# On the Breiman conjecture

## Péter Kevei<sup>1</sup> David Mason<sup>2</sup>

<sup>1</sup>TU Munich

<sup>2</sup>University of Delaware

12th GPSD Bochum

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## Breiman 1965

Coin tossing  $\longrightarrow$  random walk  $S_1, S_2, \dots$ Put  $Y_1, Y_2, \dots$  the interarrival times between the zeros of  $S_1, S_2, \dots$  $X, X_1, X_2, \dots$  iid  $\mathbf{P}\{X = 0\} = \frac{1}{2} = \mathbf{P}\{X = 1\}.$  $T_n = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i}$ 

is the proportion of the time that the random walk spends in  $[0,\infty)$ .

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## Breiman 1965

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Arc-sine law

In this case:

$$\lim_{n\to\infty} \mathbf{P}\left\{T_n \le x\right\} = \frac{2}{\pi} \arcsin\sqrt{x}$$

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## In general

# *Y*, *Y*<sub>1</sub>, *Y*<sub>2</sub>,... non-negative iid rv's with df *G* $X, X_1, X_2, \ldots$ iid with df *F*, independent from *Y*, *Y*<sub>1</sub>, *Y*<sub>2</sub>,..., $E|X| < \infty$

$$T_n = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i}$$

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## Remark

### If $\mathbf{E}Y < \infty$ , then

$$\frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} Y_i} = \frac{\frac{\sum_{i=1}^{n} X_i Y_i}{n}}{\frac{\sum_{i=1}^{n} Y_i}{n}} \xrightarrow{\text{a.s.}} \mathbf{E} X.$$

 $\mathbf{E}|X| < \infty$  implies  $(T_n)$  is tight.

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## Theorem (Breiman, 1965)

If  $T_n$  converges in distribution for every F, and the limit is non-degenerate for at least one F, then  $Y \in D(\beta)$ , for some  $\beta \in [0, 1)$ .

## Conjecture (Breiman)

If  $T_n$  has a non-degenerate limit for some F, then  $Y \in D(\beta)$  for some  $\beta \in [0, 1)$ .

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Breiman, L.

On some limit theorems similar to the arc-sin law *Teor. Verojatnost. i Primenen.* **10** 351–360, 1965.

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 $D(\beta)$ 

Domain of attraction of an  $\beta$ -stable law:

$$Y \in D(\beta) \Leftrightarrow 1 - G(x) = rac{\ell(x)}{x^{\beta}},$$

where  $\ell$  is slowly varying  $(\ell(\lambda x)/\ell(x) \to 1 \text{ for any } \lambda > 0 \text{ as } x \to \infty)$ .

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D(0)

### $Y \in D(0)$ if 1 - G(x) is slowly varying in which case (Darling, 1952)

$$\frac{\max\{Y_i: i=1,2,\ldots,n\}}{\sum_{i=1}^n Y_i} \longrightarrow 1$$

and so

$$\frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} Y_i} \xrightarrow{\mathcal{D}} X$$

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## Limits

Theorem (Breiman) Assume that  $Y \in D(\beta)$ ,  $\beta \in (0, 1)$ , and  $\mathbf{E}|X|^{\beta+\varepsilon} < \infty$ , for some  $\varepsilon > 0$ . Then  $T_n \xrightarrow{\mathcal{D}} T$ , where

$$\mathbf{P}\left\{T \le x\right\} = \frac{1}{2} + \frac{1}{\pi\beta} \arctan\left[\frac{\int |u-x|^{\beta} \operatorname{sgn}(x-u)F(\mathrm{d}u)}{\int |u-x|^{\beta}F(\mathrm{d}u)} \tan\frac{\pi\beta}{2}\right]$$

$$\mathsf{P}{T > x} \approx \mathsf{P}{X > x}$$

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## Limits

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$$\mathbf{P}\left\{T \le x\right\} = \frac{1}{2} + \frac{1}{\pi\beta} \arctan\left[\frac{\int |u-x|^{\beta} \operatorname{sgn}(x-u)F(\mathrm{d}u)}{\int |u-x|^{\beta}F(\mathrm{d}u)} \tan\frac{\pi\beta}{2}\right]$$

 $\mathsf{P}\{T > x\} \approx \mathsf{P}\{X > x\}$ 

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 $|\mathbf{E}|X|^{2+\delta} < \infty$ 

## Theorem (Mason & Zinn, 2005) Assume that $\mathbf{E}|X|^{2+\delta} < \infty$ . Then $T_n \to R$ , where R is non-degenerate, iff $Y \in D(\beta)$ , $\beta \in [0, 1)$ .

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# Studentization

Other type of self-normalization (Logan & Mallows & Rice & Shepp, 1973):



 $X, X_1, X_2, \dots$  iid. Student's *T*-statistic:

$$\frac{\sum_{i=1}^{n} X_i}{\sqrt{n}\sqrt{\frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n-1}}}$$

The two ratios are asymptotically the same.

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## Conjecture (Logan & Mallows & Rice & Shepp, 1973)

$$\frac{\sum_{i=1}^{n} X_i}{\sqrt{\sum_{i=1}^{n} X_i^2}} \xrightarrow{\mathcal{D}} W,$$

where  $P\{|W| = 1\} < 1$ , iff  $X \in D(\alpha)$ ,  $\alpha \in (0, 2]$ ; if  $\alpha > 1$ , EX = 0; if  $\alpha = 1$ ,  $X \in D(Cauchy)$ .

Giné & Götze & Mason (1997): *W* is standard normal iff  $X \in D(2)$  and  $\mathbf{E}X = 0$ Chistyakov & Götze (2004): in general

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## Recall

# *Y*, *Y*<sub>1</sub>, *Y*<sub>2</sub>,... non-negative iid rv's with df *G X*, *X*<sub>1</sub>, *X*<sub>2</sub>,... iid with df *F*, independent from *Y*, *Y*<sub>1</sub>, *Y*<sub>2</sub>,..., $\mathbf{E}|X| < \infty$ .

$$\phi_X(t) = \mathbf{E} e^{\mathrm{i} t X}$$

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### Theorem (K – Mason)

Assume that for some EX = 0,  $1 < \alpha \le 2$ , positive slowly varying function L at zero and c > 0,

$$rac{-\log\left(\Re\phi_X(t)
ight)}{|t|^lpha\,L\left(|t|
ight)}
ightarrow c, \; \textit{as}\; t
ightarrow 0.$$

Whenever

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$$\frac{\sum_{i=1}^{n} X_{i} Y_{i}}{\sum_{i=1}^{n} Y_{i}} \xrightarrow{\mathcal{D}} W \quad (W \text{ nondegenerate})$$

then  $Y \in D(\beta)$  for some  $\beta \in [0, 1)$ .

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# What does this condition mean?

For  $\alpha$  < 2 this holds iff (Pitman)

$$\mathbf{P}\left\{|X|>x\right\}\sim L(1/x)x^{-\alpha}c\Gamma(\alpha)\frac{2}{\pi}\sin\left(\frac{\pi\alpha}{2}\right)$$

If  $\mathbf{E}X = 0$  and  $X \in D(\alpha)$  then this condition is satisfied. Also if  $\mathbf{E}X = 0$ ,  $\mathbf{E}X^2 < \infty$  then the condition of the theorem is satisfied ( $\alpha = 2$ ,  $c = \sigma^2/2$ ). Partial solution

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## What does this condition mean?

$$\begin{split} \mathsf{As} &- \log \Re \phi_X(t) \sim 1 - \Re \phi_X(t), \ t \to 0, \\ &\frac{- \log \left( \Re \phi_X(t) \right)}{|t|^{\alpha} L(|t|)} \to c \ \Leftrightarrow \ \frac{1 - \Re \phi_X(t)}{|t|^{\alpha} L(|t|)} \to c. \end{split}$$

For  $\alpha$  < 2 this holds iff (Pitman)

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## Proposition

# Assume that the assumptions of the theorem hold. Then for some 0 $<\gamma \leq$ 1

$$\mathsf{E}\frac{\sum_{i=1}^{n} Y_{i}^{\alpha}}{\left(\sum_{i=1}^{n} Y_{i}\right)^{\alpha}} \to \gamma. \tag{(\star)}$$

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## Proposition

If (\*) holds with some  $\gamma \in (0, 1]$  then  $Y \in D(\beta)$ , for some  $\beta \in [0, 1)$ , where  $-\beta \in (-1, 0]$  is the unique solution of

Beta
$$(\alpha - 1, 1 - \beta) = \frac{\Gamma(\alpha - 1)\Gamma(1 - \beta)}{\Gamma(\alpha - \beta)} = \frac{1}{\gamma(\alpha - 1)}.$$

In particular,  $Y \in D(0)$  for  $\gamma = 1$ . Conversely, if  $Y \in D(\beta)$ ,  $0 \le \beta < 1$ , then (\*) holds with

$$\gamma = \frac{\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(1 - \beta)} = \frac{1}{(\alpha - 1)\operatorname{Beta}(\alpha - 1, 1 - \beta)}.$$

Extension of a result by Fuchs, Joffe and Teugels (2001), where  $\alpha = 2$ .

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$$\mathbf{E} \frac{\sum_{i=1}^{n} \mathbf{Y}_{i}^{\alpha}}{\left(\sum_{i=1}^{n} \mathbf{Y}_{i}\right)^{\alpha}} \to \gamma \tag{(*)}$$

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$$\begin{split} \mathbf{E} \frac{\sum_{i=1}^{n} Y_{i}^{\alpha}}{\left(\sum_{i=1}^{n} Y_{i}\right)^{\alpha}} &= n \mathbf{E} \frac{Y_{1}^{\alpha}}{\left(\sum_{i=1}^{n} Y_{i}\right)^{\alpha}} \\ &= \frac{n}{\Gamma(\alpha)} \mathbf{E} \int_{0}^{\infty} Y_{1}^{\alpha} e^{-t \sum_{i=1}^{n} Y_{i}} t^{\alpha-1} \mathrm{d}t \\ &= \frac{n}{\Gamma(\alpha)} \int_{0}^{\infty} t^{\alpha-1} \mathbf{E} \left( e^{-tY_{1}} Y_{1}^{\alpha} \right) (\mathbf{E} e^{-tY_{1}})^{n-1} \mathrm{d}t \\ &=: \frac{n}{\Gamma(\alpha)} \int_{0}^{\infty} t^{\alpha-1} \phi_{\alpha}(t) \phi_{0}(t)^{n-1} \mathrm{d}t. \end{split}$$

Note that for  $\alpha = 2$  we have  $\phi_{\alpha} = \phi_0''$ .

$$s \int_0^\infty t^{\alpha-1} \phi_\alpha(t) e^{s \log \phi_0(t)} \mathrm{d}t \to \gamma \Gamma(\alpha), \quad s \to \infty.$$

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Note that for  $\alpha = 2$  we have  $\phi_{\alpha} = \phi_{0}''$ .

$$s \int_0^\infty t^{lpha - 1} \phi_lpha(t) e^{s \log \phi_0(t)} \mathrm{d}t o \gamma \Gamma(lpha), \quad s o \infty.$$

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$$\phi_{\alpha}(t) = \mathbf{E} e^{-tY} Y^{\alpha}, \quad \phi_{0}(t) = \mathbf{E} e^{-tY}$$

By Karamata's Tauberian theorem

$$\lim_{t\to 0}\frac{\int_0^t y^{\alpha-1}\phi_\alpha(y)\mathrm{d}y}{1-\phi_0(t)}=\gamma\Gamma(\alpha).$$

After some further calculation

$$t^{\alpha-1} \frac{\int_0^\infty \overline{G}(u) u^{\alpha-1} e^{-ut} \mathrm{d} u}{\int_0^\infty \overline{G}(u) e^{-ut} \mathrm{d} u} \to \gamma \Gamma(\alpha), \text{ as } t \searrow 0.$$

$$u^{1-\alpha}e^{-ut} = \frac{1}{\Gamma(\alpha-1)}\int_0^\infty y^{\alpha-2}e^{-(y+t)u}\mathrm{d}y,$$

which holds for u > 0 and  $\alpha \in (1, 2]$ . Weyl-transform, or Weyl-fractional integral of the function  $e^{-ut}$ .

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#### We obtain

$$\int_{1}^{\infty} (u-1)^{\alpha-2} u^{-\alpha} \frac{g_{\infty}(x/u)}{g_{\infty}(x)} \mathrm{d}u = \frac{k \stackrel{M}{*} g_{\infty}(x)}{g_{\infty}(x)} \to [\gamma(\alpha-1)]^{-1}$$

#### with

$$g_{\infty}(x) = \int_0^{\infty} \overline{G}(ux) u^{\alpha-1} e^{-u} \mathrm{d} u.$$

$$k \stackrel{M}{*} h(x) = \int_0^\infty h(x/u) k(u) / u \mathrm{d} u$$

*Mellin-convolution* of *h* and *k*.

Drasin-Shea theorem implies that  $g_{\infty}(x)$  is regularly varying at infinity with index  $0 \ge \rho > -1$ .

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## Recall

 $Y, Y_1, Y_2, \dots$  non-negative iid rv's with df G $X, X_1, X_2, \dots$  iid with df F, independent from  $Y, Y_1, Y_2, \dots$ ,  $\mathbf{E}|X| < \infty$ 

$$T_n = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i}$$

 $\mathbf{E}|X| < \infty$  implies (*T<sub>n</sub>*) is tight.

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## Notation

 $id(a, b, \nu)$  infinitely divisible distribution on  $\mathbb{R}^d$  with characteristic exponent

$$\mathrm{i}u'b-rac{1}{2}u'au+\int\left(\mathrm{e}^{\mathrm{i}u'x}-1-\mathrm{i}u'xl(|x|\leq1)
ight)
u(\mathrm{d}x),$$

where  $b \in \mathbb{R}^d$ ,  $a \in \mathbb{R}^{d \times d}$  is a positive semidefinite matrix and  $\nu$  is the Lévy measure.

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## Theorem (K & Mason, 2012) If along a subsequence {n'}

$$\frac{1}{a_{n'}}\sum_{i=1}^{n'}Y_i \stackrel{\mathcal{D}}{\longrightarrow} W_2, \text{ as } n' \to \infty,$$

where  $W_2 \sim id(0, b, \Lambda)$ , then

$$\left(\frac{\sum_{i=1}^{n'} X_i Y_i}{a_{n'}}, \frac{\sum_{i=1}^{n'} Y_i}{a_{n'}}\right) \stackrel{\mathcal{D}}{\longrightarrow} (W_1, W_2), \ n' \to \infty,$$

where  $(W_1, W_2) \sim \operatorname{id}(\mathbf{0}, \mathbf{b}, \Pi)$ 

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## Theorem (K & Mason, 2012) *i.e. its characteristic function*

$$\Psi(\theta_1, \theta_2) = \mathbf{E} e^{i(\theta_1 W_1 + \theta_2 W_2)} = \exp \left\{ i(\theta_1 b_1 + \theta_2 b_2) + \int_0^\infty \int_{-\infty}^\infty \left( e^{i(\theta_1 x + \theta_2 y)} - 1 - (i\theta_1 x + i\theta_2 y) \mathbf{1}_{\{x^2 + y^2 \le 1\}} \right) F(dx/y) \Lambda(dy) \right\}.$$

$$H(x) = \mathbf{P}\left\{\frac{W_1}{W_2} \le x\right\} = \frac{1}{2} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Im \mathfrak{m} \Psi(u, -ux)}{u} \mathrm{d} u$$

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## Feller class

 $\xi, \xi_1, \dots$  iid with df *F*,  $S_n = \sum_{i=1}^n \xi_i$ . *F* is in the *centered Feller class*, if there exists  $B_n$ , such that every subsequence *n'* has a further subsequence *n''*, such that

$$\frac{\mathsf{S}_{n''}}{\mathsf{B}_{n''}} \xrightarrow{\mathcal{D}} W,$$

### where W is non-degenerate.

Theorem (Feller (1966), Maller (1979))

Y is in the centered Feller class, iff

$$\limsup_{x\to\infty}\frac{x^2\mathbf{P}\{|Y|>x\}+x|\mathbf{E}YI(|Y|\leq x)|}{\mathbf{E}[Y^2I(|Y|\leq x)]}<\infty.$$

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## Surprising result

## Theorem (K & Mason, 2012)

The subsequential limit distributions of

$$T_n = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i}$$

are continuous for all X with finite expectation if and only if  $Y \in \mathcal{F}_c$ .

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## Towards Lévy processes

$$(W_1, W_2) \stackrel{\mathcal{D}}{=} (a_1 + U, a_2 + V),$$
  
where  $(a_1, a_2) = \left( \left( b - \int_0^1 x \Lambda(\mathrm{d}x) \right) \mathbf{E} X, b - \int_0^1 x \Lambda(\mathrm{d}x) \right)$ 

$$\mathbf{E}\mathrm{e}^{\mathrm{i}(\theta_{1}U+\theta_{2}V)}=\exp\left\{\int_{0}^{\infty}\int_{-\infty}^{\infty}\left(\mathrm{e}^{\mathrm{i}(\theta_{1}x+\theta_{2}y)}-1\right)F\left(\mathrm{d}x/y\right)\Lambda\left(\mathrm{d}y\right)\right\}$$

Under the conditions of the theorem

$$\left(\frac{\sum_{1\leq i\leq n't}X_iY_i}{a_{n'}},\frac{\sum_{1\leq i\leq n't}Y_i}{a_{n'}}\right)_{t>0}\stackrel{\mathcal{D}}{\longrightarrow}(a_1t+U_t,a_2t+V_t)_{t>0},$$

where  $(U_t, V_t)$ ,  $t \ge 0$ , is the corresponding Lévy process.

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$$\frac{U_t}{V_t} \xrightarrow{\mathcal{D}} , \ t \to 0 \ \text{ or } \ t \to \infty$$

### 📔 Kevei, P, Mason, D.M.

Randomly Weighted Self-normalized Lévy Processes *Stochastic Processes and their Applications*, **123** (2) 2013, 490–522.

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