On the convergence of solutions of functional differential equations

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1. Introduction

The application of Lyapunov functions and functionals has proved to be useful in the study of the stability of solutions of functional differential equations. Such investigations were initiated by N. N. KRASOVSKIĬ [9] and B. S. RAZUMIKHIN [10]. The Lyapunov functions and functionals are usable for studying other properties, too. For instance, S. R. BEINFELD and J. R. HADDOCK ([1], [2], [4]) examined the existence of the limit of solutions as $t \to \infty$ by the aid of Lyapunov functions. But their method was not applicable when the right-hand side of the equation is the sum of an ordinary and a functional part of the same order. But such equations have occurred in the applications, for example in the investigation of biological populations [3]. In this case the problem was solved for certain autonomous and periodic equations only [5], [6]. In this paper we give a sufficient condition for the existence of the limit of solutions in case of non-periodic equations. Our main result guarantees the existence of the limit of a Lyapunov function along the solutions as $t \to \infty$. We present several applications in which we show that the solutions or their norm tend to a constant as $t \to \infty$. Among these, we study a stability example of N. N. KRASOVSKIĬ proving that his assumptions imply the existence of the limit of solutions in addition to the stability of the zero solution.

The main theorems are valid results for functional differential equations in any Banach space $X$. But they also yield new results for the special case $X = \mathbb{R}$ (Section 4).

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2. Notations and definitions

Let $R$ be the set of real numbers and $R^+$ the set of nonnegative real numbers. Let $X$ be a Banach space with norm $|\cdot|$ and let $C=C([-r, 0], X)$ denote the space of continuous functions which map the interval $[-r, 0]$ into $X$, where $r>0$. For $\varphi\in C$ define $\|\varphi\| = \max_{-r \leq s \leq 0} |\varphi(s)|$. If $x: [t_0-r, t_0+A) \rightarrow X$ is a continuous function $(t_0 \in R^+, 0 < A \equiv \infty)$, then for $t \in [t_0, t_0+A)$ the function $x_t \in C$ is defined by $x_t(s) = x(t+s)$, $-r \leq s \leq 0$.

We consider the nonlinear, non-autonomous functional differential equation

$$(2.1) \quad \dot{x}(t) = F(t, x_t),$$

where $F: R^+ \times C_F \rightarrow X$, $C_F \subseteq C$.

Let $t_0 \in R^+$ and $\varphi_0 \in C_F$ be given. A function $x(\cdot) = x(t_0, \varphi_0)(\cdot)$ is said to be a solution of (2.1) (with the initial function $\varphi_0$ at $t_0$) if there exists a number $A (0 < A \equiv \infty)$ such that $x(\cdot)$ is defined and continuous on $[t_0-r, t_0+A)$, absolutely continuous on the bounded intervals of $[t_0, t_0+A)$, $x_{t_0} = \varphi_0$, $x_t \in C_F$ for $t \in (t_0, t_0+A)$ and $\dot{x}(t) = F(t, x_t)$ almost everywhere on $[t_0, t_0+A)$. In this paper we suppose $A = \infty$, i.e. the solutions of (2.1) exist for $t \equiv t_0$ (see, for example, [7], [8]).

By a Lyapunov function we mean a continuous function $V: [-r, \infty) \times X \rightarrow R$. The upper right-hand derivative $D^+_{(a, t)}V$ of a Lyapunov function $V$ with respect to system (2.1) is defined by

$$D^+_{(a, t)}V(t, \varphi) = \lim_{h \to 0^+} \frac{1}{h} \left[ V(t+h, \varphi(0) + hF(t, \varphi)) - V(t, \varphi(0)) \right] \quad ((t, \varphi) \in R^+ \times C_F).$$

If $V$ is a Lyapunov function and $(t, \varphi) \in R^+ \times C_F$, then let

$$\overline{V}(t, \varphi) = \sup_{-r \leq s \leq 0} V(t+s, \varphi(s)), \quad \underline{V}(t, \varphi) = \inf_{-r \leq s \leq 0} V(t+s, \varphi(s)).$$

Finally, for a Lyapunov function $V$ and given numbers $0 < \eta \equiv \varepsilon$ define

$$\mathcal{S}(V, \eta, \varepsilon) = \{(t, \varphi) \in R^+ \times C_F: V(t, \varphi(0)) \equiv \varepsilon, \overline{V}(t, \varphi) \equiv 2\varepsilon, \underline{V}(t, \varphi) - V(t, \varphi(0)) \leq \eta\},$$

$$\mathcal{S}(V, \eta, \varepsilon) = \{(t, \varphi) \in R^+ \times C_F: V(t, \varphi(0)) \equiv -\varepsilon, \overline{V}(t, \varphi) - V(t, \varphi(0)) \leq \eta\}.$$

3. The main result

The main result guarantees the existence of the limit of a Lyapunov function along the solutions of (2.1) as $t \rightarrow \infty$.

Theorem 3.1. Suppose that for a nonnegative Lyapunov function $V$ there exists a functional $W: R^+ \times C_F \rightarrow R$ with the following property: for every $\varepsilon > 0$ there exist $\eta = \eta(\varepsilon) > 0$ and $\zeta = \zeta(\varepsilon) > 0$ such that
(i) if \((t, \varphi) \in \mathcal{S}(V, \eta, \varepsilon)\), then

\[
W(t, \varphi) \equiv \xi [V(t, \varphi) - V(t, \varphi(0))],
\]

(ii) if \(x(\cdot)\) is a solution of (2.1) and \((t, x_0) \in \mathcal{S}(V, \eta, \varepsilon)\) for every \(t \in [t_1, t_2]\) \((t_0 \leq t_1 \leq t_2)\), then

\[
V(t_2, x(t_2)) - V(t_1, x(t_1)) \leq \int_{t_1}^{t_2} W(t, x_0) \, dt.
\]

Then for each solution \(x(\cdot)\) of (2.1) \(\lim_{t \to \infty} V(t, x(t))\) exists.

We first prove the following lemma.

Lemma. If the conditions of Theorem 3.1 are satisfied, then for each solution \(x(\cdot)\) of (2.1) the function \(V(\cdot, x)\) is non-increasing.

Proof. Assume that (2.1) has a solution \(x(\cdot)\) such that \(V(\cdot, x)\) is not a non-increasing function. Then there exists a \(t_1 \equiv t_0\) and in any right-hand side neighbourhood of \(t_1\) there exists a \(t\) such that \(V(t, x_0) = V(t_1, x_1) = V(t_1, x(t_2)) > 0\). Let \(\varepsilon\) be chosen such that \(0 < \varepsilon < V(t_1, x(t_2)) - 2\varepsilon\) and choose \(\eta = \eta(\varepsilon), \xi = \xi(\varepsilon)\) according to assumptions of the lemma. Obviously there exist \(t_2, t_3\) such that \(t_3 > t_2 \equiv t_1, t_3 - t_2 < \frac{1}{\xi}\), \(V(t_3, x(t_3)) = V(t_1, x(t_1)) < V(t_3, x(t_3)) \equiv 2\varepsilon, V(t_3, x(t_3)) - V(t_2, x(t_2)) < \eta\) and if \(t \in [t_2, t_3]\), then \(V(t, x(t)) \equiv V(t, x(t))\) and \(V(t, x(t)) \equiv V(t_3, x(t_3))\). For such \(t_2, t_3\) we have \((t, x_0) \in \mathcal{S}(V, \eta, \varepsilon)\) provided \(t \in [t_2, t_3]\). Also

\[
V(t, x(t)) - V(t, x(t)) \equiv V(t_2, x(t_2)) - V(t_2, x(t_2)) \quad (t \in [t_2, t_3]).
\]

It follows from (3.1), (3.2) and (3.3) that

\[
V(t_3, x(t_3)) - V(t_2, x(t_2)) \equiv \int_{t_2}^{t_3} W(t, x_0) \, dt \equiv
\]

\[
\int_{t_2}^{t_3} \xi [V(t, x(t)) - V(t, x(t))] \, dt \equiv \int_{t_2}^{t_3} \xi [V(t_3, x(t_3)) - V(t_2, x(t_2))] \, dt =
\]

\[
(t_3 - t_2) \cdot \xi [V(t_3, x(t_3)) - V(t_2, x(t_2))].
\]

Hence \(t_3 - t_2 \equiv \frac{1}{\xi}\). This is a contradiction. The lemma is proved.

Proof of Theorem 3.1. Suppose that (2.1) has a solution \(x(\cdot)\) such that the limit \(\lim_{t \to \infty} V(t, x(t))\) does not exist. Then \(\lim_{t \to \infty} V(t, x(t)) = \alpha > 0\) (this limit exists by Lemma). Let \(\varepsilon\) be chosen so that \(0 < \varepsilon < \alpha < 2\varepsilon\) and choose \(\eta = \eta(\varepsilon)\) and \(\xi = \xi(\varepsilon)\) according to the assumptions of the theorem. Then we can find a constant \(\beta\)
\(0 < \beta < \min \left\{ \frac{\eta}{3}, \frac{\alpha - \eta}{3} \right\} \) and numbers \( t_1, t_2 \) such that
\[
(3.4) \quad t_2 > t_1 \equiv t_0, \quad t_2 - t_1 \equiv r,
\]
\[
(3.5) \quad V(t_2, x(t_2)) - V(t_1, x(t_1)) = \beta,
\]
\[
(3.6) \quad V(t_1, x(t_1)) - \alpha \leq \frac{\eta}{3}, \quad |\alpha - V(t_2, x(t_2))| \leq \frac{\eta}{3},
\]
\[
0 \leq V(t_2, x(t_2)) - V(t, x(t)) \equiv \beta \quad (t \in [t_1, t_2]),
\]
\[
(3.7) \quad V(t_1, x(t_1)) \equiv \epsilon, \quad V(t_1, x_\epsilon) \equiv 2\epsilon,
\]
\[
(3.8) \quad V(t_1, x_\epsilon) - V(t_2, x(t_2)) \equiv \gamma,
\]
where \( \gamma > 0 \) and
\[
(3.9) \quad \frac{1}{\xi} \sum_{k=1}^{[\beta/\gamma]} \frac{1}{k+1} > r.
\]

From (3.8) and the monotonicity of \( V(\cdot, x_\cdot) \) it follows that
\[
(3.10) \quad V(t, x_r) - V(t_2, x(t_2)) \equiv \gamma \quad (t \in [t_1, t_2]).
\]

Since \( \beta < \frac{\eta}{3} \), by (3.6) we have
\[
(3.11) \quad V(t, x_r) - V(t, x(t)) \equiv V(t, x_r) - \alpha + |\alpha - V(t_2, x(t_2))| + V(t_2, x(t_2)) - V(t, x(t)) <
\]
\[
\leq \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta \quad (t \in [t_1, t_2]).
\]

From (3.5), (3.6), (3.7) and (3.11) we obtain \( (t, x_t) \in S(V, \eta, \epsilon) \) for \( t \in [t_1, t_2] \). Thus, (3.1) and (3.2) hold for \( (t, x_t) \) as \( t \in [t_1, t_2] \). Let \( \tau_k \in [t_1, t_2] \) be the greatest number for which
\[
(3.12) \quad V(t_2, x(t_2)) - V(\tau_k, x(\tau_k)) = k\gamma \quad \left( k = 1, 2, \ldots, \left[ \frac{\beta}{\gamma} \right] \right).
\]

From (3.10) and the choice of \( \tau_k \) it follows that
\[
(3.13) \quad V(t, x_r) - V(t, x(t)) \equiv (k+1)\gamma \quad \left( t \in [\tau_k, t_2]; k = 1, 2, \ldots, \left[ \frac{\beta}{\gamma} \right] \right).
\]

By (3.1), (3.2), (3.12) and (3.13) we have
\[
\gamma = V(\tau_k, x(\tau_k)) - V(\tau_k, x(\tau_k)) \equiv \int_{\tau_k}^{\tau_k} W(t, x_t) dt \equiv
\]
\[
\equiv \int_{\tau_k}^{\tau_k} \xi[V(t_2, x(t_2)) - V(t, x(t))] dt \equiv \int_{\tau_k}^{\tau_k} \xi(k+1)\gamma dt =
\]
\[
= (\tau_k - \tau_k)\xi(k+1)\gamma \quad \left( k = 1, 2, \ldots, \left[ \frac{\beta}{\gamma} \right]; \tau_0 = t_2 \right).
\]
Hence $\tau_{k-1} - \tau_k \leq \frac{1}{\xi} \frac{1}{k+1}$. Thus, by (3.9)

$$t_2 - t_1 \leq t_2 - \tau_{k+1} + (\tau_1 - \tau_2) + \cdots + (\tau_{k} - \tau_{k+1}) \leq \frac{1}{\xi} \sum_{k=1}^{r} \frac{1}{k+1} > r,$$

which contradicts (3.4). This completes the proof.

**Remark 3.1.** If the Lyapunov function $V$ in Theorem 3.1 is locally Lipschitzian and $W = D_{(\xi, k)}^+ V$, then for each solution $x(\cdot)$ of (2.1) the assumption (3.2) is satisfied and even

$$V(t_2, x(t_2)) - V(t_1, x(t_1)) = \int_{t_1}^{t_2} D_{(\xi, k)}^+ V(t, x(t))\, dt \quad (t_0 \leq t_1 \leq t_2).$$

This can be shown as follows. If $\dot{x}(t) = F(t, x(t))$, then $x(t+h) = x(t) + hF(t, x(t)) + o(h)$ $(h \to 0+)$. From the Lipschitz condition for $V$ we obtain

$$V(t+h, x(t+h)) - V(t, x(t)) \equiv V(t+h, x(t) + hF(t, x(t))) + L|o(h)| - V(t, x(t))$$

$$(h \to 0+),$$

where $L$ is the Lipschitz constant for $V$ on a neighbourhood of $(t, x(t))$. Hence $D^+ V(t, x(t)) \equiv D_{(\xi, k)}^+ V(t, x(t))$, where $D^+ V(t, x(t))$ is the upper right-hand derivative of $V$ along the solution $x(t)$ of (2.1), that is

$$D^+ V(t, x(t)) = \lim_{h \to 0+} \frac{1}{h} [V(t+h, x(t+h)) - V(t, x(t))].$$

Likewise we can prove $D^+ V(t, x(t)) \equiv D_{(\xi, k)}^+ V(t, x(t))$ and we obtain

(3.14) \hspace{1cm} D^+ V(t, x(t)) = D_{(\xi, k)}^+ V(t, x(t)).

(3.14) was proved by T. YOSHIZAWA [11] for ordinary differential equations in the case $X = R^m$. Since $V$ is locally Lipschitzian, $V(\cdot, x(\cdot))$ is absolutely continuous on every bounded interval of $[t_0, \infty)$ and thus

(3.15) \hspace{1cm} V(t_2, x(t_2)) - V(t_1, x(t_1)) = \int_{t_1}^{t_2} D^+ V(t, x(t))\, dt \quad (t_0 \leq t_1 \leq t_2).

From (3.14) and (3.15) it follows that our statement holds.

**Corollary 3.1.** If for every $\varepsilon > 0$ there exists an $\eta = \eta(\varepsilon) > 0$ such that $(t, \varphi) \in S((\varphi(0)), \eta, \varepsilon)$ implies $D_{(\xi, k)}^+|\varphi| \leq 0$, then for each solution $x(\cdot)$ of (2.1)

$$\lim_{t \to \infty} |x(t)| \text{ exists.}$$

**Proof.** We apply Theorem 3.1. Let $V(t, x) = |x|$ and $W(t, \varphi) = D_{(\xi, k)}^+|\varphi|$. Since the condition of Corollary 3.1 is stronger than condition (i) of Theorem 3.1 and the
function $V$ is locally Lipschitzian, from Remark 3.1 it is obvious that the limit exists.

Corollary 3.1 is due to J. R. HADDOCK [4].

In the next corollary of Theorem 3.1 we do not assume that the Ljapunov function is nonnegative.

**Corollary 3.2.** Suppose that for a Ljapunov function $V$ there exist functionals $W_1, W_2: R^+ \times C_{\mathbb{R}} \to R$ with the following property: for every $\varepsilon > 0$ there exist $\eta = \eta(\varepsilon) > 0$ and $\xi = \xi(\varepsilon) > 0$ such that

(i) $(t, \varphi) \in S(V, \eta, \varepsilon)$ implies $W_1(t, \varphi) \leq \xi [V(t, \varphi) - V(t, \varphi(0))]$,

(ii) $(t, \varphi) \in S(V, \eta, \varepsilon)$ implies $W_2(t, \varphi) \leq \xi [V(t, \varphi(0)) - V(t, \varphi)]$,

(iii) if $(t, x_t) \in S(V, \eta, \varepsilon) \ (t \in [t_1, t_2], \ t_0 \leq t_1 \leq t_2)$, then

$$V(t_2, x(t_2)) - V(t_1, x(t_1)) \leq \int_{t_1}^{t_2} W_1(t, x(t)) \, dt,$$

(iv) if $(t, x_t) \in S(V, \eta, \varepsilon) \ (t \in [t_1, t_2], \ t_0 \leq t_1 \leq t_2)$, then

$$V(t_1, x(t_1)) - V(t_2, x(t_2)) \leq \int_{t_1}^{t_2} W_2(t, x(t)) \, dt,$$

where $x(\cdot)$ is a solution of (2.1).

Then for each solution $x(\cdot)$ of (2.1) $\lim_{t \to \infty} V(t, x(t))$ exists.

**Proof.** Let $V_1(t, x) = \max \{V(t, x, 0), \ V_2(t, x) = \min \{V(t, x, 0). \}$ From conditions (i), (ii), (iii), (iv) of Corollary 3.2 it follows that $V_1, W_1$ and $V_2, W_2$ satisfy conditions (i), (ii) of Theorem 3.1. This implies that for every solution $x(\cdot)$ of (2.1) the limits $\lim_{t \to \infty} V_1(t, x(t))$ and $\lim_{t \to \infty} V_2(t, x(t))$ exist. Thus the corollary is proved.

**4. Applications and examples**

I. Consider the equation

$$\dot{x}(t) = f(t, x(t)) + g(t, x), \quad (4.1)$$

where $f: R^+ \times X_f \to X$, $X_f \subset X$, $g: R^+ \times C_{\mathbb{R}} \to X$, $C_{\mathbb{R}} \subset C$. (4.1) is the special case of the equation (2.1), when

$$F(t, \varphi) = f(t, \varphi(0)) + g(t, \varphi).$$

**Theorem 4.1.** Suppose that for a nonnegative, locally Lipschitzian Ljapunov function $V$ there exist functions $\alpha, p: R^+ \to R^+$ with the following properties:
(i) $\alpha(t)$ is bounded for $t \geq t_0$, 
(ii) the function $p$ is locally Lipschitzian on $(0, \infty)$, 
(iii) $V(t, x+y) \leq V(t, x) + V(t, y)$ for all $(t, x), (t, y) \in R^+ \times X_f$, 
(iv) $\lim_{h \to 0^+} \frac{1}{h} \left[ V(t+h, x+hf(t, x)) - V(t, x) \right] \leq \alpha(t)p(t, x)$ 
for all $(t, x) \in R^+ \times X_f$, 
(v) for every $\varepsilon > 0$ there exists an $\eta = \eta(\varepsilon) > 0$ such that $(t, \phi) \in S(V, \eta, \varepsilon)$ implies $\lim_{h \to 0^+} \frac{1}{h} V(t+h, h\phi(t, \phi)) \leq \alpha(t)p(t, \phi)$. 

Then for each solution $x(\cdot)$ of (4.1) $\lim_{t \to \infty} V(t, x(t))$ exists.

Proof. We apply Theorem 3.1. Let $W = D_{(4.1)}^+ V$. By Remark 3.1 it is sufficient to prove that condition (i) of Theorem 3.1 is satisfied. Let $\varepsilon > 0$ be given and choose $\eta = \eta(\varepsilon)$ according to assumption (v). If $(t, \phi) \in S(V, \eta, \varepsilon)$, then from conditions (i)–(v) we obtain

$$D_{(4.1)}^+ V(t, \phi) = \lim_{h \to 0^+} \frac{1}{h} \left[ V(t+h, \phi(t, \phi)) + hf(t, \phi(t, \phi)) - V(t, \phi(t, \phi)) \right] \leq$$

$$\leq \lim_{h \to 0^+} \frac{1}{h} \left[ V(t+h, \phi(t, \phi)) - V(t, \phi(t, \phi)) \right] + \lim_{h \to 0^+} \frac{1}{h} V(t+h, h\phi(t, \phi)) \leq$$

$$\leq \alpha(t) [p(V(t, \phi)) - p(V(t, \phi(t, \phi)))] \leq K \|V(t, \phi) - V(t, \phi(t, \phi))\| =$$

$$= \varepsilon \|V(t, \phi) - V(t, \phi(t, \phi))\|,$$

where $L$ is the Lipschitz constant of $p$ on $[\varepsilon, 2\varepsilon]$ and $K$ is an upper bound for $\alpha$ on $[t_0, \infty)$. This completes the proof.

II. We now apply Theorem 4.1 to obtain a result for equation (4.1) in the case $X = R$.

Theorem 4.2. Let $X = R$. If $f(t, 0) \equiv 0$, $xf(t, x) \equiv -\alpha(t)x^2$ for all $(t, x) \in R^+ \times X_f$, $|g(t, \phi)| \leq \alpha(t) \|\phi\|$ for all $(t, \phi) \in R^+ \times C_\theta$ and $a(t)$ is bounded for $t \geq t_0$, then for each solution $x(\cdot)$ of (4.1) $\lim_{t \to \infty} x(t)$ exists.

Proof. In Theorem 4.1, let $V(t, x) = |x|$, $\alpha(t) = a(t)$, $p(t) = 1$, $\eta(\varepsilon) = \varepsilon$. Thus, $V$ is a Lipschitzian function and conditions (i), (ii), (iii) and in the case $x = 0$ condition (iv) in Theorem 4.1 are obviously satisfied. We have

$$\lim_{h \to 0^+} \frac{1}{h} (|x+hf(t, x)| - |x|) \leq \lim_{h \to 0^+} \frac{1}{h} |x| \left(1 + h \frac{f(t, x)}{x} - 1\right) \leq -a(t) |x|,$$

if $x \neq 0$ and

$$\lim_{h \to 0^+} \frac{1}{h} |hg(t, \phi)| = |g(t, \phi)| \leq a(t) \|\phi\|.$$
that assures also conditions (iv), (v) in Theorem 4.1 to be satisfied. This completes the proof.

**Example 4.1.** Let us consider the scalar equation

\[ \dot{x}(t) = -ax(t) + b(t)x(t - \tau(t)), \]

where \( a > 0, \ b(t) \) and \( \tau(t) \) are continuous for \( t \geq t_0, \ |b(t)| \equiv a, \ 0 \leq \tau(t) \leq r. \)

For equation (4.2) in this case N. N. KRASOVSKII [9] proved that the zero solution is uniformly stable. Applying Theorem 4.2 we obtain that \( x(t) \) tends to a constant as \( t \to \infty, \) where \( x(\cdot) \) is a solution of (4.2).

III. Let us consider the following special form of equation (4.1):

\[ \dot{x}(t) = -a(t)x(t) + \sum_{k=1}^{n} b_k(t)x(t - \tau_k(t)), \]

where \( a, b_k, \tau_k : R^+ \to R \) are continuous functions and \( 0 \leq \tau_k(t) \leq r \) \((k = 1, 2, \ldots, n)\):

**Theorem 4.3.** Let \( k : [-r, \infty) \to (0, \infty) \) be a continuous and locally Lipschitzian function. If there exists a \( K \in R^+ \) such that

\[ k(t) \sum_{k=1}^{n} \frac{|b_k(t)|}{k(t - \tau_k(t))} \leq a(t) - \frac{D^+ k(t)}{k(t)} \leq K \quad (t \in R^+), \]

then for each solution \( x(\cdot) \) of (4.3) \( \lim_{t \to \infty} |k(t)x(t)| \) exists.

**Proof.** Apply Theorem 4.1 setting \( V(t, x) = |k(t)x|, \ \alpha(t) = a(t) - \frac{D^+ k(t)}{k(t)}, \)

\( p(u) = 1, \eta(\varepsilon) = \varepsilon. \) It is clear that conditions (i), (ii), (iii) in Theorem 4.1 are satisfied. Using (4.4) we can check conditions (iv), (v) in Theorem 4.1 as follows

\[ \lim_{h \to 0^+} \frac{1}{h} \left( |k(t+h)(x-ha(t)x)| - |k(t)x| \right) = |k(t)x| \lim_{h \to 0^+} \frac{1}{h} \frac{k(t+h)(1-ha(t))-k(t)}{k(t)} = \]

\[ = |k(t)x| \left( \frac{D^+ k(t)}{k(t)} - a(t) \right) = -\alpha(t)V(t, x), \]

\[ \lim_{h \to 0^+} \frac{1}{h} \left| k(t+h) \sum_{k=1}^{n} b_k(t)x(t - \tau_k(t)) \right| = k(t) \left| \sum_{k=1}^{n} b_k(t)x(t - \tau_k(t)) \right| \leq \]

\[ \leq k(t) \sum_{k=1}^{n} \frac{|b_k(t)|}{k(t - \tau_k(t))} V(t, x_i) \leq \left( a(t) - \frac{D^+ k(t)}{k(t)} \right) V(t, x_i) = \alpha(t)V(t, x_i). \]

This completes the proof.
Remark 4.1. If for equation (4.3) the inequality \( \sum_{k=1}^{n} |b_k(t)| \leq a(t) \leq K \) holds, then \( k\) satisfies (4.4) and by Theorem 4.3 \( \lim_{t \to \infty} |x(t)| \) exists. But Theorem 4.3 can be used even if this inequality does not hold, as the following example shows.

Example 4.2. Let us consider the equation

\[
\dot{x}(t) = -a(t)x(t) + b(t)x(t - \tau(t)),
\]

where \( a(t), b(t) \) and \( \tau(t) \) are continuous for \( t \equiv t_0, 0 \equiv \tau(t) \equiv r \) and there exists a \( K \in \mathbb{R}^+ \) such that \( a(t) \equiv |b(t)| \equiv K \) for \( t \equiv t_0 \). Let \( k(t) = \exp \left( \int_0^t (a(s) - |b(s)|) ds \right) \).

We have

\[
\frac{D^+ k(t)}{k(t)} = a(t) - |b(t)|,
\]

\[
\exp \left( \int_0^t (a(s) - |b(s)|) ds \right) \frac{|b(t)|}{\exp \left( \int_0^t (a(s) - |b(s)|) ds \right)} \equiv |b(t)| \equiv K.
\]

Thus, from Theorem 4.3 it follows that for each solution \( x(\cdot) \) of (4.5)

\[
\lim_{t \to \infty} |x(t)\exp \left( \int_0^t (a(s) - |b(s)|) ds \right)|
\]

exists.

IV. Let us consider the equation

\[
\dot{x}(t) = -h(x(t)) + h(x(t - \tau(t))),
\]

where \( \tau(t) \) is continuous for \( t \equiv t_0, 0 \equiv \tau(t) \equiv r \) and \( h(s) \) is continuous for \( s \in \mathbb{R} \).

Theorem 4.4. If the function \( h \) is increasing and locally Lipschitzian on \((\infty, 0)\) and \((0, \infty)\), then for each solution \( x(\cdot) \) of (4.6) \( \lim_{t \to \infty} x(t) \) exists.

Proof. Apply Corollary 3.2 setting \( V(t, x) = x, \ W_1(t, \varphi) = -W_2(t, \varphi) = -D^+_{t_0 \varphi} V(t, \varphi) \). Let \( \varepsilon > 0 \) be given, \( \eta(\varepsilon) = \varepsilon \) and \( \xi(\varepsilon) = \max \{L_1, L_2\} \), where \( L_1, L_2 \) are the Lipschitz constants of \( h \) on \([\varepsilon, 2\varepsilon], [-2\varepsilon, -\varepsilon] \), respectively. Since \( W_1(t, x_i) = -W_2(t, x_i) = \dot{x}(t) \) it is obvious that conditions (iii), (iv) in Corollary 3.2 are satisfied. If \( (t, x) \in S(x(t), \eta, \varepsilon) \) then

\[
\dot{x}(t) = -h(x(t)) + h(x(t - \tau(t))) \leq -h(x(t)) + h(x_i) \leq \xi(x_i - x(t)).
\]

If \( (t, x) \in S(x(t), \eta, \varepsilon) \), then

\[
-\dot{x}(t) = h(x(t)) - h(x(t - \tau(t))) \leq h(x(t)) - h(x_i) \leq \xi(x(t) - x_i).
\]

Thus conditions (i), (ii) in Corollary 3.2 are satisfied and the theorem is proved.
Remark 4.2. Applying Theorem 4.4 to case \( h(u)=u^{1/3} \), \( \tau(t) \equiv r \) we get a new proof for the following conjecture of S. R. Bernfeld and J. R. Haddock [1], which was solved by C. Ježu [5]: each solution of the scalar equation \( \dot{x}(t) = -x^{1/3}(t) + x^{1/3}(t-r) \) tends to a constant as \( t \to \infty \).

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References

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