k-3}$. By Case (i).2, we can not have $|L/\theta| = n-2$.

(ii) Assume that $|\mathrm{Con}(L)| < 5 \cdot 2^{n-5}$, and assume by absurdum that $|\mathrm{Con}(L)| > 2^{n-3}$, so that $|\mathrm{Con}(L/\theta)| > 2^{n-4} = 2^{(n-3)-1}$, hence $|L/\theta| > n-3$ by Theorem 5.5, (i), thus $|L/\theta| \in \{n-1, n-2\}$. By Cases (i).1 and (i).2 above, in both of these situations we obtain that $|\mathrm{Con}(L)| \geq 5 \cdot 2^{n-5}$, contradicting the current assumption. Therefore $|\mathrm{Con}(L)| \leq 2^{n-3}$.

Now assume that $|\mathrm{Con}(L)| = 2^{n-3}$, so that $|\mathrm{Con}(L/\theta)| \geq 2^{n-4} = 2^{(n-3)-1}$, hence $|L/\theta| \geq n-3$ by Theorem 5.5, (i), thus $|L/\theta| \in \{n-1, n-2, n-3\}$.

Case (ii).1: Assume that $|L/\theta| = n-1$. Then, since $|\mathrm{Con}(L/\theta)| \geq 2^{n-4} = 2^{(n-1)-3}$, Theorem 5.5 and the induction hypothesis ensure us that $|\mathrm{Con}(L/\theta)| \in \{2^{n-2}, 2^{n-3}, 5 \cdot 2^{n-6}, 2^{n-4}\}$. By Case (i).1, we can not have $|\mathrm{Con}(L/\theta)| \in \{2^{n-2}, 2^{n-3}\}$.

Subcase (ii).1.1: Assume that $|\mathrm{Con}(L/\theta)| = 5 \cdot 2^{n-6}$, which, by the induction hypothesis, means that $L/\theta \cong \mathcal{L}_k \dotplus N_5 \dotplus \mathcal{L}_{n-k-4}$ for some $k \in [1, n-5]$, so that L is in one of the following situations, that we separate as above, where the elements of L/θ are denoted as in the rightmost diagram above, with $x, y, z, t, u \in L$:

- if $a/\theta = b/\theta \leq x/\theta$, then $a < b \leq x$ and $L \cong \mathcal{L}_2 \dotplus L/\theta \cong \mathcal{L}_{k+1} \dotplus N_5 \dotplus \mathcal{L}_{n-k-4}$, while, if $a/\theta = b/\theta \geq u/\theta$, then $u \leq a < b$ and $L \cong L/\theta \dotplus \mathcal{L}_2 \cong \mathcal{L}_k \dotplus N_5 \dotplus \mathcal{L}_{n-k-3}$, hence $|\mathrm{Con}(L)| = 2 \cdot |\mathrm{Con}(L/\theta)| = 5 \cdot 2^{n-5}$;

- if $x/\theta < a/\theta = b/\theta < u/\theta$, then $x < a < b < u$ and: either $\{a,b\} \cap \{z,t\} \neq \emptyset$, case in which a,b,z,t are pairwise comparable, because otherwise a would be meet–reducible or b would be join–reducible, thus $L \cong \mathcal{L}_k \dotplus (\mathcal{L}_3 \boxplus \mathcal{L}_5) \dotplus \mathcal{L}_{n-k-4}$, or $y \in \{a,b\}$, so that $L \cong \mathcal{L}_k \dotplus (\mathcal{L}_4 \boxplus \mathcal{L}_4) \dotplus \mathcal{L}_{n-k-4}$, hence $|\mathrm{Con}(L)| = 2^{k-1} \cdot (2^2+3) \cdot 2^{n-k-5} = 7 \cdot 2^{n-6}$, which contradicts the current assumption.

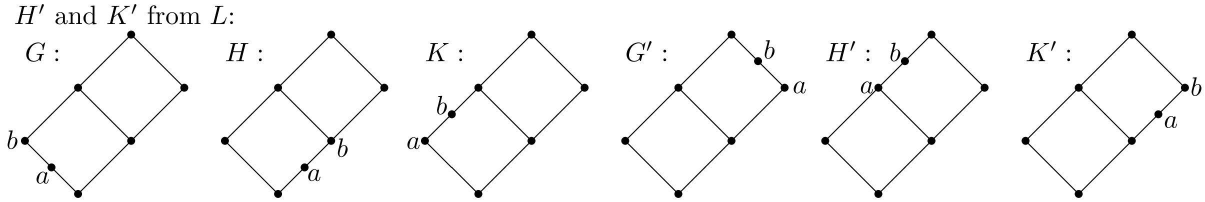
The following subcases can be treated exactly as above. For brevity, we shall only indicate the shapes of the lattices in the remaining part of the proof.

Subcase (ii).1.2: Assume that $|\mathrm{Con}(L/\theta)| = 2^{n-4} = 2^{(n-1)-3}$, which, by the induction hypothesis, means that either $L/\theta \cong \mathcal{L}_r \dotplus (\mathcal{L}_2 \times \mathcal{L}_3) \dotplus \mathcal{L}_{n-r-5}$ for some $r \in [1, n-6]$, or $L/\theta \cong \mathcal{L}_k \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_m \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_{n-k-m-5}$ for some $k,m \in \mathbb{N}^*$ such that $k+m \leq n-6$, so that L is in one of the following situations:

- $L \cong \mathcal{L}_{r+1} \dotplus (\mathcal{L}_2 \times \mathcal{L}_3) \dotplus \mathcal{L}_{n-r-5}$ or $L \cong \mathcal{L}_r \dotplus (\mathcal{L}_2 \times \mathcal{L}_3) \dotplus \mathcal{L}_{n-r-4}$ or $L \cong \mathcal{L}_{k+1} \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_m \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_{n-k-m-5}$ or $L/\theta \cong \mathcal{L}_k \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_{m+1} \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_{n-k-m-5}$ or $L/\theta \cong \mathcal{L}_k \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_m \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_{n-k-m-4}$, so that $|\mathrm{Con}(L)| = 2 \cdot |\mathrm{Con}(L/\theta)| = 2^{n-3}$;

- $L \cong \mathcal{L}_k \dotplus N_5 \dotplus \mathcal{L}_m \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_{n-k-m-5}$ or $L \cong \mathcal{L}_k \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_m \dotplus N_5 \dotplus \mathcal{L}_{n-k-m-5}$, case in which $|\mathrm{Con}(L)| = 5 \cdot 2^2 \cdot 2^{k-1+m-1+n-k-m-6} = 5 \cdot 2^{n-6} < 7 \cdot 2^{n-6} < 8 \cdot 2^{n-6} = 2^{n-3}$, contradicting the current assumption;

- $L \cong \mathcal{L}_r \dotplus G \dotplus \mathcal{L}_{n-r-5}$ or $L \cong \mathcal{L}_r \dotplus G' \dotplus \mathcal{L}_{n-r-5}$ or $L \cong \mathcal{L}_r \dotplus H \dotplus \mathcal{L}_{n-r-5}$ or $L \cong \mathcal{L}_r \dotplus H' \dotplus \mathcal{L}_{n-r-5}$ or $L \cong \mathcal{L}_r \dotplus K \dotplus \mathcal{L}_{n-r-5}$ or $L \cong \mathcal{L}_r \dotplus K' \dotplus \mathcal{L}_{n-r-5}$, where G, H and K are the following gluings of a pentagon with a rhombus and G', H' and K' are the duals of G, H and K, respectively, so that $|\mathrm{Con}(L)| = 9 \cdot 2^{r-1+n-r-6} = 9 \cdot 2^{n-7} < 14 \cdot 2^{n-7} = 7 \cdot 2^{n-6} < 2^{n-3}$, contradicting the current assumption, since $|\mathrm{Con}(G)| = |\mathrm{Con}(H)| = |\mathrm{Con}(K)| = 9$, which is simple to verify, and thus $|\mathrm{Con}(G')| = |\mathrm{Con}(H')| = |\mathrm{Con}(K')| = 9$, as well; below we are indicating the positions of a and b in these copies of G, H, K, G', H' and K' from L:

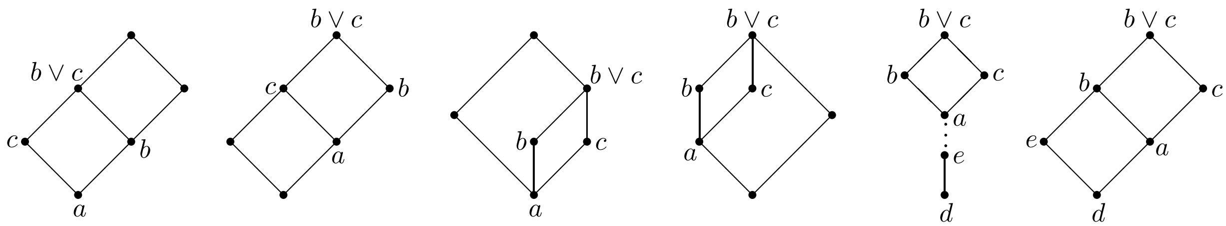


Case (ii).2: Assume that $|L/\theta| = n-2$. Then, since $|\mathrm{Con}(L/\theta)| \geq 2^{n-4} = 2^{(n-2)-2}$, Theorem 5.5 ensures us that $|\mathrm{Con}(L/\theta)| \in \{2^{n-3}, 2^{n-4}\}$. By Case (i).2, we can not have $|\mathrm{Con}(L/\theta)| = 2^{n-3}$, thus $|\mathrm{Con}(L/\theta)| = 2^{n-4}$, hence $L/\theta \cong \mathcal{L}_k \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_{n-k-5}$ for some $k \in [1, n-6]$, according to Theorem 5.5, (ii). We are in the situation from Lemma 5.1, (iv), hence a is meet–reducible or b is join–reducible. We shall assume that a is meet–reducible, so that $a \prec c$ for some $c \in L \setminus \{b\}$, and we apply Lemma 5.1, (iv), and Remark 5.2; the case when b is join–reducible shall follow by duality. Since $\{a,b\} = a/\theta \prec c/\theta = \{c, b \vee c\}$ and, for all $x \in L \setminus (a/\theta \cup c/\theta) = L \setminus \{a,b,c,b \vee c\}$, $x/\theta = \{x\}$ and $x \notin [a, b \vee c]_L$, hence L has one of the following forms:

- $L \cong \mathcal{L}_s \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_t \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_{n-s-t-4}$ for some $s,t \in \mathbb{N}^*$ such that $s+t \leq n-5$; in this case, one of the two copies of $\mathcal{L}_2^2$ from L is $\{a,b,c,b \vee c\}$, $r \in \{s, s+t+2\}$, and, indeed, $|\mathrm{Con}(L)| = 2^{n-3}$;

- $L \cong \mathcal{L}_r \dotplus (\mathcal{L}_2 \times \mathcal{L}_3) \dotplus \mathcal{L}_{n-r-4}$, case in which, indeed, $|\mathrm{Con}(L)| = 2^{n-3}$, and a, b, c, $b \vee c$ belong to the copy of $\mathcal{L}_2 \times \mathcal{L}_3$ from L, in which they are situated as in one of the following two leftmost diagrams, since $\theta = \mathrm{con}(a,b)$ only collapses a,b and $c, b \vee c$;

- $L \cong \mathcal{L}_r \dotplus (\mathcal{L}_3 \boxplus (\mathcal{L}_2^2 \dotplus \mathcal{L}_2)) \dotplus \mathcal{L}_{n-r-4}$ or $L \cong \mathcal{L}_r \dotplus (\mathcal{L}_3 \boxplus (\mathcal{L}_2 \dotplus \mathcal{L}_2^2)) \dotplus \mathcal{L}_{n-r-4}$, in which a, b, c and $b \vee c$ would be positioned in the copy of $\mathcal{L}_3 \boxplus (\mathcal{L}_2^2 \dotplus \mathcal{L}_2)$, respectively $\mathcal{L}_3 \boxplus (\mathcal{L}_2 \dotplus \mathcal{L}_2^2)$, as in the following two center diagrams, but then $|\mathrm{Con}(L)| = (2+2^2) \cdot 2^{r-1+n-r-5} = 6 \cdot 2^{n-6} < 7 \cdot 2^{n-6} < 8 \cdot 2^{n-6} = 2^{n-3}$, which contradicts the current hypothesis that $|\mathrm{Con}(L)| = 2^{n-3}$.



Case (ii).3: Assume that $|L/\theta| = n-3$. Then, since $|\mathrm{Con}(L/\theta)| \geq 2^{n-4} = 2^{(n-3)-1}$, by Theorem 5.5, (i), it follows that $|\mathrm{Con}(L/\theta)| = 2^{n-4}$, so that $L/\theta \cong \mathcal{L}_{n-3}$. We are in the case from Lemma 5.3; assume that a is meet–reducible, so that $a \prec c$ for some $c \in L \setminus \{b\}$; the case when b is join–reducible follows by duality.

In the situation from Lemma 5.3, (i), since L/θ is a chain, it follows that, for any $x, y \in L \setminus \{a, b, c, b \vee c\}$, $\{x\} = x/\theta < a/\theta = \{a, b, c, b \vee c\}$ or $a/\theta = (b \vee c)/\theta < x/\theta$, and $x/\theta \leq y/\theta = \{y\}$ or $y/\theta \leq x/\theta$, so that $x \leq y$ or $y \leq x$, and $x < z$ for every $z \in \{a, b, c, b \vee c\}$ or $z < x$ for every $z \in \{a, b, c, b \vee c\}$. Therefore $L \cong \mathcal{L}_k \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_{n-k-2}$ for some $k \in [1, n-3]$, so that $|\mathrm{Con}(L)| = 2^{n-2}$, which contradicts the hypothesis that $|\mathrm{Con}(L)| < 2^{n-2}$ of the present theorem.

In the same way, in the situations from Lemma 5.3, (ii) and (iv), we obtain that $L \cong \mathcal{L}_k \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_{n-k-2}$ for some $k \in [1, n-3]$, so that $|\mathrm{Con}(L)| = 2^{n-2}$, which contradicts both the hypothesis that $|\mathrm{Con}(L)| < 2^{n-2}$ of the theorem and the fact that $\theta = \mathrm{con}(a, b)$. Simlilarly, in the situations from Lemma 5.3, (iii) and (v), we get that $L \cong \mathcal{L}_k \dotplus N_5 \dotplus \mathcal{L}_{n-k-3}$ for some $k \in [1, n-4]$, so that $|\mathrm{Con}(L)| = 5 \cdot 2^{n-5}$, which contradicts the current assumption that $|\mathrm{Con}(L)| = 2^{n-3}$.

Now assume we are in the situation from Lemma 5.3, (vi), with d and e as in the enunciation of the lemma. Since L/θ is a chain, without loss of generality, we may assume that $d/\theta < a/\theta \prec c/\theta$, because the other case is dual to this one. So, in L/θ, we will have $\{d, e\} < \{a, b\} \prec \{c, b \vee c\}$ and, for all $x \in L \setminus \{a, b, c, b \vee c, d, e\}$: either $x/\theta < a/\theta \prec c/\theta$ or $a/\theta \prec c/\theta = (b \vee c)/\theta < x/\theta$, and either $x/\theta \leq d/\theta$ or $d/\theta = e/\theta < x/\theta$, therefore, since $x/\theta = \{x\}$, Remark 5.2 ensures us that we have either $x < a$ or $b \vee c < x$, and either $x < d$ or $e < x$.

If we had $e < a$, then $L \cong \mathcal{L}_k \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_{n-k-2}$ for some $k \in [3, n-3]$, because $d, e, a, b, c, b \vee c$ would be positioned in L as in the fifth diagram above, thus $|\mathrm{Con}(L)| = 2^{n-2}$, which contradicts the hypothesis that $|\mathrm{Con}(L)| < 2^{n-2}$ of the theorem, as well as the fact that $\theta = \mathrm{con}(a, b)$. We have $\{d, e\} = d/\theta < a/\theta = \{a, b\}$. Since a/θ and $d/\theta = e/\theta$ are convex, we can not have $e > a$. Hence e and a are incomparable, $d < a$ and $e < b$. So $d \leq a \wedge e \leq e$, thus $a \wedge e \in d/\theta = \{d, e\}$ by the convexity of d/θ, hence $a \wedge e = d$ since $e \nleq a$ by the above. Analogously, $a \vee e = b$. Hence $\{d, e, a, b, c, b \vee c\} \cong \mathcal{L}_2 \times \mathcal{L}_3$.

Recall that $d \prec e$, $a \prec b \prec b \vee c$, $a \prec c \prec b \vee c$ and $[a, b \vee c]_L = \{a, b, c, b \vee c\}$. Assume by absurdum that $[d, b]_L \neq \{d, e, a, b\}$, so that $x \in [d, b]_L$ for some $x \in L \setminus \{d, e, a, b, c, b \vee c\} = L \setminus (d/\theta \cup a/\theta \cup c/\theta)$. If x is comparable to neither e, nor a, then $\{d, e, x, a, b\} \cong M_3$, so that $(a, d) \in \mathrm{con}(a, b) = \theta$, which contradicts the fact that $a/\theta \neq d/\theta$. If x is comparable to a, then $d < x < a$, while, if x is comparable to e, then $e < x < b$, since $d < x < b$, $d \prec e$ and $a \prec b$; in each of these cases, $\{d, e, x, a, b\} \cong N_5$, so $x \in a/\mathrm{con}(a, b) = a/\theta$ in the first of these two cases, and $x \in d/\mathrm{con}(a, b) = d/\theta$ in the second, and each of these situations contradicts the fact that $x \notin d/\theta \cup a/\theta \cup c/\theta$. Therefore $[d, b]_L = \{d, e, a, b\}$ and thus $[d, b \vee c]_L = \{d, e, a, b, c, b \vee c\} \cong \mathcal{L}_2 \times \mathcal{L}_3$, so that d, e, a, b, c, $b \vee c$ are positioned in L as in the rightmost diagram above, and, since L/θ is a chain, for all $x \in L \setminus \{d, e, a, b, c, b \vee c\}$, we have $\{x\} = x/\theta < d/\theta \prec a/\theta \prec c/\theta$ or $d/\theta \prec a/\theta \prec c/\theta < x/\theta$, so that $x < d$ or $b \vee c < x$ by Remark 5.2. Hence $L \cong \mathcal{L}_k \dotplus (\mathcal{L}_2 \times \mathcal{L}_3) \dotplus \mathcal{L}_{n-k-4}$ for some $k \in [1, n-5]$, which, indeed, has $|\mathrm{Con}(L)| = 2^{n-3}$.

(iii) Assume that $|\mathrm{Con}(L)| < 2^{n-3}$ and assume by absurdum that $|\mathrm{Con}(L)| > 7 \cdot 2^{n-6}$, so that $|\mathrm{Con}(L/\theta)| > 7 \cdot 2^{n-7} > 5 \cdot 2^{n-7} > 4 \cdot 2^{n-7} = 2^{n-5} = 2^{(n-4)-1}$ by Lemma 5.1, (ii), hence $|L/\theta| > n-4$ by Theorem 5.5, (i), thus $|L/\theta| \in \{n-1, n-2, n-3\}$.

Case (iii).1: Assume that $|L/\theta| = n-1$. Since $|\mathrm{Con}(L/\theta)| > 7 \cdot 2^{n-7} = 7 \cdot 2^{(n-1)-6}$, by Theorem 5.5 and the induction hypothesis it follows that $|\mathrm{Con}(L/\theta)| \in \{2^{n-2}, 2^{n-3}, 5 \cdot 2^{n-6}, 2^{n-4}\}$. By Case (i).1, $|\mathrm{Con}(L/\theta)| \notin \{2^{n-2}, 2^{n-3}\}$. By Subcase (ii).1.1, since $|\mathrm{Con}(L)| > 7 \cdot 2^{n-6}$, it follows that $|\mathrm{Con}(L/\theta)| \neq 5 \cdot 2^{n-6}$. Finally, by Subcase (ii).1.2, since $2^{n-3} > |\mathrm{Con}(L/\theta)| > 7 \cdot 2^{n-7}$, it follows that $|\mathrm{Con}(L/\theta)| \neq 2^{n-4}$.

Case (iii).2: Assume that $|L/\theta| = n-2$. Since $|\mathrm{Con}(L/\theta)| > 7 \cdot 2^{n-7} > 5 \cdot 2^{n-7}$, by Theorem 5.5 and the induction hypothesis it follows that $|\mathrm{Con}(L/\theta)| \in \{2^{n-3}, 2^{n-4}\}$. By Case (i).2, $|\mathrm{Con}(L/\theta)| \neq 2^{n-3}$. By Case (ii).2, $|\mathrm{Con}(L/\theta)| \neq 2^{n-4}$.

Case (iii).3: Assume that $|L/\theta| = n-3$. Since $|\mathrm{Con}(L/\theta)| > 7\cdot 2^{n-7} > 4\cdot 2^{n-7} = 2^{n-5} = 2^{(n-3)-2}$, by Theorem 5.5 it follows that $|\mathrm{Con}(L/\theta)| = 2^{n-4} = 2^{(n-3)-1}$ and hence $L/\theta \cong \mathcal{L}_{n-3}$. By Case (ii).3, it follows that we can not have $2^{n-3} > |\mathrm{Con}(L)| > 7\cdot 2^{n-6}$.

Therefore $|\mathrm{Con}(L)| \leq 7\cdot 2^{n-6}$.

Now assume that $|\mathrm{Con}(L)| = 7\cdot 2^{n-6}$, so that $|\mathrm{Con}(L/\theta)| \geq 7\cdot 2^{n-7} > 5\cdot 2^{n-7} > 4\cdot 2^{n-7} = 2^{n-5} = 2^{(n-4)-1}$ by Lemma 5.1, (ii), hence $|L/\theta| > n-4$ by Theorem 5.5, (i), thus $|L/\theta| \in \{n-1, n-2, n-3\}$.

Case 1: Assume that $|L/\theta| = n-1$. Since $|\mathrm{Con}(L/\theta)| \geq 7\cdot 2^{n-7} = 7\cdot 2^{(n-1)-6}$, by Theorem 5.5 and the induction hypothesis it follows that $|\mathrm{Con}(L/\theta)| \in \{2^{n-2}, 2^{n-3}, 5\cdot 2^{n-6}, 2^{n-4}, 7\cdot 2^{n-7}\}$. By Case (i).1, $|\mathrm{Con}(L/\theta)| \notin \{2^{n-2}, 2^{n-3}\}$.

Subcase 1.1: Assume that $|\mathrm{Con}(L/\theta)| = 5\cdot 2^{n-6}$. Since we also have $|\mathrm{Con}(L)| = 7\cdot 2^{n-6}$, by Subcase (i).1.1 it follows that, for some $k \in [1, n-3]$, $L/\theta \cong \mathcal{L}_k \dotplus N_5 \dotplus \mathcal{L}_{n-k-4} \cong \mathcal{L}_k \dotplus (\mathcal{L}_3 \boxplus \mathcal{L}_4) \dotplus \mathcal{L}_{n-k-4}$ and either $L \cong \mathcal{L}_k \dotplus (\mathcal{L}_3 \boxplus \mathcal{L}_5) \dotplus \mathcal{L}_{n-k-4}$ or $L \cong \mathcal{L}_k \dotplus (\mathcal{L}_4 \boxplus \mathcal{L}_4) \dotplus \mathcal{L}_{n-k-4}$.

Subcase 1.2: Assume that $|\mathrm{Con}(L/\theta)| = 7\cdot 2^{n-7}$, so that, by the induction hypothesis, for some $k \in [1, n-6]$, $L/\theta \cong \mathcal{L}_k \dotplus (\mathcal{L}_3 \boxplus \mathcal{L}_5) \dotplus \mathcal{L}_{n-k-5}$ or $L/\theta \cong \mathcal{L}_k \dotplus (\mathcal{L}_4 \boxplus \mathcal{L}_4) \dotplus \mathcal{L}_{n-k-5}$, thus, since $[a,b]_L$ is a narrows in this Case 1, we have one of the following situations:

- $L \cong \mathcal{L}_{k+1} \dotplus (\mathcal{L}_3 \boxplus \mathcal{L}_5) \dotplus \mathcal{L}_{n-k-5}$ or $L \cong \mathcal{L}_k \dotplus (\mathcal{L}_3 \boxplus \mathcal{L}_5) \dotplus \mathcal{L}_{n-k-4}$ or $L \cong \mathcal{L}_{k+1} \dotplus (\mathcal{L}_4 \boxplus \mathcal{L}_4) \dotplus \mathcal{L}_{n-k-5}$ or $L \cong \mathcal{L}_k \dotplus (\mathcal{L}_4 \boxplus \mathcal{L}_4) \dotplus \mathcal{L}_{n-k-4}$;

- $L \cong \mathcal{L}_k \dotplus (\mathcal{L}_3 \boxplus \mathcal{L}_6) \dotplus \mathcal{L}_{n-k-5}$ or $L \cong \mathcal{L}_k \dotplus (\mathcal{L}_4 \boxplus \mathcal{L}_5) \dotplus \mathcal{L}_{n-k-5}$, but in these cases $\mathrm{Con}(L) \cong \mathcal{L}_2^{n-7} \times (\mathcal{L}_2^3 \dotplus \mathcal{L}_2^2)$, so that $|\mathrm{Con}(L)| = 2^{n-7}\cdot(2^3+3) = 11\cdot 2^{n-7} < 14\cdot 2^{n-7} = 7\cdot 2^{n-6}$, which contradicts the current assumption that $|\mathrm{Con}(L)| = 7\cdot 2^{n-6}$.

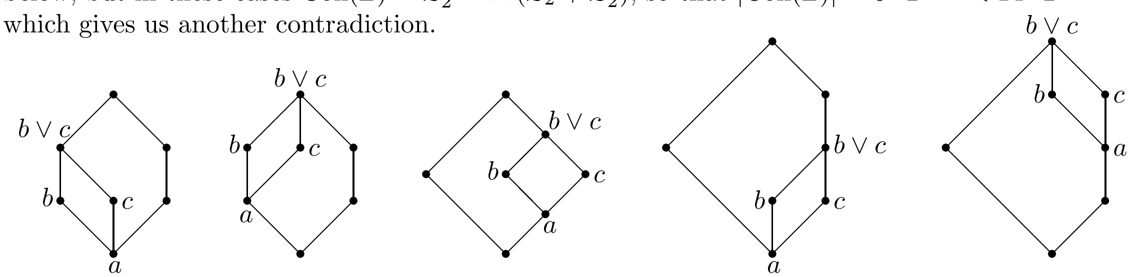
Case 2: Assume that $|L/\theta| = n-2$. Then, by Lemma 5.1, (iv), we can assume that a is meet–reducible, so that $a \prec c$ for some $c \in L \setminus \{b\}$, since the other case is dual to this one. Since $|\mathrm{Con}(L/\theta)| \geq 7\cdot 2^{n-7} > 5\cdot 2^{n-7} = 5\cdot 2^{(n-2)-5}$, by Theorem 5.5 and the induction hypothesis it follows that $|\mathrm{Con}(L/\theta)| \in \{2^{n-3}, 2^{n-4}, 5\cdot 2^{n-7}\}$. By Case (i).2, $|\mathrm{Con}(L/\theta)| \neq 2^{n-3}$. By Case (ii).2, since $|\mathrm{Con}(L)| = 7\cdot 2^{n-6}$, it follows that $|\mathrm{Con}(L/\theta)| \neq 2^{n-4}$. Hence $|\mathrm{Con}(L/\theta)| = 5\cdot 2^{n-7}$, so that, by the induction hypothesis, for some $k \in [1, n-6]$, $L/\theta \cong \mathcal{L}_k \dotplus N_5 \dotplus \mathcal{L}_{n-k-5}$, so that L is in one of the following situations, as shown by Lemma 5.1, (iv):

- either $k \geq 4$ and, for some $r, s \in \mathbb{N}^*$ such that $r+s+2 \leq k$, $L \cong \mathcal{L}_r \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_s \dotplus N_5 \dotplus \mathcal{L}_{n-k-5}$, or $n \geq k+9$ and, for some $r, s \in \mathbb{N}^*$ such that $r+s+2 \leq n-k-5$, $L \cong \mathcal{L}_k \dotplus N_5 \dotplus \mathcal{L}_r \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_s$, but in these cases $\mathrm{Con}(L) \cong \mathcal{L}_2^{n-6} \times (\mathcal{L}_2 \dotplus \mathcal{L}_2^2)$, so that $|\mathrm{Con}(L)| = 5\cdot 2^2\cdot 2^{n-8} = 5\cdot 2^{n-6} < 7\cdot 2^{n-6}$, which contradicts the current assumption that $|\mathrm{Con}(L)| = 7\cdot 2^{n-6}$;

- $L \cong \mathcal{L}_k \dotplus ((\mathcal{L}_2^2 \dotplus \mathcal{L}_2) \boxplus \mathcal{L}_4) \dotplus \mathcal{L}_{n-k-5}$ or $L \cong \mathcal{L}_k \dotplus ((\mathcal{L}_2 \dotplus \mathcal{L}_2^2) \boxplus \mathcal{L}_4) \dotplus \mathcal{L}_{n-k-5}$, with the positions of a, b, c and $b \vee c$ in the copy of $(\mathcal{L}_2^2 \dotplus \mathcal{L}_2) \boxplus \mathcal{L}_4$, respectively $(\mathcal{L}_2 \dotplus \mathcal{L}_2^2) \boxplus \mathcal{L}_4$ from L as depicted in the two leftmost diagrams below, but in these cases $\mathrm{Con}(L) \cong \mathcal{L}_4 \times \mathcal{L}_2^{n-7}$, so that $|\mathrm{Con}(L)| = 2^{n-5} = 2\cdot 2^{n-6} < 7\cdot 2^{n-6}$, which gives us another contradiction to the current assumption;

- $L \cong \mathcal{L}_k \dotplus (\mathcal{L}_3 \boxplus (\mathcal{L}_2 \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_2)) \dotplus \mathcal{L}_{n-k-5}$, with the positions of a, b, c and $b \vee c$ in the copy of $\mathcal{L}_3 \boxplus (\mathcal{L}_2 \dotplus \mathcal{L}_2^2 \dotplus \mathcal{L}_2)$ from L as depicted in the center diagram below, but in this case $\mathrm{Con}(L) \cong (\mathcal{L}_2^2 \dotplus \mathcal{L}_2^2) \times \mathcal{L}_2^{n-7}$, so that $|\mathrm{Con}(L)| = 7\cdot 2^{n-7} < 7\cdot 2^{n-6}$, and we obtain a contradiction again;

- $L \cong \mathcal{L}_k \dotplus (\mathcal{L}_3 \boxplus (\mathcal{L}_2^2 \dotplus \mathcal{L}_3)) \dotplus \mathcal{L}_{n-k-5}$ or $L \cong \mathcal{L}_k \dotplus (\mathcal{L}_3 \boxplus (\mathcal{L}_3 \dotplus \mathcal{L}_2^2)) \dotplus \mathcal{L}_{n-k-5}$, with the positions of a, b, c and $b \vee c$ in the copy of $\mathcal{L}_3 \boxplus (\mathcal{L}_2^2 \dotplus \mathcal{L}_3)$, respectively $\mathcal{L}_3 \boxplus (\mathcal{L}_3 \dotplus \mathcal{L}_2^2)$ from L as depicted in the two rightmost diagrams below, but in these cases $\mathrm{Con}(L) \cong \mathcal{L}_2^{n-7} \times (\mathcal{L}_2 \dotplus \mathcal{L}_2^2)$, so that $|\mathrm{Con}(L)| = 5\cdot 2^{n-7} < 14\cdot 2^{n-7} = 7\cdot 2^{n-6}$, which gives us another contradiction.



Case 3: Assume that $|L/\theta| = n-3$. Since $|\mathrm{Con}(L/\theta)| \geq 7\cdot 2^{n-7} > 4\cdot 2^{n-7} = 2^{n-5} = 2^{(n-3)-2}$, Theorem 5.5 ensures us that $|\mathrm{Con}(L/\theta)| = 2^{n-4} = 2^{(n-3)-1}$, so that $L/\theta \cong \mathcal{L}_{n-3}$. But Case (iii).3 shows us that, in this case, $|\mathrm{Con}(L)| \neq 7\cdot 2^{n-6}$, so we have a contradiction to the current assumption. □

Corollary 5.7. *(i)* $|\mathrm{Con}(L)| = 2^{n-1}$ *iff* $\mathrm{Con}(L) \cong \mathcal{L}_2^{n-1}$.

(ii) $|\mathrm{Con}(L)| = 2^{n-2}$ *iff* $n \geq 4$ *and* $\mathrm{Con}(L) \cong \mathcal{L}_2^{n-2}$.

(iii) $|\mathrm{Con}(L)| = 5\cdot 2^{n-5}$ *iff* $n \geq 5$ *and* $\mathrm{Con}(L) \cong \mathcal{L}_2^{n-5} \times (\mathcal{L}_2 \dotplus \mathcal{L}_2^2)$.

(iv) $|\mathrm{Con}(L)| = 2^{n-3}$ *iff* $n \geq 6$ *and* $\mathrm{Con}(L) \cong \mathcal{L}_2^{n-3}$.

(v) $|\mathrm{Con}(L)| = 7\cdot 2^{n-6}$ *iff* $n \geq 6$ *and* $\mathrm{Con}(L) \cong \mathcal{L}_2^{n-6} \times (\mathcal{L}_2^2 \dotplus \mathcal{L}_2^2)$.

Proof. The converse implications are trivial, and the direct implications follow from Theorems 5.5 and 5.6. □

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