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ON FREE ALGEBRAS IN VARIETIES GENERATED BY ITERATED SEMIDIRECT PRODUCTS OF SEMILATTICES

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We present a new solution of the word problem of free algebras in varieties generated by iterated semidirect products of semilattices. As a consequence, we provide asymptotical bounds for free spectra of these varieties. In particular, each finite $\mathcal{R}$-trivial (and, dually, each finite $\mathcal{L}$-trivial) semigroup has a free spectrum whose logarithm is bounded above by a polynomial function.

Keywords: Free spectra; semigroup; semilattice; semidirect product.

Mathematics Subject Classification 2010: 20M05, 20M07, 08B20

1. Introduction

Let $\mathcal{V}$ be a variety and $n \geq 1$ an integer. The free algebra of $\mathcal{V}$ over an n-element set will be denoted by $\mathbf{F}_n(\mathcal{V})$. If $\mathcal{V}$ is locally finite, then the sequence $f_n(\mathcal{V}) = |\mathbf{F}_n(\mathcal{V})|$, $n \geq 1$, consists of positive integers; it is called the *free spectrum* of $\mathcal{V}$. If $\mathbf{A}$ is an algebra generating a locally finite variety (for example, if $\mathbf{A}$ is finite), then the free spectrum of $\mathbf{A}$, $f_n(\mathbf{A})$, is just the free spectrum of the variety $\mathbf{A}$ generates. By elementary facts from universal algebra, $f_n(\mathbf{A})$ is in fact the number of all n-ary operations on A, the carrier set of $\mathbf{A}$, induced by terms in the signature of $\mathbf{A}$. These operations are called the *term operations* of $\mathbf{A}$. For example, n-ary term operations of a semigroup are just the operations of its underlying set induced by non-empty words over an n-element alphabet. Therefore, if $|\mathbf{A}| = m$, then $f_n(\mathbf{A}) \leq m^{m^n}$. This upper bound is attained for the two-element Boolean algebra: The free spectrum of the variety of Boolean algebras is 2^{2^n}.

A closely related invariant of an algebra $\mathbf{A}$ generating a locally finite variety is its *p_n-sequence*, $p_n(\mathbf{A})$. Namely, an n-ary term operation of $\mathbf{A}$ is said to be *essentially n-ary* if it depends on all of its variables, and $p_n(\mathbf{A})$ counts all such operations. The p_n-sequence of a locally finite variety $\mathcal{V}$ is just the p_n-sequence of any algebra generating $\mathcal{V}$. We refer the reader to the survey [6] for an overview of basic results on free spectra and p_n-sequences and the development of the theory up to the early nineties.

An important milestone in the theory of free spectra of locally finite varieties came with the paper of Kearnes [10], who showed that the free spectrum of a general finite algebra $\mathbf{A}$ is a great deal governed by the free spectrum of an associated monoid, called the *twin monoid* of $\mathbf{A}$. This clearly emphasized the need to focus the attention to free spectra of (finite) semigroups and monoids and to try to classify all possible spectra according to their asymptotic behavior. A first attempt in this direction was made by Seif [13], who formulated an intriguing conjecture demarcating between finite monoids whose free spectra are doubly exponential and those which do not have this property. The quest for such a boundary traces back to an old result of Higman [7] and Neumann [11], which states that for a finite group $\mathbf{G}$ we have $\log f_n(\mathbf{G}) \in \mathcal{O}(n^c)$ if and only if $\mathbf{G}$ is step-c nilpotent, while $f_n(\mathbf{G})$ is doubly exponential if $\mathbf{G}$ is not nilpotent.

Today, the theory of free spectra of semigroups and semigroup varieties shapes into a steadily growing subject, see, for example, [3–5, 8, 9, 12]. In this paper, we consider free objects in semigroup varieties $\mathcal{SL}^t$, $t \geq 1$, generated by t times iterated semidirect products of semilattices. Previously, these varieties and the word problem of their free algebras were thoroughly investigated by Almeida [1] (see also [2, Sec. 10.3]). Here we provide an alternative solution to the equational problem of $\mathcal{SL}^t$ (that is, of word problems of $\mathbf{F}_n(\mathcal{SL}^t)$) that is much more suitable for our main goal of obtaining (polynomial) upper bounds for $\log f_n(\mathcal{SL}^t)$. We present a procedure for an effective computation of a normal form for a word from X^+ with respect to the fully invariant congruence of X^+ corresponding to the variety $\mathcal{SL}^t$. Also, we give asymptotic estimates for $f_n(\mathcal{SL}^t)$ and $p_n(\mathcal{SL}^t)$. Finally, we note that by an old result of Stiffler [14] our results imply that $\log f_n(\mathbf{S})$ is bounded above by a polynomial function whenever $\mathbf{S}$ is a finite semigroup (or a monoid) in which one of Green's relations $\mathscr{R}$, $\mathscr{L}$ is trivial, thereby confirming Seif's conjecture in this particular case.

2. Preliminaries

Let $\mathbf{S}$ and $\mathbf{T}$ be semigroups. A *left action* of $\mathbf{T}$ on $\mathbf{S}$ is a monoid homomorphism $\varphi : \mathbf{T}^1 \to \operatorname{End}\mathbf{S}$. If $t \in T^1$ and $s \in S$ we write ts as a short-hand for $[\varphi(t)](s)$, and if we write the operation of $\mathbf{S}$ additively (which does not mean that $\mathbf{S}$ is necessarily commutative), then for all $s, s' \in S$ and $t, t' \in T^1$ we have

$$t(s + s') = ts + ts',$$
$$(tt')s = t(t's),$$
$$1s = s.$$

If $\mathbf{S}$ happens to be a monoid with an identity element 0 then it is also required that $t0 = 0$ for all $t \in T^1$. For each left action φ of $\mathbf{T}$ on $\mathbf{S}$ we define the *semidirect product* $\mathbf{S} *_\varphi \mathbf{T}$ (which is often abbreviated to $\mathbf{S} * \mathbf{T}$ if the action is clear from the context) as the semigroup defined on the set $S \times T$ by

$$(s, t)(s', t') = (s + ts', tt').$$

If $\mathcal{U}$ and $\mathcal{V}$ are semigroup varieties, then $\mathcal{U} * \mathcal{V}$ is defined to be the variety generated by all semidirect products $\mathbf{S} * \mathbf{T}$ such that $\mathbf{S} \in \mathcal{U}$, $\mathbf{T} \in \mathcal{V}$. Even though the construction of the semidirect product is in general not associative (that is, we do not need to have $(\mathbf{R} * \mathbf{S}) * \mathbf{T} \cong \mathbf{R} * (\mathbf{S} * \mathbf{T})$), it is nevertheless an associative operation on the set of all semigroup varieties.

If $\mathcal{SL}$ denotes the variety of all semilattices, we define a sequence of varieties $\mathcal{SL}^t$, $t \geq 1$, by $\mathcal{SL}^1 = \mathcal{SL}$ and $\mathcal{SL}^{i+1} = \mathcal{SL} * \mathcal{SL}^i$ for all $i \geq 1$. These varieties (and the corresponding pseudovarieties of finite semigroups, obtained by taking their finite members), generated by t times iterated semidirect products of semilattices, were thoroughly studied by Almeida [1], see also [2]. In this paper, we supply an alternative solution of word problems for their free algebras, which will allow us to construct systems of normal forms of elements of these free algebras and calculate the free spectra and p_n-sequences of varieties of the form $\mathcal{SL}^t$.

First of all, we recall the identity basis of $\mathcal{SL}^t$ provided in [1]. As usual, X^+ (X^*) denotes the free semigroup (free monoid) on the set X, and for $w \in X^*$, $c(w)$ is the *content* of w, the set of letters that occur in w.

Theorem 2.1 (Almeida [1, 2]). *Let the set Σ_{t-1}, $t \geq 1$, consist of the following two types of identities over a countably infinite alphabet X containing the letters x, x_1, x_2:*

$$u_{t-1} \cdots u_1 x^2 = u_{t-1} \cdots u_1 x, \tag{1}$$

where $u_1, \ldots, u_{t-1} \in X^+$ are such that $x \in c(u_1) \subseteq \cdots \subseteq c(u_{t-1})$, and

$$u_{t-1} \cdots u_1 x_1 x_2 = u_{t-1} \cdots u_1 x_2 x_1, \tag{2}$$

where $u_1, \ldots, u_{t-1} \in X^+$ are such that $x_1, x_2 \in c(u_1) \subseteq \cdots \subseteq c(u_{t-1})$. Then the variety $\mathcal{SL}^t$ is defined by Σ_{t-1}.

As it turns out, $\mathcal{SL}^t$ is not finitely based whenever $t \geq 3$.

Throughout the remainder of the paper, let X be a countably infinite alphabet. We say that

$$u = w_0, w_1, \ldots, w_r = v \in X^+ \tag{3}$$

is a *deduction* of the identity $u = v$ from a set of identities Σ if for each $j \in \{0, \ldots, r-1\}$ there exist factorizations

$$w_j = a_j \phi_j(u_j) b_j \quad \text{and} \quad w_{j+1} = a_j \phi_j(v_j) b_j, \tag{4}$$

where each $\phi_j : X^+ \to X^+$ is a substitution, $a_j, b_j \in X^*$, and at least one of the identities $u_j = v_j$, $v_j = u_j$ belongs to Σ. A deduction is *left absorbing* if each prefix a_j occurring in (4) is the empty word. We say that the deduction (3) *involves no substitutions* if all endomorphisms ϕ_j are the identity mapping. Some of the relevant results of [1], which refine the previous theorem, may be summarized in the following way.

Proposition 2.2. *Let $u, v \in X^+$. The identity $u = v$ holds in $\mathcal{SL}^t$ if and only there exists a deduction of $u = v$ from Σ_{t-1} which is left absorbing and involves no substitutions.*

In other words, $\mathcal{SL}^t$ satisfies $u = v$ if and only if there is a deduction $u = w_0, w_1, \ldots, w_r = v$ such that each identity $w_j = w_{j+1}$ in the deduction (or $w_{j+1} = w_j$) is of one of the following forms:

$$u_{t-1} \cdots u_1 x^2 w = u_{t-1} \cdots u_1 x w, \tag{5}$$

with $x \in c(u_1) \subseteq \cdots \subseteq c(u_{t-1})$, $w \in X^*$, and

$$u_{t-1} \cdots u_1 x_1 x_2 w = u_{t-1} \cdots u_1 x_2 x_1 w, \tag{6}$$

such that $x_1, x_2 \in c(u_1) \subseteq \cdots \subseteq c(u_{t-1})$ and $w \in X^*$. We call each identity of the form (5) and (6) an *elementary step on level t*.

3. The Word Problem and the Recurrence Formula

For $u, v \in X^+$, let us write $u \sim_t v$ if $\mathcal{SL}^t$ satisfies the identity $u = v$. By elementary universal algebraic facts, $\sim_t$ is a fully invariant congruence on X^+, and $X^+/{\sim_t}$ is the $\mathcal{SL}^t$-free semigroup on X. Our goal is to give a characterization of relations $\sim_t$, $t \geq 1$, which will suit our needs. Since $\mathcal{SL} \subseteq \mathcal{SL}^t$ for all $t \geq 1$, we have that $u \sim_t v$ implies $c(u) = c(v)$; in fact, $u \sim_1 v$ if and only if $c(u) = c(v)$. Also, the following fact observed by Almeida [1, 2] will be used in the sequel.

Lemma 3.1. *Let $u, v, w \in X^+$ such that $u \sim_t v$ for some $t \geq 1$ and $c(w) \supseteq c(u) = c(v)$. Then $wu \sim_{t+1} wv$.*

For a word $u \in X^+$ let $m_u \in X$ be the letter that is last to occur in u from the left. Let f_u be the longest prefix of u that does not contain m_u; then $c(f_u) = c(u)\backslash\{m_u\}$. Accordingly, u factorizes as

$$u = f_u m_u b_u$$

for some word $b_u \in X^*$. This is the notation that we are going to use in the following result, which describes the relation $\sim_t$, $t \geq 2$, and thus, by our earlier remarks, provides a solution of the equational problem of varieties $\mathcal{SL}^t$.

Theorem 3.2. *Let $t \geq 2$ and $u, v \in X^+$ such that $c(u) = c(v)$. Then $u \sim_t v$ if and only if the following conditions hold*:

(i) $m_u = m_v$,
(ii) $f_u \sim_t f_v$,
(iii) $b_u \sim_{t-1} b_v$.

Proof. ($\Leftarrow$) Since, by definition, $c(f_w m_w) = c(w) \supseteq c(b_w)$ holds for any word $w \in X^+$ the assumption $b_u \sim_{t-1} b_v$ implies $u = f_u m_u b_u \sim_t f_u m_u b_v = f_u m_v b_v$, by (i), the previous lemma and the fact that $c(u) = c(v)$. However, since $\sim_t$ is a congruence, we have that (ii) implies $f_u m_v b_v \sim_t f_v m_v b_v = v$. Hence, $u \sim_t v$.

($\Rightarrow$) It suffices to prove this implication only for the case when $u = v$ is an elementary step on level t, because then a routine induction on the length of the deduction of an identity holding in $\mathcal{SL}^t$ proves the implication in the general case.

In fact, we consider only the case of an elementary step of type (5), while the case of (6) is handled similarly. We have

$$u = u_{t-1} \cdots u_1 x^2 w,$$
$$v = u_{t-1} \cdots u_1 x w,$$

where $x \in X$, $u_1, \ldots, u_{t-1} \in X^+$, $w \in X^*$, such that $x \in c(u_1) \subseteq \cdots \subseteq c(u_{t-1})$. We distinguish two cases.

Case 1. $c(u_{t-1}) = c(u)$. Then m_u occurs in u_{t-1}, so $f_u m_u$ is a prefix of u_{t-1}; similarly, $f_v m_v$ is a prefix of u_{t-1}. This implies that $m_u = m_v = m_{u_{t-1}}$ and, consequently, $f_u = f_v = f_{u_{t-1}}$. Therefore, $u_{t-1} = f_u m_u s = f_v m_v s$ for some $s \in X^*$. Now the pair of words

$$\begin{aligned} b_u &= (su_{t-2}) \cdots u_1 x^2 w, \\ b_v &= (su_{t-2}) \cdots u_1 x w, \end{aligned}$$

forms an elementary step on level $t-2$, which shows that $b_u \sim_{t-1} b_v$.

Case 2. $c(u_{t-1}) \neq c(u)$. Then, by the given conditions, $c(u_{t-1} \cdots u_1 x^2) = c(u_{t-1} \cdots u_1 x) = c(u_{t-1}) \neq c(u) = c(v)$, so both letters m_u and m_v occur in w. If w' is the shortest prefix of w with the property that $c(u_{t-1}) \cup c(w') = c(u)$, then $u_{t-1} \cdots u_1 x^2 w'$ is the shortest prefix of u with the same content as u, while $u_{t-1} \cdots u_1 x w'$ is the shortest prefix of v with the same content as v. Thus the rightmost letter of w' coincides with both m_u and m_v, and if $w = w'w''$, then $b_u = b_v = w''$. Also, if $w' = sm_u = sm_v$, then we have

$$\begin{aligned} f_u &= u_{t-1} \cdots u_1 x^2 s, \\ f_v &= u_{t-1} \cdots u_1 x s. \end{aligned}$$

Since the latter pair of words form an elementary step on level t, we conclude $f_u \sim_t f_v$, as required. $\square$

For the sake of brevity, denote by $f_n(t) = f_n(\mathcal{SL}^t)$ and $p_n(t) = p_n(\mathcal{SL}^t)$ the free spectrum and the p_n-sequence of $\mathcal{SL}^t$, respectively, where $n, t \geq 1$. Recall (e.g. from [6]) that these two sequences are connected by the following simple combinatorial formula:

$$f_n(t) = \sum_{k=0}^{n} \binom{n}{k} p_k(t). \tag{7}$$

Also, the $\sim_t$-class of a word $w \in X^+$ will be denoted by $[w]_t$. Then $X^+/{\sim_t} = \{[w]_t : w \in X^+\}$ is a model for the free algebra of $\mathcal{SL}^t$ over X (the same is true for any finite subset of X), while $p_n(t)$ counts the classes $[w]_t$ such that $c(w)$ is equal to some fixed n-element subset of X.

Theorem 3.3. *We have $p_n(1) = 1$ and $f_n(1) = 2^n - 1$ for all $n \geq 1$. For $n, t \geq 2$, the following recurrence formula holds*:

$$p_n(t) = np_{n-1}(t)(f_n(t-1) + 1),$$

while $p_1(t) = t$.

Proof. Let X_n be an arbitrary but fixed subset of X such that $|X_n| = n$. Let $E_n(t) = \{[w]_t : c(w) = X_n\}$; by our previous remarks, $p_n(t) = |E_n(t)|$. On the other hand, define the set $M_n(t) = \{(x, [w]_t) : x \in X_n, c(w) = X_n \backslash \{x\}\}$; clearly, we have $|M_n(t)| = np_{n-1}(t)$, so that the direct product $M_n(t) \times (X_n^*/{\sim_{t-1}})$ has precisely $np_{n-1}(t)(f_n(t-1)+1)$ elements (recall that $X_n^*/{\sim_{t-1}}$ has an extra element not in $X_n^+/\sim_{t-1}$, which is the $\sim_{t-1}$-class of the empty word). So, to prove the theorem, it suffices to establish a bijection $\psi : E_n(t) \to M_n(t) \times (X_n^*/{\sim_{t-1}})$. Indeed, if we define

$$\psi([w]_t) = (m_w, [f_w]_t, [b_w]_{t-1}),$$

for all $w \in X^+$ such that $c(w) = X_n$, then the previous theorem asserts that ψ is a well-defined injection. Moreover, if $x \in X_n$, $f \in X_n^*$ such that $c(f) = X_n \backslash \{x\}$ and $b \in X_n^*$, then for $u = fxb$ we have $f_u = f$, $m_u = x$ and $b_u = b$, so $\psi([u]_t) = (x, [f]_t, [b]_{t-1})$, which means that ψ is surjective.

Finally, $x^i \sim_t x^j$ cannot be true for $1 \le i < j \le t$, since an elementary step is applicable only to words of length $\ge t$, so no elementary step can be applied to the word x^i. On the other hand, $x^{t+1} \sim_t x^t$, which follows from a step of the form (5) with $u_{t-1} = \cdots = u_1 = x$. This suffices to establish that $p_1(t) = t$. □

4. Computing the Normal Form

We are going to define a sequence of functions $\varphi_t : X^+ \to X^+$, $t \ge 1$, which will yield our normal forms of words with respect to $\sim_t$. For this purpose, we need to fix a total order of X, and the simplest way to do this is to enumerate the letters from X as $X = \{x_1, x_2, \ldots\}$ (whence we set $X_n = \{x_1, \ldots, x_n\}$). The mappings φ_t are defined recursively by the following rules applied to a word $w \in X^+$.

(1) If $c(w) = \{x_{i_1}, \ldots, x_{i_k}\}$ such that $i_1 < \cdots < i_k$, then

$$\varphi_1(w) = x_{i_1} \cdots x_{i_k}.$$

(2) For any $x \in X$ we have

$$\varphi_t(x^k) = x^{\min(k,t)}$$

for all $k, t \ge 1$.

(3) If $t \ge 2$ and $|c(w)| \ge 2$, then

$$\varphi_t(w) = \varphi_t(f_w) m_w \varphi_{t-1}(b_w).$$

There is a particularly convenient way of visualizing the process of computing $\varphi_t(w)$ by using trees. Some of the vertices of the tree will be labeled only by words, and some of them will be labeled by pairs (s, u), where $s \le t$ is a positive integer and u is a word. Initially, the root of the tree is labeled by (t, w). Now, whenever a new vertex (s, u) is added to the tree such that $s \ge 2$ and $|c(u)| \ge 2$, we attach three child-vertices to it, labeled by (s, f_u), m_u (no integer label) and $(s-1, b_u)$, respectively. At the end of this procedure, we are left with a tree in which each leaf is labeled either by a single letter, or by a pair of the form (s, u) such that $u = x_i^k$ for some $k \ge 1$ and $x_i \in X$, or by a pair of the form $(1, u)$. We can finish this process by attaching a single child-vertex labeled by $x_i^{\min(k,s)}$ to each leaf labeled by (s, x_i^k), and a single child-vertex labeled by $\varphi_1(u)$ (see rule (1)) to any leaf labeled by $(1, u)$. Now, each leaf in the modified tree is labeled just by a word, and if we read these labeling words together from the left to the right, the result will be

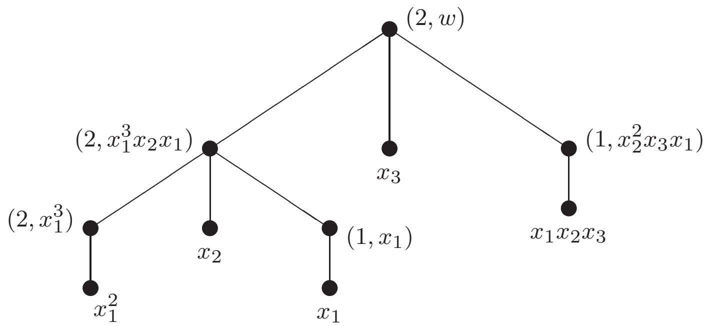

Fig. 1. The labeled tree associated to the computation of $\varphi_2(x_1^3x_2x_1x_3x_2^2x_3x_1)$.

$\varphi_t(w)$. This is illustrated by an example in Fig. 1 for $t = 2$, $w = x_1^3x_2x_1x_3x_2^2x_3x_1$; the corresponding labeled tree easily shows that $\varphi_2(w) = x_1^2x_2x_1x_3x_1x_2x_3$.

The following theorem shows that $\varphi_t(w)$ is indeed a normal form for a word w with respect to $\sim_t$.

Theorem 4.1. (1) *Let $u, v \in X^+$. Then $u \sim_t v$ if and only if $\varphi_t(u) = \varphi_t(v)$.*

(2) *For each $w \in X^+$ we have $w \sim_t \varphi_t(w)$; in fact, $\varphi_t(w)$ is the shortest word in $[w]_t$.*

Proof. (1) First of all, it is quite easy to verify that $c(\varphi_t(w)) = c(w)$ holds for all $w \in X^+$; so, if $c(u) \neq c(v)$, then neither $u \sim_t v$, nor $\varphi_t(u) = \varphi_t(v)$ can hold. Hence, we may assume that $c(u) = c(v)$. The statement is now proved by induction on t and $n = |c(u)| = |c(v)|$.

If either $t = 1$ or $n = 1$, the assertion is evident; thus we assume that $n, t \geq 2$. By Theorem 3.2, $u \sim_t v$ is equivalent to $f_u \sim_t f_v$, $m_u = m_v$ and $b_u \sim_{t-1} b_v$. By the induction hypothesis, $f_u \sim_t f_v$ holds if and only if $\varphi_t(f_u) = \varphi_t(f_v)$, while $b_u \sim_{t-1} b_v$ is equivalent to $\varphi_{t-1}(b_u) = \varphi_{t-1}(b_v)$. So, by rule (3) of the definition of φ_t, we obtain $\varphi_t(u) = \varphi_t(f_u)m_u\varphi_{t-1}(b_u) = \varphi_t(f_v)m_v\varphi_{t-1}(b_v) = \varphi_t(v)$.

Conversely, assume that $\varphi_t(u) = \varphi_t(v)$. Notice that for any $w \in X^+$ we have $m_{\varphi_t(w)} = m_w$ and, consequently, $f_{\varphi_t(w)} = \varphi_t(f_w)$ and $b_{\varphi_t(w)} = \varphi_{t-1}(b_w)$. Therefore, we have $m_u = m_v$, $\varphi_t(f_u) = \varphi_t(f_v)$ and $\varphi_{t-1}(b_u) = \varphi_{t-1}(b_v)$. By the induction hypothesis, we have $f_u \sim_t f_v$ and $b_u \sim_{t-1} b_v$. By Theorem 3.2, this implies $u \sim_t v$.

(2) As in part (1) we note that the assertion is clear if $n = |c(w)| = 1$ or $t = 1$, and proceed by induction on n and t, so that $n, t \geq 2$. By induction hypothesis, $f_w \sim_t \varphi_t(f_w) = f_{\varphi_t(w)}$ and $b_w \sim_{t-1} \varphi_{t-1}(b_w) = b_{\varphi_t(w)}$. Also, we have already noted that $m_w = m_{\varphi_t(w)}$. By Theorem 3.2, we must have $w \sim_t \varphi_t(w)$. Now let $v \in [u]_t$, that is, $v \sim_t u$. Then, again by Theorem 3.2, $f_v \in [f_u]_t$, so $\varphi_t(f_u) = f_{\varphi_t(u)} \in [f_u]_t$ is not longer than f_v. Similarly, $\varphi_{t-1}(b_u) = b_{\varphi_t(u)} \in [b_u]_t$ is not longer than b_v. Of course, $m_u = m_v$. Therefore, $\varphi_t(u) = \varphi_t(f_u)m_u\varphi_{t-1}(b_u) \in [u]_t$ is not longer than $v = f_vm_vb_v$, as wanted. $\square$

Because of the previous theorem, a model of the $\mathcal{SL}^t$-free algebra on X can be constructed on the set $\{\varphi_t(w) : w \in X^+\}$ of all normal forms, where the operation $\circ$ is given by $u \circ v = \varphi_t(uv)$ for any two normal forms u, v. Now we estimate the time complexity of the algorithm for computing the normal form of a product of two normal forms.

Proposition 4.2. *For any $w \in X_n^+$ we have $|\varphi_t(w)| \leq \binom{n+t}{t} - 1$. Given two normal forms u, v (with respect to $\sim_t$) such that $c(u) = c(v) = X_n$, the normal form of their product can be computed in time $\mathcal{O}(n^{2t-1})$.*

Proof. Let $\mu(n, t)$ denote the maximal length of the normal form of a word $w \in X_n^+$ with respect to $\sim_t$. If $n, t \geq 2$, by the rule (3) and the arguments given in the proof of the previous theorem, we have

$$\mu(n, t) = \mu(n - 1, t) + \mu(n, t - 1) + 1,$$

while $\mu(1, t) = t$ and $\mu(n, 1) = n$. The solution of this recurrence is $\mu(n, t) = \binom{n+t}{t} - 1 \in \mathcal{O}(n^t)$.

Now let $\ell(n, t)$ denote the number of leaves on the tree obtained in the process of computing the normal form $\varphi_t(w)$ for a word w such that $c(w) = X_n$. Again, it is not difficult to extract the following recurrence formula from our previous considerations:

$$\ell(n, t) = \ell(n - 1, t) + \ell(n, t - 1) + 1,$$

where $\ell(n, 1) = \ell(1, t) = 1$. The solution of this recurrence is $\ell(n, t) = 2\binom{n+t-2}{t-1} - 1 \in \mathcal{O}(n^{t-1})$. Since every non-leaf vertex of the tree is the parent-vertex of some leaf, and no two leaves have the same parent, the number of vertices in the tree is exactly $2\ell(n, t) \in \mathcal{O}(n^{t-1})$. The number of non-leaf vertices coincides with the number of rules (1)–(3) applied in the course of the algorithm computing $\varphi_t(w)$, and performing each such step requires a linear time with respect to the length of the word that is processed at a particular step. We already know that $|u|, |v| \in \mathcal{O}(n^t)$, so that $|uv| \in \mathcal{O}(n^t)$, and the same holds for any word processed in the course of computing $\varphi_t(uv)$. Since there are $\mathcal{O}(n^{t-1})$ steps involved, the total running time required for calculating $\varphi_t(uv)$ is $\mathcal{O}(n^{2t-1})$. $\square$

5. Upper Bounds for Free Spectra and p_n-Sequences

Throughout this brief section, let "log" refer to the base-2 logarithm. We have already remarked that $f_n(\mathcal{SL}^1) = 2^n - 1$ (so that $\log f_n(\mathcal{SL}^1) \in \mathcal{O}(n)$) and $p_n(\mathcal{SL}^1) = 1$ holds for $n \geq 1$. Using the formula given in Theorem 3.3, we conclude that $p_n(2) = p_{n-1}(2) \cdot n \cdot 2^n$, which easily yields

$$p_n(2) = n! \cdot 2^{\binom{n+1}{2}}.$$

Of course, it is not realistic to expect to obtain nice closed formulae for $f_n(t)$ and $p_n(t)$ when $t \geq 3$. Instead, we aim to determine only the asymptotic behavior of these sequences.

Theorem 5.1. *For each $t \geq 1$ both $\log f_n(t)$ and $\log p_n(t)$ belong to the asymptotic class $\mathcal{O}(n^t)$.*

Proof. First of all, it is not difficult to see (by an argument belonging essentially to elementary calculus) that if $\log p_n(t) \in \mathcal{O}(n^k)$ for some $k \geq 1$, then we have $\log f_n(t) \in \mathcal{O}(n^k)$ as well. We prove the statement of the theorem by induction on t. For $t = 1$, the assertion clearly holds; therefore, let $t \geq 2$.

Rewrite the recurrence relation given in Theorem 3.3 as

$$\log p_n(t) = \log p_{n-1}(t) + \log n + \log(f_n(t-1) + 1).$$

By the induction hypothesis, $\log n + \log(f_n(t-1) + 1) \in \mathcal{O}(n^{t-1})$, so for n large enough we have

$$\log p_n(t) \leq \log p_{n-1}(t) + Cn^{t-1}$$

for some constant $C > 0$. By a simple telescoping of the above inequality, we conclude that for some positive integer n_0, all $n \geq n_0$ and a positive constant B, the following inequality holds:

$$\log p_n(t) \leq B + C \sum_{k=n_0}^{n} k^{t-1}.$$

Hence, $\log p_n(t) \in \mathcal{O}(n^t)$ and thus $\log f_n(t) \in \mathcal{O}(n^t)$. □

Recall (e.g. from [2]) that a *pseudovariety* of finite semigroups is a class of finite semigroups closed under taking homomorphic images, subsemigroups and finite direct products. Let Sl denote the pseudovariety of all finite semilattices, and for $t \geq 1$, let Sl^t be the pseudovariety of finite semigroups generated by all t times iterated semidirect products of members of Sl. In fact, Sl^t is just the class of all finite members of $\mathcal{SL}^t$ (see [1]). By a result of Stiffler [14], $\bigcup_{t\geq 1} \mathsf{Sl}^t$ coincides with the pseudovariety R of all finite $\mathscr{R}$-trivial semigroups, where $\mathscr{R}$ is the Green relation on a semigroup relating elements which generate the same principal right ideal (see [2]). Therefore, each finite $\mathscr{R}$-trivial semigroup belongs to a variety of the form $\mathcal{SL}^t$ for a suitable t, which immediately implies the following conclusion.

Corollary 5.2. *For any finite $\mathscr{R}$-trivial semigroup $\mathbf{S}$, $\log f_n(\mathbf{S}) \in \mathcal{O}(n^k)$ for some $k \geq 1$.*

The same assertion holds if we replace the word "semigroup" by "monoid" since the free spectrum of the semigroup variety generated by a monoid $\mathbf{M}$ and that of the monoid variety generated by $\mathbf{M}$ are asymptotically equivalent. Also, the above corollary holds for finite $\mathscr{L}$-trivial semigroups as well, since the dual semigroup of a $\mathscr{L}$-trivial semigroup is $\mathscr{R}$-trivial, and dual semigroups have identical free spectra.

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