Generalized n-Paul paradox[1]

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Dedicated to Professor Sándor Csörgő for his sixtieth birthday

Abstract

The paradoxical results of Csörgő and Simons for mutually beneficial sharing among any fixed number of St. Petersburg gamblers are extended to games played by a possibly biased coin, with p as the probability of 'heads.' The extension is not straightforward because, unlike in the classical case with $p = 1/2$, admissibly pooled winnings generally fail to stochastically dominate individual ones for more than two gamblers. Best admissible pooling strategies are determined when p is rational, and the algebraic depth of the problem for an irrational p is illustrated by an example.

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1. Results and discussion

Peter offers to let Paul toss a possibly biased coin until it lands heads and pays him r^k ducats if this happens on the k^{th} toss, $k \in \mathbb{N} = \{1, 2, \ldots\}$, where $r = 1/q$ for $q = 1 - p$ and $p \in (0, 1)$ is the probability of 'heads' at each throw. This is the generalized St. Petersburg(p) game, in which $\boldsymbol{P}\{X = r^k\} = q^{k-1}p$, $k \in \mathbb{N}$, for Paul's gain X. Following Csörgő and Simons (2002, 2006), we assume that Peter plays exactly one such game with each of $n \geq 2$ players, Paul_1, Paul_2, … , Paul_n, whose independent individual winnings are $X_1, X_2, \ldots, X_n$.

The n players may agree to use a pooling strategy $\boldsymbol{p}_n = (p_{1,n}, p_{2,n}, \ldots, p_{n,n})$, before they play, where $p_{1,n}, p_{2,n}, \ldots, p_{n,n} \geq 0$ and $\sum_{j=1}^{n} p_{j,n} = 1$. Under this strategy Paul_1 receives $p_{1,n}X_1 + p_{2,n}X_2 + \cdots + p_{n,n}X_n$, Paul_2 receives $p_{n,n}X_1 + p_{1,n}X_2 + \cdots + p_{n-1,n}X_n$, Paul_3 receives $p_{n-1,n}X_1 + p_{n,n}X_2 + p_{1,n}X_3 + \cdots + p_{n-2,n}X_n$, …, and Paul_n receives $p_{2,n}X_1 + p_{3,n}X_2 + \cdots + p_{n,n}X_{n-1} + p_{1,n}X_n$ ducats. This strategy is fair to every Paul in the sense that their winnings are equally distributed and each receives the same *added value* equal to

$$\begin{aligned} A_p(\boldsymbol{p}_n) &= \boldsymbol{E}[\,p_{1,n}X_1 + \cdots + p_{n,n}X_n, X_1] \\ &= \int_0^\infty \big[\boldsymbol{P}\{p_{1,n}X_1 + \cdots + p_{n,n}X_n > x\} - \boldsymbol{P}\{X_1 > x\}\big]\,\mathrm{d}x, \end{aligned} \tag{1}$$

whenever the integral is defined, so that comparison is possible. We refer to Csörgő and Simons (2002, 2006) for a detailed exposition and discussion of the comparison operator $\boldsymbol{E}[\,\cdot\,,\cdot]$. We call a strategy $\boldsymbol{p}_n = (p_{1,n}, \ldots, p_{n,n})$ *admissible* if each of its components is either zero or a nonnegative integer power of $q = 1 - p$. Individualistic strategies $(1, 0, \ldots, 0)$ are

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thus admissible for each p, otherwise the powers in nonzero components are positive integers. The entropy of a pooling strategy is $H_r(\boldsymbol{p}_n) = \sum_{j=1}^{n} p_{j,n} \log_r 1/p_{j,n}$, where $\log_r$ denotes the base r logarithm and $0 \log_r 1/0 = 0$. We say that the random variable U is stochastically larger than the random variable V, written $U \geq_{\mathcal{D}} V$, if $\boldsymbol{P}\{U > x\} \geq \boldsymbol{P}\{V > x\}$ for all $x \in \mathbb{R}$.

Theorem 1. *For any $p \in (0,1)$ and $n \in \mathbb{N}$, the added value $A_p(\boldsymbol{p}_n)$ exists as an improper Riemann integral if and only if $\boldsymbol{p}_n$ is admissible, in which case $A_p(\boldsymbol{p}_n) = \frac{p}{q} H_r(\boldsymbol{p}_n)$.*

Csörgő and Simons (2006) proved this theorem for the classical St. Petersburg(1/2) game, played with an unbiased coin. However, in that case they proved the following stronger result: the independent St. Petersburg(1/2) variables $X_1, \ldots, X_n$ can be defined on a rich enough probability space that carries, for each admissible strategy $\boldsymbol{p}_n = (p_{1,n}, \ldots, p_{n,n})$, a St. Petersburg(1/2) random variable $X_{\boldsymbol{p}_n}$ and a nonnegative random variable $Y_{\boldsymbol{p}_n}$ such that $T_{\boldsymbol{p}_n} = p_{1,n}X_1 + \cdots + p_{n,n}X_n = X_{\boldsymbol{p}_n} + Y_{\boldsymbol{p}_n}$ almost surely. This implies the stochastic inequality $T_{\boldsymbol{p}_n} \geq_{\mathcal{D}} X_1$. Hence the integrand in $A_{1/2}(\boldsymbol{p}_n)$ is nonnegative and thus $A_{1/2}(\boldsymbol{p}_n)$ is trivially finite as a Lebesgue integral. As the next result shows, stochastic dominance is preserved for two players for an arbitrary St. Petersburg parameter $p \in (0,1)$.

Theorem 2. *For any $p \in (0,1)$, if $\boldsymbol{p}_2 = (q^a, q^b)$ is an admissible pooling strategy for some $a, b \in \mathbb{N}$, then $T_{\boldsymbol{p}_2} = q^a X_1 + q^b X_2$ is stochastically larger than X_1.*

Surprisingly, however, for $n \geq 3$ gamblers stochastic dominance generally fails to hold for admissible strategies. Our example to demonstrate this is when $p = (n-1)/n$, $q = 1-p = 1/n$, so that $r = 1/q = n$ is also the number of Pauls. Then $\boldsymbol{P}\{X = n^k\} = (n-1)/n^k$, $k \in \mathbb{N}$, and the averaging pooling strategy $\boldsymbol{p}_n = \boldsymbol{p}_n^{\diamond} = (1/n, 1/n, \ldots, 1/n)$ is admissible. For this strategy the weighted sum is $T_{\boldsymbol{p}_n^{\diamond}} = (X_1 + \cdots + X_n)/n$, so that for $n = 2$ Theorem 2 says in particular that $S_2 = 2T_{\boldsymbol{p}_2^{\diamond}} = X_1 + X_2$ is stochastically larger than $2X_1$. This is not true for $n \geq 3$.

Theorem 3. *If $p = (n-1)/n$, $q = 1/n$ and $n \geq 3$, then neither $S_n = X_1 + \cdots + X_n$ nor nX_1 is stochastically larger than the other.*

In view of Theorem 2, the integrand in (1) is nonnegative whenever $\boldsymbol{p}_2$ is admissible, so that the integral $A_p(\boldsymbol{p}_2)$ described in Theorem 1 strengthens to that of a Lebesgue integral when $n = 2$. While the same conclusion holds for $n \geq 3$, Theorem 3 rules out so simple a line of reasoning.

Theorem 4. *For every $p \in (0,1)$ and every admissible strategy $\boldsymbol{p}_n = (p_{1,n}, \ldots, p_{n,n})$ the integral $A_p(\boldsymbol{p}_n)$ in (1) is finite as a Lebesgue integral.*

Theorem 1 characterizes the pooling strategies that yield added values. However, admissible strategies do *not* exist for all, in fact, for most parameters $p \in (0,1)$. Call a parameter p *admissible*, if for p there exists an admissible strategy which is not individualistic. Theorem

1 then says that p is admissible if and only if for $q = 1 - p$ there exist positive integers $a_1 \geq a_2 \geq \cdots \geq a_k$, for some $k \in \mathbb{N}$, such that $q^{a_1} + q^{a_2} + \cdots + q^{a_k} = 1$. In this case, $r = 1/q$ is an algebraic integer. If $a_1 > a_2$, then q is also an algebraic integer, thus q is an algebraic unit. The set of algebraic numbers is countable, so there are at most a countable number of admissible parameters p. When $q = 1-p$ is rational for an admissible $p \in (0,1)$, the equation implies $q = 1/m$ for some integer $m \geq 2$. Thus the set of rational admissible parameters is $\{(m-1)/m : m \geq 2\}$. In particular, it is interesting that the classical $p = 1/2$ is the smallest such St. Petersburg parameter. It follows that the set of all admissible parameters p is countable. Nevertheless, it can be shown that this set is dense in the interval $(0,1)$.

When a given number of our Pauls happen to have admissible strategies, a natural question is: which is the best? In the latter rational case when $p = (m-1)/m$ for some integer $m \geq 2$, and so $r = 1/q = m \geq 2$ is an integer, the answer is given by the next result, in which $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$ is the integer part, $\lceil x \rceil = \min\{k \in \mathbb{Z} : k \geq x\}$ is the integer ceiling and $\langle x \rangle = x - \lfloor x \rfloor = x + \lceil -x \rceil$ is the fractional part of a number $x \in \mathbb{R}$.

Theorem 5. *If $p = (r-1)/r$ and $n = r^{\lfloor \log_r n \rfloor} + (r-1)r_n$ for some integers $r \geq 2$ and $0 \leq r_n \leq r^{\lfloor \log_r n \rfloor} - 1$, then*

$$A_p(\boldsymbol{p}_n) = \frac{p}{q}\, H_r(\boldsymbol{p}_n) \leq \frac{p}{q}\, \log_r n - \delta_p(n) =: A^*_{p,n} \tag{2}$$

for every admissible strategy $\boldsymbol{p}_n$, where $\delta_p(u) = 1 + (r-1)\langle \log_r u \rangle - r^{\langle \log_r u \rangle}$, $u > 0$. Moreover, the bound $A^*_{p,n}$ is attainable by means of the admissible strategy

$$\boldsymbol{p}^*_n = (p^*_{1,n}, \ldots, p^*_{n,n}) = (rp^*_n, \ldots, rp^*_n, p^*_n, \ldots, p^*_n) \quad \textit{with} \quad p^*_n = \frac{1}{r^{\lceil \log_r n \rceil}},$$

where the number of p^*_n s and rp^*_n s are, respectively,

$$m_{1,p}(n) = \frac{rn - r^{\lceil \log_r n \rceil}}{r-1} \quad \textit{and} \quad m_{2,p}(n) = \frac{r^{\lceil \log_r n \rceil} - n}{r-1}.$$

Apart from reorderings of the components of $\boldsymbol{p}^*_n$, the point of maximum is unique.

It is easy to see that if n is not in the form $r^{\lfloor \log_r n \rfloor} + (r-1)r_n$, then 0 must be included among the components of the strategy, which does not increase the entropy. So it is enough to investigate the number of players in the form above. The continuous function $\delta_p(\cdot)$ is nonnegative, its maximum is given in formula (3.4) of Csörgő and Simons (1996).

Theorem 5 is not applicable for an irrational p. On the other hand, in every admissible situation $A_p(\boldsymbol{p}_n) = (r-1)H_r(\boldsymbol{p}_n)$ by Theorem 1, and the trivial upper bound $H_r(\boldsymbol{p}_n) \leq \log_r n = H_r(\boldsymbol{p}^\diamond_n)$ is valid for the entropy of every $\boldsymbol{p}_n$, where $\boldsymbol{p}^\diamond_n = (1/n, 1/n, \ldots, 1/n)$. However equality cannot hold in general because $\boldsymbol{p}^\diamond_n$ is not admissible for every admissible parameter p. Apart from those cases which can be reduced to the rational case, that is when $q = 1-p = 1/\sqrt[k]{m}$ for some integers $m, k \geq 2$, the problem of the best admissible strategy is unsolved.

For the irrational case the simplest example is the equation $q^2 + q = 1$, the solution of which is $q = \tau := (\sqrt{5} - 1)/2 \approx 0.618$, the ratio of golden section. Thus, pertaining to the irrational parameter $p^\star = (3 - \sqrt{5})/2 \approx 0.382$, the vector (τ^2, τ) is an admissible strategy for two players. From this strategy we can generate admissible strategies for an arbitrary number of players. Indeed, substituting $\tau^3 + \tau^2 = \tau$ for τ, and $\tau^4 + \tau^3 = \tau^2$ for τ^2, we obtain (τ^3, τ^2, τ^2) and (τ^4, τ^3, τ), both admissible strategies for three Pauls. Continuing this algorithm, each time substituting $\tau^{m+2} + \tau^{m+1}$ for τ^m if the exponent m is present, after l steps we obtain admissible strategies for $2 + l$ gamblers, $l \in \mathbb{N}$. However, even if we allow all possible branches generated by this algorithm, the result is incomplete in the sense that there are admissible strategies, such as $(\tau^8, \tau^5, \tau^5, \tau^5, \tau^3, \tau^3, \tau^3)$ for seven Pauls, that are avoided. Consider all the strategies that can be generated by the branching algorithm from (τ^2, τ), and for every $n \geq 2$ call the best among these *conditionally optimal*, denoted by $\boldsymbol{p}_n^\star$. Let f_n be the n^{th} Fibonacci number, so that with $f_0 = 1$, $f_1 = 1$ and $f_{n+1} = f_{n-1} + f_n$, $n \in \mathbb{N}$. We can show that the conditionally optimal strategy for $f_n + k$ players, $k \in \{0, 1, \ldots, f_{n-1} - 1\}$, each playing a St. Petersburg($p^\star$) game, is

$$\boldsymbol{p}_{f_n+k}^\star = \big(\underbrace{\tau^{n+1}, \ldots, \tau^{n+1}}_{k \text{ times}}, \underbrace{\tau^n, \ldots, \tau^n}_{f_{n-2}+k \text{ times}}, \underbrace{\tau^{n-1}, \ldots, \tau^{n-1}}_{f_{n-1}-k \text{ times}}\big),$$

with the corresponding added value $A_p(\boldsymbol{p}_{f_n+k}^\star) = \tau^n[k(2-\tau) + n f_{n-2}\, \tau + (n-1) f_{n-1}]$. Because of the inherent number-theoretic difficulties, we do not know whether these conditionally optimal strategies are optimal in general.

Finally, we show that an extended form of our branching algorithm has an interesting property concerning stochastic domination. For any admissible parameter $p \in (0, 1)$, let $(q^{a_1}, q^{a_2}, \ldots, q^{a_n})$ and $(q^{b_1}, q^{b_2}, \ldots, q^{b_m})$ be admissible strategies for n and m Pauls for any $n, m \geq 2$. Substituting $q^{a_k+b_1} + q^{a_k+b_2} + \cdots + q^{a_k+b_m} = q^{a_k}$ for q^{a_k}, where $k \in \{1, \ldots, n\}$ is arbitrary, we obtain a strategy $(q^{d_1}, q^{d_2}, \ldots, q^{d_{n+m-1}})$ for $n + m - 1$ gamblers, where the sequence $d_1 \geq d_2 \geq \cdots \geq d_{n+m-1}$ is a nonincreasing rearrangement of the sequence $a_1, \ldots, a_{k-1}, a_{k+b_1}, \ldots, a_{k+b_m}, a_{k+1}, \ldots, a_n$. We say that a strategy $\boldsymbol{p}_n = (p_{1,n}, \ldots, p_{n,n})$ is *stochastically dominant* if $p_{1,n}X_1 + \cdots + p_{n,n}X_n \geq_{\mathcal{D}} X_1$. The last theorem states that the branching algorithm preserves stochastic dominance. Choosing first $n = m = 2$, it may be used in conjunction with Theorem 2 as a starting point.

Theorem 6. *If the admissible strategies* $(q^{a_1}, q^{a_2}, \ldots, q^{a_n})$ *and* $(q^{b_1}, q^{b_2}, \ldots, q^{b_m})$ *are both stochastically dominant, then so is the generated strategy* $(q^{d_1}, q^{d_2}, \ldots, q^{d_{n+m-1}})$.

All our results here are for fixed numbers of players. Csörgő and Simons (2005) proved for an arbitrary sequence of strategies $\boldsymbol{p}_n = (p_{1,n}, \ldots, p_{n,n})$ that $(p_{1,n}X_1 + \cdots + p_{n,n}X_n)/H_r(\boldsymbol{p}_n)$ converges in probability to p/q, as $n \to \infty$, whenever $H_r(\boldsymbol{p}_n) \to \infty$.

2. Proofs

The first two lemmas are needed for the proof of Theorem 1, while the third lemma is used in the proof of Theorem 5. These lemmas are the generalizations of Lemmas 3, 4 and 5 in Csörgő and Simons [hereafter abbreviated as Cs–S] (2006).

Lemma 1. *If X_1, X_2 are independent St. Petersburg(p) random variables and c_1 and c_2 positive constants, then* $\boldsymbol{E}(\min(c_1X_1, c_2X_2)) < \infty$.

Proof. We know from Cs–S (1996, 2005) that $1 - F_p(x) = \boldsymbol{P}\{X > x\} = q^{\lfloor \log_r x \rfloor} = r^{\langle \log_r x \rangle}/x$ for all $x \geq r$, and 1 otherwise. Hence, if $x \geq r \max(c_1, c_2)$, then the inequality $\boldsymbol{P}\{\min(c_1X_1, c_2X_2) > x\} = \boldsymbol{P}\{c_1X_1 > x\}\, \boldsymbol{P}\{c_2X_2 > x\} < c_1c_2r^2/x^2$ holds and, therefore, $\boldsymbol{E}(\min(c_1X_1, c_2X_2)) = \int_0^\infty \boldsymbol{P}\{\min(c_1X_1, c_2X_2) > x\}\, \mathrm{d}x < \infty$. ■

Lemma 2. *If X is a St. Petersburg(p) random variable and $b \geq 1$, then*

$$\int_0^b P\{X > x\}\, \mathrm{d}x = (r-1)\lfloor \log_r b \rfloor + r^{\langle \log_r b \rangle} = 1 + (r-1)\log_r b - \delta_p(b),$$

where the function $\delta_p(\cdot)$ is defined in Theorem 5.

Proof. Notice that $1 = 1 - F_p(x) = \boldsymbol{P}\{X > x\} = q^{\lfloor \log_r x \rfloor}$ even for $x \in [1, r)$. So what we need to prove is that $\int_1^b q^{\lfloor \log_r x \rfloor}\, \mathrm{d}x = (r-1)\lfloor \log_r b \rfloor + r^{\langle \log_r b \rangle} - 1$ for $b > 1$. If $c = \log_r b > 0$, then

$$\int_1^{r^c} q^{\lfloor \log_r x \rfloor}\, \mathrm{d}x = \int_0^c q^{\lfloor y \rfloor} r^y \log r\, \mathrm{d}y = (\log r)\int_0^c r^{\langle y \rangle} \mathrm{d}y = (\log r)\left[\lfloor c \rfloor \int_0^1 r^y \mathrm{d}y + \int_{\lfloor c \rfloor}^c r^{\langle y \rangle} \mathrm{d}y\right]$$

$$= \lfloor c \rfloor (r-1) + (\log r)\int_0^{\langle c \rangle} r^y\, \mathrm{d}y = (r-1)\lfloor c \rfloor + r^{\langle c \rangle} - 1,$$

where $\log = \log_{\mathrm{e}}$ is the natural logarithm, which is the desired equation. ■

Lemma 3. *If $r \in \{2, 3, \ldots\}$, then the number of the smallest strictly positive components of an admissible strategy $\boldsymbol{p}_n = (p_{1,n}, \ldots, p_{n,n})$ is divisible by r.*

Proof. Let the smallest strictly positive component be $1/r^k$ for some $k \in \mathbb{N}$. Since $\sum_{j=1}^n p_{j,n} r^k = r^k$, the sum must be divisible by r, so the number of terms equal to 1 in the sum, which is the number of the components $1/r^k$ in $\boldsymbol{p}_n$, is also divisible by r. ■

Proof of Theorem 1. With the extended Lemmas 1 and 2, the proof is an easy generalization of that in the classical case $p = 1/2$ in Cs–S (2006), so we only sketch the differences.

For a given strategy $\boldsymbol{p}_n = (p_{1,n}, \ldots, p_{n,n})$, the integral $A_p(\boldsymbol{p}_n)$ in (1) is defined in the improper Riemann sense if and only if $A_p(\boldsymbol{p}_n, b) \to A_p(\boldsymbol{p}_n)$ as $b \to \infty$, where

$$A_p(\boldsymbol{p}_n, b) = \int_0^b \big[\boldsymbol{P}\{p_{1,n}X_1 + \cdots + p_{n,n}X_n > x\} - \boldsymbol{P}\{X_1 > x\}\big]\mathrm{d}x.$$

It can be shown that $A_p(\boldsymbol{p}_n, b) = (r-1)H_r(\boldsymbol{p}_n) + R_r(\boldsymbol{p}_n, b) + o(1)$ as $b \to \infty$, where $R_r(\boldsymbol{p}_n, b) = \delta_p(b) - \sum_{j=1}^n p_{j,n}\, \delta_p(b/p_{j,n})$. Notice that $\delta_p(ur^k) = \delta_p(u)$, $u > 0$, for every $k \in \mathbb{Z}$. Thus if $\boldsymbol{p}_n$ is admissible, then

$$R_r(\boldsymbol{p}_n, b) = \delta_p(b) - \sum_{j=1}^n p_{j,n}\, \delta_p\bigg(\frac{b}{p_{j,n}}\bigg) = \delta_p(b) - \sum_{j=1}^n p_{j,n}\, \delta_p(b) = 0,$$

and hence $A_p(\boldsymbol{p}_n, b) = (r-1)H_r(\boldsymbol{p}_n) + o(1)$ as $b \to \infty$, which is the "if part" of the theorem.

Conversely, suppose that $A_p(\boldsymbol{p}_n)$ in (1) exists, so that $A_p(\boldsymbol{p}_n, b) \to A_p(\boldsymbol{p}_n)$ as $b \to \infty$. Using the above periodicity property of $\delta_p(\cdot)$, we get $R_r(\boldsymbol{p}_n, r^k b) = R_r(\boldsymbol{p}_n, b)$ for every $k \in \mathbb{Z}$. Fixing $b > 0$ and letting $k \to \infty$, so that $r^k b \to \infty$, we get $R_r(\boldsymbol{p}_n, b) = A_p(\boldsymbol{p}_n) - (r-1)H_r(\boldsymbol{p}_n)$. Let $D = D_+ - D_-$, where D_+ and D_- are the right-side and left-side differential operators, respectively. Then one can prove that

$$D\delta_p(s) = \begin{cases} \frac{r-1}{r^k}, & \text{for } s = r^k \text{ when } k \in \mathbb{Z}, \\ 0, & \text{for all other } s > 0, \end{cases}$$

from which, for all $j \in \{1, \ldots, n\}$ for which $p_{j,n} > 0$, we find that

$$D\, p_{j,n}\, \delta_p\biggl(\frac{b}{p_{j,n}}\biggr) = \begin{cases} \frac{r-1}{r^k}, & \text{for } b = r^k\, p_{j,n} \text{ when } k \in \mathbb{Z}, \\ 0, & \text{for all other } b > 0. \end{cases}$$

Consequently, we have

$$DR_r(\boldsymbol{p}_n, b)\big|_{b=1} = r - 1 - (r-1) \sum_{j \in A} p_{j,n},$$

where A is the set of indices $j \in \{1, \ldots, n\}$ for which $p_{j,n}$ is an integer power of r. Since, on the other hand, $DR_r(\boldsymbol{p}_n, b) = 0$, this implies $\sum_{j \in A} p_{j,n} = 1$, and thus completes the proof. ■

Proof of Theorem 2. Let $q^a + q^b = 1$ for some $a, b \in \mathbb{N}$. Then $\boldsymbol{P}\{X_1 \le r^k\} = F_p(r^k) = 1 - q^k$ for every $k \in \mathbb{N}$. We estimate the probability $\boldsymbol{P}\{T_2 \le r^k\}$, where $T_2 = q^a X_1 + q^b X_2$. If $T_2 \le r^k$, then

(1) $X_1, X_2 \le r^k$, or
(2) $X_1 = r^{k+1}, \ldots, r^{k+a-1}$ and $X_2 \le r^{k-1}$, or
(3) $X_2 = r^{k+1}, \ldots, r^{k+b-1}$ and $X_1 \le r^{k-1}$.

We obtain

$$\begin{aligned} \boldsymbol{P}\bigl\{T_2 \le r^k\bigr\} &\le \bigl(1 - q^k\bigr)^2 + \bigl(1 - q^{k-1}\bigr) q^k \bigl(1 - q^{a-1}\bigr) + \bigl(1 - q^{k-1}\bigr) q^k \bigl(1 - q^{b-1}\bigr) \\ &= \bigl(1 - q^k\bigr)^2 + \bigl(1 - q^{k-1}\bigr) q^k \left(2 - \frac{1}{q}\right) = 1 - q^{k-1} + q^{2k} \left(\frac{1}{q} - 1\right)^2. \end{aligned}$$

Since the distribution function of X_1 jumps only in the points $x = r^k$, it is enough to show that $\boldsymbol{P}\bigl\{T_2 < r^k\bigr\} \le \boldsymbol{P}\{X_1 < r^k\} = \boldsymbol{P}\{X_1 \le r^{k-1}\} = 1 - q^{k-1}$. This is true, because

$$\begin{aligned} \boldsymbol{P}\bigl\{T_2 < r^k\bigr\} &= \boldsymbol{P}\bigl\{T_2 \le r^k\bigr\} - \boldsymbol{P}\bigl\{T_2 = r^k\bigr\} \\ &\le 1 - q^{k-1} + q^{2k} \left(\frac{1}{q} - 1\right)^2 - \boldsymbol{P}\bigl\{X_1 = r^k,\ X_2 = r^k\bigr\} \\ &= 1 - q^{k-1}, \end{aligned}$$

completing the proof. ■

Proof of Theorem 3. We prove that stronger statement that the graphs of the distribution functions of S_n and nX_1 cross each other infinitely often. More exactly we show that both $\boldsymbol{P}\{nX_1 \le n^k\} > \boldsymbol{P}\{S_n \le n^k\}$ and $\boldsymbol{P}\{nX_1 < n^k\} < \boldsymbol{P}\{S_n < n^k\}$ hold whenever $k \ge 3$.

Notice that the inequality $S_n \le n^k$ holds if and only if $X_1 \le n^{k-1}$, $X_2 \le n^{k-1}, \dots$, $X_n \le n^{k-1}$. This implies for arbitrary $k \ge 2$ that

$$\boldsymbol{P}\big\{S_n \le n^k\big\} = \boldsymbol{P}\left\{\bigcap_{j=1}^{n} \{X_j \le n^{k-1}\}\right\} = \left(\frac{n-1}{n} + \frac{n-1}{n^2} + \cdots + \frac{n-1}{n^{k-1}}\right)^n = \left(1 - \frac{1}{n^{k-1}}\right)^n.$$

Clearly, $\boldsymbol{P}\big\{nX_1 \le n^k\big\} = \boldsymbol{P}\big\{X_1 \le n^{k-1}\big\} = 1 - 1/n^{k-1}$, so $\boldsymbol{P}\big\{nX_1 \le n^k\big\} > \boldsymbol{P}\big\{S_n \le n^k\big\}$.

Now consider the probabilities $\boldsymbol{P}\{nX_1 < n^k\}$ and $\boldsymbol{P}\{S_n < n^k\}$. When $k = 2$, both of them are zero. So, assume $k \ge 3$. Noticing that $\boldsymbol{P}\{S_n < n^k\} = \boldsymbol{P}\{S_n \le n^k\} - \boldsymbol{P}\{S_n = n^k\}$,

$$\boldsymbol{P}\big\{S_n = n^k\big\} = \boldsymbol{P}\big\{X_1 = n^{k-1}, X_2 = n^{k-1}, \dots, X_n = n^{k-1}\big\} = \left(\frac{n-1}{n^{k-1}}\right)^n,$$

and $\boldsymbol{P}\{nX_1 < n^k\} = \boldsymbol{P}\{nX_1 \le n^{k-1}\} = \boldsymbol{P}\{X_1 \le n^{k-2}\}$, we have

$$\boldsymbol{P}\big\{S_n < n^k\big\} = \left(1 - \frac{1}{n^{k-1}}\right)^n - \left(\frac{n-1}{n^{k-1}}\right)^n > 1 - \frac{1}{n^{k-2}} = \boldsymbol{P}\{nX_1 < n^k\},$$

where the inequality holds for $n \ge 3$ and $k \ge 3$ by elementary calculations. ■

Proof of Theorem 4. Let $n \ge 2$ be the number of Pauls. By Theorem 1, for every admissible strategy $q = 1-p$ satisfies the equation $q^{a_1} + q^{a_2} + \cdots + q^{a_m} = 1$, where $a_1, a_2, \dots, a_m \in \mathbb{N}$ and $m \in \{2, 3, \dots, n\}$. Without loss of generality we assume that the zeros, if any, are the last components of the strategy, so that $\boldsymbol{p}_n = (q^{a_1}, q^{a_2}, \dots, q^{a_m}, 0, \dots, 0)$. Then $T_m := \sum_{j=1}^{n} p_{j,n} X_j = q^{a_1} X_1 + \cdots + q^{a_m} X_m$. We estimate the probability $\boldsymbol{P}\{T_m \le r^k\}$. If the event $\{T_m \le r^k\}$ occurs, then we must have all the inequalities $X_1 \le r^{k+a_1-1}, X_2 \le r^{k+a_2-1}, \dots, X_m \le r^{k+a_m-1}$. Hence,

$$\begin{aligned}
\boldsymbol{P}\big\{T_m \le r^k\big\} &\le \big(1 - q^{k+a_1-1}\big)\big(1 - q^{k+a_2-1}\big) \cdots \big(1 - q^{k+a_m-1}\big) \\
&= 1 - q^k\big(q^{a_1-1} + q^{a_2-1} + \cdots + q^{a_m-1}\big) + q^{2k} C_2 + \cdots + q^{mk} C_m \\
&= 1 - q^{k-1} + q^{2k} C_2 + \cdots + q^{mk} C_m\,,
\end{aligned}$$

where the constants $C_2, C_3, \dots, C_m$ do not depend on k.

Since $\boldsymbol{p}_n$ is admissible, the integral $\int_0^\infty \big[\boldsymbol{P}\{T_m > x\} - \boldsymbol{P}\{X_1 > x\}\big]\,\mathrm{d}x$ exists as an improper Riemann integral. Hence it suffices to show that the integral of the negative part $g_m^-(x)$ of the function $g_m(x) := \boldsymbol{P}\{T_m > x\} - \boldsymbol{P}\{X_1 > x\}$ is finite. Notice that

$$g_m(x) = \boldsymbol{P}\{X_1 \le x\} - \boldsymbol{P}\{T_m \le x\} \ge \boldsymbol{P}\{X_1 < x\} - \boldsymbol{P}\{T_m \le x\} =: h_m(x)$$

for all $x > 0$. Clearly, the function $h_m(x)$ takes a minimum value on the interval $(r^{k-1}, r^k]$ at $x = r^k$, for which the estimate above yields

$$\begin{aligned}
h_m(r^k) = \boldsymbol{P}\big\{X_1 < r^k\big\} - \boldsymbol{P}\big\{T_m \le r^k\big\} &\ge 1 - q^{k-1} - \big(1 - q^{k-1} + q^{2k} C_2 + \cdots + q^{mk} C_m\big) \\
= -\big(q^{2k} C_2 + \cdots + q^{mk} C_m\big).
\end{aligned}$$

Therefore, setting $C_1 = \int_0^1 h_m^-(x)\,\mathrm{d}x$, we obtain

$$\int_0^\infty g_m^-(x)\,\mathrm{d}x \le \int_0^\infty h_m^-(x)\mathrm{d}x \le C_1 + \sum_{k=1}^\infty \int_{r^{k-1}}^{r^k} \big(q^{2k}|C_2| + \cdots + q^{mk}|C_m|\big)\mathrm{d}x$$
$$= C_1 + \sum_{k=1}^\infty r^k\left(1-\frac{1}{r}\right)\big(q^{2k}|C_2| + \cdots + q^{mk}|C_m|\big)$$
$$= C_1 + (1-q)\sum_{k=1}^\infty \big(q^k|C_2| + \cdots + q^{(m-1)k}|C_m|\big) < \infty,$$

which proves the theorem. ■

Proof of Theorem 5. This is based on the proof of Theorem 2 in Cs–S (2006), so we skip the details. Without loss of generality we assume that $\boldsymbol{p}_n$ is ordered: $p_{1,n} \ge p_{2,n} \ge \cdots \ge p_{n,n}$. The proof is by induction on r_n. For $r_n = 0$ the statement is true. Now suppose that all the statements of the theorem hold for $r_n - 1 \ge 0$, and consider the case $n = r^{\lfloor \log_r n\rfloor} + (r-1)r_n$. If $p_{n,n} = 0$, then we have at least $r-1$ zeros. Deleting them, we get a strategy $\hat{\boldsymbol{p}}_{n-(r-1)}$, and we are done in view of the fact that the bound $A^*_{p,n}$ in (2) is nondecreasing in n. In the other case, when $p_{n,n} = 1/r^k$ for some $k \in \mathbb{N}$, we have at least r of these smallest components by Lemma 3. Changing r of these to a single component $1/r^{k-1}$, we obtain a strategy $\hat{\boldsymbol{p}}_{n-(r-1)}$ for which $H_r(\boldsymbol{p}_n) - H_r\big(\hat{\boldsymbol{p}}_{n-(r-1)}\big) = 1/r^{k-1}$. Using the induction hypothesis and the formula $A^*_{p,n} = (r-1)\lfloor \log_r n\rfloor + (r-1)r_n/r^{\lfloor \log_r n\rfloor}$, the proof can be completed as in Cs–S (2006). The uniqueness assertion of the theorem follows by the same induction argument. ■

Lemma 4. *If U, V, W are independent random variables and $U \ge_{\mathcal{D}} V$, then $U+W \ge_{\mathcal{D}} V+W$.*

Proof. Let F, G and H be the distribution functions of U, V and W, respectively. By assumption, $F(x) \le G(x)$ for all $x \in \mathbb{R}$. The random variables $U+W$ and $V+W$ have the distribution functions $F{*}H(\cdot)$ and $G{*}H(\cdot)$, where $*$ denotes Lebesgue–Stieltjes convolution. Thus

$$F{*}H(x) = \int_{-\infty}^\infty F(x-y)\mathrm{d}H(y) \le \int_{-\infty}^\infty G(x-y)\mathrm{d}H(y) = G{*}H(x),$$

which proves the statement. ■

Proof of Theorem 6. Let $Y_1, \ldots, Y_m, X_1, \ldots, X_n$ be independent St. Petersburg(p) variables. From the assumption we get $q^{a_k+b_1}Y_1 + q^{a_k+b_2}Y_2 + \cdots + q^{a_k+b_m}Y_m \ \ge_{\mathcal{D}}\ q^{a_k}X_k$. By Lemma 6 this implies

$$q^{a_1}X_1 + \cdots + q^{a_{k-1}}X_{k-1} + q^{a_k+b_1}Y_1 + \cdots + q^{a_k+b_m}Y_m + q^{a_{k+1}}X_{k+1} + \cdots + q^{a_n}X_n$$
$$\ge_{\mathcal{D}}\ q^{a_1}X_1 + \cdots + q^{a_{k-1}}X_{k-1} + q^{a_k}X_k + q^{a_{k+1}}X_{k+1} + \cdots + q^{a_n}X_n.$$

Now the assumption and obvious transitivity together imply the theorem. ■

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