THESES

1. Introduction

Lattices are very important related algebraic structures. They often appear in many branches of algebra, they are clear enough to consider easily, and rich enough to characterize many types of algebraic properties. Here lattices occur in connection with diagrammatic schemes and Maltsev conditions. Moreover, we carry out lattice theoretic investigations on the shift of a lattice identity.

Traditionally in mathematics: " An **invariant** is something that does not change under a set of transformations. The property of being an invariant is **invariance.** "(Wikipedia [Inv1].)

However, beside its strict meaning outlined above, the word 'invariant' has also a more general meaning in universal algebra. We obtain this meaning by replacing transformation, which is a selfmap $A \to A$ of a set A by the notion of algebraic operations. Thus we arrive at the notion of an **invariant relation** ([PK]).

2. Invariance groups of threshold functions

A *threshold function* is a Boolean function, i.e. a mapping $\{0,1\}^n \to \{0,1\}$ with the following property: There exist real numbers $w_1, ..., w_n$, t such that

$$f(x_1, \ldots, x_n) = 1 \text{ iff } \sum_{i=1}^{n} w_i x_i \geq t,$$

where w_i is called the *weight* of x_i for $i = 1, 2, \ldots, n$, and t is a constant called the *threshold value.*

There is a geometrical interpretation of threshold functions. The set $\{0,1\}^n$ can be considered to span a hypercube in the Euclidean space $\mathbf{R}^n$. A Boolean function is defined by assigning either 0 or 1 to the 2^n vertices of the hypercube $\{0,1\}^n$. In the n-dimensional space $\mathbf{R}^n$ the set of vertices where the value of the function is 1 can be separated by a hyperplane from those vertices where the value is 0. This is why threshold functions are also called linearly separable functions ([Sh]).

Threshold functions are useful to study because they are not only models of neurons for example, but also it is easy and relatively cheap to realize them by electrical network ([Sh], [Mu]).

THEOREM 2.1 ([Ho1]). *For every n-ary threshold function f there exists a partition C_f of $\mathbf{n}$ such that the invariance group G of f consists exactly of those permutations of S_n which preserve each block of C_f. Conversely, for every partition C of $\mathbf{n}$ there exists a threshold function f_C such that the invariance group G of f_C consists exactly of those permutations of S_n that preserve each block of C.*

The proof of Theorem 2.1 is based on the following subsidiary statements, and contains only elementary considerations. Let us define the relation $\sim$ on the set $\mathbf{n}$ as follows: $i \sim j$ iff $i = j$ or the transposition $(i\ j)$ leaves f invariant.

Claim 2.1. The partition C_f defined by $\sim$ is convex.

Claim 2.2. Let $\gamma = (j_1\ j_2 \dots j_{k-1}\ l\ j_k \dots j_m) \in S_n$ be a cycle of length $m+1$ with $j_s \in C_p$, $1 \le s \le m$, $l \in C_q$, $p \ne q$. Then $\gamma \notin G$.

Lemma 2.1 ([Ho1]). *If a cycle $\beta \in S_n$ has entries from at least two blocks of C_f, then $\beta \notin G$.*

Claim 2.3. For $\pi \in S_X$, let $\pi = \gamma_1 \dots \gamma_r$ where the γ_i are disjoint cycles. If there exists a γ_j with $1 \le j \le r$ and $\gamma_j \notin G$, then $\pi \notin G$.

Lemma 2.2 ([Ho1]). *Let $\pi \in S_X$ be of the form $\pi = \pi_2\pi_1$, where $\pi_1, \pi_2 \in S_X$, with $M(\pi_1) \cap M(\pi_2) = \emptyset$ and $\pi_1 \notin G$. Then $\pi \notin G$.*

Claim 2.3. For $\pi \in S_X$, let $\pi = \gamma_1 \dots \gamma_r$ where the γ_i are disjoint cycles. If there exists a γ_j with $1 \le j \le r$ and $\gamma_j \notin G$, then $\pi \notin G$.

The following corollary is worth formulating.

Corollary 2.1 ([Ho1]). *The invariance group of any threshold function is isomorphic to a direct product of symmetric groups.*

3. Proving primality by the operation-relation duality

We consider a k-ary relation as a set of unary functions $r\colon \mathbf{k} \to A$, $\mathbf{k} = \{1, 2, \dots, k\}$. We say that a k-ary relation D is *diagonal,* if there exists an equivalence relation ρ_D on $\mathbf{k}$ such that

$$D = \{r\colon \mathbf{k} \to A \mid r(u) = r(v) \text{ if } u \rho_D v,\ u, v \in \mathbf{k}\}.$$

The collection of all diagonal relations on A forms the minimal closed class of relations on A.

The following Proposition 3.1 and Lemma 3.1 and Lemma 3.1' enable us to present new proofs for primality theorems. (The dissertation details only one new proof.)

Proposition 3.1 (Bodnarčuk–Kalužnin–Kotov–Romov [BKKR], Geiger [Gei], Krauss [Kr1], [Kr2]). *A finite algebra* $\mathbf{A} = (A, F)$ *is primal, iff every relation preserved by all operations in* F *is diagonal.*

Lemma 3.1 ([Ho2]). *Given an algebra* $\mathbf{A} = (A, F)$*, the following two conditions are equivalent:*

(i) For each $R \subseteq A^k$*, the relation* $[R]$ *is diagonal.*

(ii) For each $x, y \in A^k$*, the relation* $[x, y]$ *is diagonal.*

Lemma 3.1′ ([Ho2]). *The following three conditions are equivalent:*

(i) The algebra $\mathbf{A} = (A, F)$ *is primal.*

(ii) For each $x, y, z \in A^k$*, we have* $z \in [x, y]$ *whenever*

$$((\forall u, v \in \mathbf{k}\ )(x(u) = x(v) \wedge y(u) = y(v) \rightarrow z(u) = z(v))).$$

(iii) For each $k \geq 1$ $x, y, z \in A^k$*, and for any equivalence* ρ *on* $\mathbf{k}$ *if* $\rho \supseteq \rho_x \cap \rho_y$*, then* $D_\rho \subseteq [x, y]$.

By Lemma 3.1 the problem of proving a primality theorem simplifies to the investigation of some suitably chosen matrices. We demonstrate our method on the Słupecki Criterion in detail.

THEOREM 3.1 (Słupecki [Sl]). *Let* A *be a finite set with* $|A| > 2$*. If* F *contains an essential operation* f *and all the unary operations, then the algebra* $\mathbf{A} = (A, F)$ *is primal.*

We can prove the functional completeness of the ternary discriminator ([Sz]), the dual discriminator (for $|A| \geq 3$) ([FP]), the n-ary ($n \geq 3$) near-projections ([Cs1]) as well as the primality theorem of Foster ([F]) this way.

4. Diagrammatic schemes

Motivated by Gumm's Shifting Lemma ([Gu1]), which asserts that congruence modular varieties satisfy a nice rectangular congruence scheme, Chajda ([ChH1], Subdivision 4.2) investigated a triangular scheme, which is a consequence of congruence distributivity. Congruence distributive varieties satisfy this scheme not only for arbitrary three congruences, but also for one tolerance (i.e., compatible, reflexive and symmetric binary relation) and two congruences; i.e., the analogue of Gumm's Shifting Principle is valid.

Definition 4.3. An algebra $\mathbf{A} = (A, F)$ satisfies the *Triangular Principle* if for each tolerance Φ and congruences β, γ the implication depicted in Figure 5 holds.

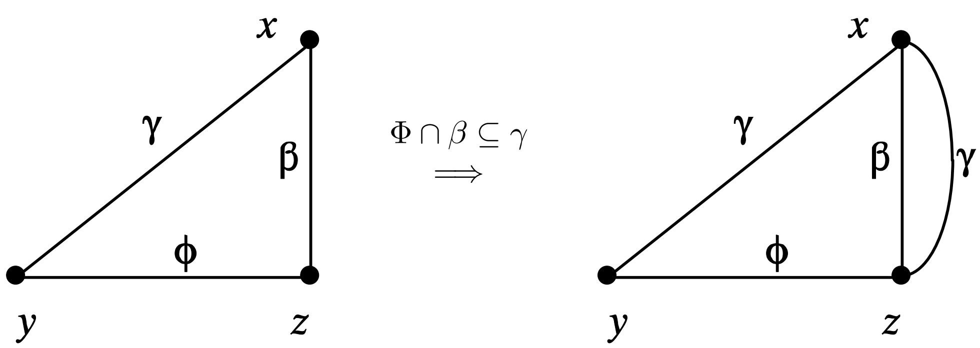

Figure 1

THEOREM 4.3 ([ChH1]). *In congruence distributive varieties (i. e. in the algebras of such varieties) the Triangular Principle holds.*

While the triangular scheme is not known to characterize congruence distributivity, an appropriate generalization called trapezoid scheme does ([CCH2], Subdivision 4.3).

We introduce a new condition under the name *Trapezoid Lemma* as follows: for any $\alpha, \beta, \gamma \in \mathbf{Con\,A}$ (where $\mathbf{A} = (A, F)$ is an algebra) if $\alpha \cap \beta \subseteq \gamma$, $(x, u), (y, v) \in \alpha$, $(x, y) \in \beta$ and $(u, v) \in \gamma$, then $(x, y) \in \gamma$. The Trapezoid Lemma is depicted in Figure 7.

The corresponding condition called *Trapezoid Principle* is defined similarly, the only difference is that α should be replaced by Φ, which stands for an arbitrary *tolerance* of $\mathbf{A}$.

The following Proposition 4.2 presents some connections among our conditions in case of a single algebra; for varieties of algebras in Theorem 4.4 we state more.

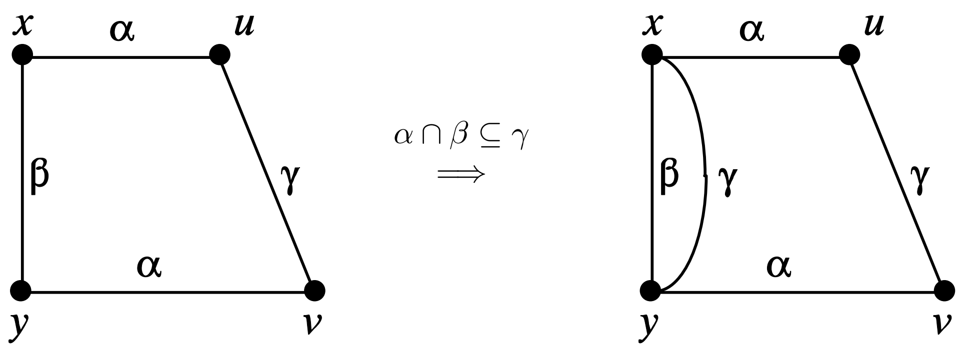

Figure 2

Proposition 4.2 [(CCH1)]. *Let* **A** *be an algebra.*

(1) If **A** *satisfies the Trapezoid Lemma resp. the Trapezoid Principle, then it satisfies the Rectangular Lemma and the Triangular Lemma resp. the Rectangular Principle and the Triangular Principle. Moreover, each of the three principles implies the corresponding lemma.*

(2) If **Con A** *is distributive, then* **A** *satisfies the Trapezoid Lemma (and therefore the other two lemmas as well).*

(3) If **A** *satisfies the Trapezoid Principle, then* **Con A** *is distributive.*

(4) If **A** *satisfies the Rectangular Principle, then* **Con A** *is modular (cf. [Gu2], Lemma 4.2).*

(5) If **A** *is congruence permutable, then* **Con A** *is distributive if and only if* **A** *satisfies the Triangular Lemma (cf.* [CzH1], *Cor. 2).*

THEOREM 4.4 ([CCH1]). *Let $\mathcal{V}$ be a variety of algebras. Then the following five conditions are equivalent.*

(a) $\mathcal{V}$ is congruence distributive;

(b) the Trapezoid Principle holds in $\mathcal{V}$;

(c) the Trapezoid Lemma holds in $\mathcal{V}$;

(d) the Rectangular Lemma and the Triangular Lemma hold in $\mathcal{V}$;

(e) there is a positive integer n and there are quaternary terms $d_0, d_1, \ldots, d_n$ such that the identities

(e1) $d_0(x, y, u, v) = x, \quad d_n(x, y, u, v) = y,$

(e2) $d_i(x, y, x, y) = d_{i+1}(x, y, x, y)$ *for i even,*

(e3) $d_i(x, y, z, z) = d_{i+1}(x, y, z, z)$ *for i odd, and*

(e4) $d_i(x, x, y, z) = x$ *for all i*

hold in $\mathcal{V}$.

These examples show that instead of identities in congruence lattices, diagrammatic statements are reasonable to consider. This phenomenon can be extended to lattice Horn sentences as well (congruence semidistributivity, ([ChH2], Subdivision 4.4).

5. Shifting lattice identities

Let

$$\lambda : \quad p(x_1, \ldots, x_n) \le q(x_1, \ldots, x_n)$$

be a lattice identity. (Notice that by a lattice identity we always mean an inequality, i.e. we use $\le$ but never $=$.) If y is a variable, then let $S(\lambda, y)$ denote the Horn sentence

$$q(x_1, \ldots, x_n) \le y \Longrightarrow p(x_1, \ldots, x_n) \le y.$$

If $y \notin \{x_1, \ldots, x_n\}$, then λ is clearly equivalent to $S(\lambda, y)$. However, we are interested in the case when $y \in \{x_1, \ldots, x_n\}$, say $y = x_i$ $(1 \le i \le n)$. Then $S(\lambda, x_i)$ is a consequence of λ. When $S(\lambda, x_i)$ happens to be equivalent to λ, then $S(\lambda, x_i)$ will be called *a shift of* λ. If $S(\lambda, x_i)$ is equivalent to λ only within a lattice variety $\mathcal{V}$, then we say that $S(\lambda, x_i)$ *is a shift of* λ *in* $\mathcal{V}$. In this chapter some known lattice identities will be shown to have a shift while some others have no shift.

Following Huhn ([Hu1]) and ([Hu2]), a lattice $\mathbf{L}$ is said to be n-distributive $(n \ge 1)$ if the identity

$$\mathrm{dist}_n : \quad \beta \sum_{i=0}^{n} \alpha_i \le \sum_{j=0}^{n} \left(\beta \sum_{i \in \{0,\ldots,n\} \setminus \{j\}} \alpha_i \right)$$

holds in $\mathbf{L}$.

THEOREM 5.1 ([CCH2]). *$S(dist_n, \alpha_0)$ is a shift of $dist_n$ in the variety of modular lattices. However, if $n \ge 2$, then $dist_n$ has no shift (in the variety of all lattices).*

The next group of lattice identities we consider is taken from McKenzie [Mc].

These identities are as follows:

$$\begin{aligned}
\zeta_0:&\quad (x+y(z+xy))(z+xy)\le y+(x+z(x+y))(y+z),\\
\zeta_1:&\quad x(xy+z(w+xyz))\le xy+(z+w)(x+w(x+z)),\\
\zeta_2:&\quad (x+y)(x+z)\le x+(x+y)(x+z)(y+z),\\
\zeta_3:&\quad (x+yz)(z+xy)\le z(x+yz)+x(z+xy),\text{and}\\
\zeta_4:&\quad y(z+y(x+yz))\le x+(x+y)(z+x(y+z)).
\end{aligned}$$

Notice that ζ_3 is Gedeonová's p-modularity, ([Ged1]).

THEOREM 5.2 ([CCH2]). *$S(\zeta_0, y)$, $S(\zeta_1, y)$, $S(\zeta_2, x)$, and $S(\zeta_3, y)$ are shifts of ζ_0, ζ_1, ζ_2 and ζ_3, respectively. On the other hand, ζ_4 has no shift.*

The Fano identity (cf. e.g. Herrmann and Huhn ([HH])) is the following:

$$\chi_2: \quad (x+y)(z+t)\le (x+z)(y+t)+(x+t)(y+z)$$

THEOREM 5.3 ([CCH2]). *The Fano identity has no shift — not even in the variety of modular lattices.*

6. Tolerances and tolerance lattices

Let $\operatorname{dist}(x,y,z)$ resp. $\operatorname{mod}(x,y,z)$ denote the distributive law

$$x\wedge(y\vee z)\le(x\wedge y)\vee(x\wedge z)$$

resp. the modular law

$$x\wedge(y\vee(x\wedge z))\le(x\wedge y)\vee(x\wedge z).$$

For an algebra $\mathbf{A}$, the *set* of tolerances and the *lattice* of congruences of $\mathbf{A}$ will be denoted by $\mathbf{Tol\,A}$ and $\mathbf{Con\,A}$, respectively. We say that dist(tol,tol,tol) holds in $\mathbf{A}$ if $\Gamma\wedge(\Phi\vee\Psi)\subseteq(\Gamma\wedge\Phi)\vee(\Gamma\wedge\Psi)$ is valid for any $\Gamma,\Phi,\Psi\in\mathbf{Tol\,A}$, where the meet resp. the join is the intersection resp. the transitive closure of the union. The meaning of mod(tol,tol,tol) is analogous. We should emphasize here that $\Phi\vee\Psi$ is **not** the join in $\mathbf{Tol\,A}$, the lattice of tolerance relations of $\mathbf{A}$. With the help of Jónsson terms ([J1]) we proved the next theorem:

THEOREM 6.1 ([CzH2]). *If $\mathcal{V}$ is a congruence distributive resp. congruence modular variety, then dist(tol,tol,tol) resp. mod(tol,tol,tol) holds in all algebras of $\mathcal{V}$.*

Two important consequences are formulated in Corollary 6.1 and Proposition 6.1.

Corollary 6.1 (Gumm [Gu1]). *If $\mathcal{V}$ is a congruence modular variety, then it satisfies Gumm's Shifting Principle, i.e. for any $\mathbf{A} \in \mathcal{V}$, $\alpha, \gamma \in$ **Con A** and $\Phi \in$ **Tol A** if $(x,y),(u,v) \in \alpha$, $(x,u),(y,v) \in \Phi$, $(u,v) \in \gamma$ and $\alpha \cap \Phi \subseteq \gamma$, then $(x,y) \in \gamma$.*

Denoting the transitive closure by *, the following proposition is an essential step towards the Maltsev conditions in Chapter 7:

Proposition 6.1 ([CzH2]). *If mod(tol,tol,tol) or dist(tol,tol,tol) holds in an algebra $\mathbf{A}$, then $\Gamma \cap \Phi^* \subseteq (\Gamma \cap \Phi)^*$ for any $\Gamma, \Phi \in$ **Tol A**.*

A lattice **L** with 0 is called *0-modular,* cf. Stern ([St]), if there is no N_5 sublattice of **L** including 0. The lattice **L** with 0 satisfies the *general disjointness property* (GD) if $a \wedge b = 0$ and $(a \vee b) \wedge c = 0$ imply $a \wedge (b \vee c) = 0$. If for each $a \in \mathbf{L}$ the set $\{x \in \mathbf{L} : \mathrm{a} \wedge \mathrm{x} = 0\}$ has greatest element, then **L** is called a *pseudocomplemented* lattice.

The following Theorem 6.2 and 6.3 are the main results about tolerance lattices in congruence modular varieties.

THEOREM 6.2 ([CHR]). A be an algebra in a congruence modular variety $\mathcal{V}$. Then the following statements hold:

(i) The map h: **Tol A** $\to$ **Con A**, $\Phi \mapsto \Phi^*$, *is a surjective lattice homomorphism and* **Tol A** *is a* 0-1 *modular lattice having the (GD) property.*

(ii) **Tol A** *is pseudocomplemented if and only if* **Con A** *is pseudocomplemented.*

THEOREM 6.3 ([CHR]). *Let* **A** *be an algebra. If* **A** *has a majority term, then:*

(i) **Tol A** *is a 0-modular pseudocomplemented lattice.*

(ii) The tolerances Γ, Φ are complements of each other in **Tol A** *if and only if they form a factor congruence pair of A.*

7. Maltsev conditions for congruence lattice identities in modular varieties

It is a 34 year old problem if all congruence lattice identities are equivalent to Maltsev conditions. In other words, we say that a lattice identity λ can be characterized by a Maltsev condition if there exists a Maltsev condition M such that, for any variety $\mathcal{V}$, λ holds in congruence lattices of all algebras in $\mathcal{V}$ if and only if M holds in $\mathcal{V}$; and the problem is if all lattice identities can be characterized this way. This problem was raised first in Grätzer ([Gr1]), where the notion of a Maltsev condition was defined. A *strong Maltsev condition* for varieties is a condition of the form "there exist terms $h_0, \dots, h_k$ satisfying a set Σ of identities" where k is fixed and the form of Σ is independent of the type of algebras considered. By a *Maltsev condition* we mean a condition of the form "there exists a natural number n such that P_n holds" where the P_n are strong Maltsev conditions and P_n implies P_{n+1} for every n. The condition "P_n implies P_{n+1}" is usually expressed by saying that a Maltsev condition must be weakening in its parameter. (For a more precise definition of Maltsev conditions cf. [T].) The problem was repeatedly asked by several authors, including Taylor ([T]), Jónsson ([J2]) and Freese and McKenzie ([FM]).

Certain lattice identities have known characterizations by Maltsev conditions. The first two results of this kind are Jónsson's characterization of (congruence) distributivity by the existence of Jónsson terms, cf. Jónsson ([J1]), and Day's characterization of (congruence) modularity by the existence of Day terms, cf. Day ([D1]). Since Day's result will be needed in the sequel, we formulate it now. For $n \geq 2$ let $(\mathbf{D}_n)$ denote the strong Maltsev condition "there are quaternary terms $m_0, \dots, m_n$ satisfying the identities

$$\begin{gathered} m_0(x,y,z,u) = x, \qquad m_n(x,y,z,u) = u, \\ m_i(x,y,y,x) = x \quad \text{for } i = 0,1,\dots,n, \\ m_i(x,x,y,y) = m_{i+1}(x,x,y,y) \quad \text{for } i = 0,1,\dots,n, \quad i \text{ even}, \\ m_i(x,y,y,z) = m_{i+1}(x,y,y,z) \quad \text{for } i = 0,1,\dots,n, \quad i \text{ odd}". \end{gathered}$$

Now Day's celebrated result says that a variety $\mathcal{V}$ is congruence modular iff the Maltsev condition "$(\exists n)(\mathbf{D}_n)$" holds in $\mathcal{V}$.

Jónsson terms and Day terms were soon followed by some similar characterizations for other lattice identities, given for example by Gedeonová ([Ge2]) and Mederly ([Me]), but Nation ([N]) and Day ([Da2]) showed that these Maltsev conditions are equivalent to the existence of Day terms or Jónsson terms; the reader is referred to Jónsson ([Jo2]) and Freese and McKenzie ([FM]) for more details.

The next milestone is Chapter XIII in Freese and McKenzie's book ([FM]). Let us call a lattice identity λ in n^2 variables a *frame identity* if λ implies modularity and λ holds in a modular lattice iff it holds for the elements of every (von Neumann) n-frame of the lattice. Freese and McKenzie showed that frame identities can be characterized by Maltsev conditions. Although that time there was a hope that their method combined with [HC] gives a Maltsev condition for each λ that implies modularity, cf. [FM], Pálfy and Szabó ([PSz]) destroyed this expectation.

The goal of the present chapter is to prove that each lattice identity implying modularity is equivalent to a Maltsev condition. Moreover, this Maltsev condition is very easy to construct. A lattice identity λ is said to imply modularity in congruence varieties, in notation $\lambda \models_c$ mod if for any variety $\mathcal{V}$ if all the congruence lattices **Con A**, $\mathbf{A} \in \mathcal{V}$, satisfy λ, then all these lattices are modular.

Given a lattice term p and $k \geq 2$, we define p_k via induction as follows. If p is a variable, then let $p_k = p$. If $p = r \wedge s$, then let $p_k = r_k \cap s_k$. Finally, if $p = r \vee s$, then let $p_k = r_k \circ s_k \circ r_k \circ s_k \circ \dots$ with k factors on the right. When congruences or, more generally, reflexive compatible relations are substituted for the variables of p_k, then the operations $\cap$ and $\circ$ will be interpreted as intersection and relational product, respectively.

Our first result about Maltsev conditions is Theorem 7.1.

THEOREM 7.1 ([CzH3]). *Let $\lambda : p \leq q$ be a lattice identity such that $\lambda \models_c$ modularity. Then for any variety $\mathcal{V}$ the following two conditions are equivalent.*

(a) For all $\mathbf{A} \in \mathcal{V}$, λ holds in the congruence lattice of $\mathbf{A}$.

(b) $\mathcal{V}$ satisfies the Maltsev condition "there is an $n \geq 2$ such that $M(p_3 \subseteq q_n)$ and $(\mathbf{D}_n)$ hold".

An algebra **A** is said to satisfy the *tolerance intersection property* , TIP for short, if for any two tolerances (i.e. reflexive symmetric compatible relations) Γ and Φ of **A** we have

$$\Gamma^* \cap \Phi^* = (\Gamma \cap \Phi)^*$$

where * stands for transitive closure. In the proof of Theorem 7.1 we already proved the following statement:

THEOREM 7.2 ([CHL]). *Every algebra in a congruence modular variety satisfies TIP.*

Next we improve Theorem 7.1 by giving the simplest (and in this sense hopefully the best) Maltsev condition associated with λ when $\lambda \models_c$ modularity.

For a term $p = p(x_1, \ldots, x_k)$ in the binary operations $\cap, \vee, \circ$, in short for a $\{\cap, \vee, \circ\}$-term, and for $n \geq 2$ we define two kinds of derived $\{\cap, \circ\}$-terms, p_n and $p_{2,2}$ via induction as follows. If p is a variable, then let $p_n = p_{2,2} = p$. If $p = r \cap s$, then let $p_n = r_n \cap s_n$ and $p_{2,2} = r_{2,2} \cap s_{2,2}$. Similarly, if $p = r \circ s$, then let $p_n = r_n \circ s_n$ and $p_{2,2} = (r_{2,2} \circ s_{2,2}) \cap (s_{2,2} \circ r_{2,2})$. Finally, if $p = r \vee s$, then let $p_n = r_n \circ s_n \circ r_n \circ s_n \circ \cdots$ with n factors on the right and $p_{2,2} = (r_{2,2} \circ s_{2,2}) \cap (s_{2,2} \circ r_{2,2})$.

THEOREM 7.4 ([CHL]). *Let $p \subseteq q$ be a congruence inclusion formula with q being $\circ$-free. (I.e. p is a $\{\cap, \vee, \circ\}$-term and q is a lattice term.) Then for any congruence modular variety $\mathcal{V}$ the following conditions are equivalent.*

(i) $p \subseteq q$ holds for congruences of $\mathcal{V}$,
(ii) $p_2 \subseteq q$ holds for congruences of $\mathcal{V}$,
(iii) $p_{2,2} \subseteq q$ holds for congruences of $\mathcal{V}$,
(iv) the Maltsev condition

$$(\exists n \geq 2) \big(M(p_2 \subseteq q_2 \circ q_2 \circ \cdots \circ q_2)\big)$$

(where $q_2 \circ q_2 \circ \cdots \circ q_2$ denotes a product of n factors) holds in $\mathcal{V}$.

As a corollary, we obtain the desired improvement of Theorem 7.1:

Corollary 7.2 ([CHL]). *Let $\lambda : p \leq q$ be a lattice identity such that $\lambda \models_c$ modularity. Then for any variety $\mathcal{V}$ the following three conditions are equivalent.*

(a) For all $\mathbf{A} \in \mathcal{V}$, λ holds in the congruence lattice of $\mathbf{A}$.

(b') $\mathcal{V}$ satisfies the Maltsev condition "there is an $n \geq 2$ such that $M(p_2 \subseteq q_n)$ and $(\mathbf{D}_n)$ hold".

(c) $\mathcal{V}$ satisfies the Maltsev condition "there is an $n \geq 2$ such that $M(p_2 \subseteq q_2 \circ q_2 \circ \cdots \circ q_2)$ (with n factors) and and $(\mathbf{D}_n)$ hold".

REFERENCES

[BKKR] V. G. Bodnarčuk, L. A. Kalužnin, V. N. Kotov, B. A. Romov: Galois theory for Post algebras, I–II (Russian), *Kibernetika*, **5** (1969), 1–10; 1–9..

[CCH1] I. Chajda, G. Czédli, E. K. Horváth.: Trapezoid lemma and congruence distributivity, *Math Slovaca*, **53** (2003), No.3, 247–253.

[CCH2] I. Chajda, G. Czédli, E. K. Horváth: The shifting lemma and shifting lattice identities, *Algebra Universalis*, **50** (2003), 51–60.

[ChH1] I. Chajda, E. K. Horváth: A triangular scheme for congeruence distributivity, *Acta Sci. Math. (Szeged)*, **68** (2002), 29–35.

[ChH2] I. Chajda, E. K. Horváth: A scheme for congruence semidistributivity, *Discussiones Mathematicae, General Algebra and Applications*, **23** (2003), 13–18.

[Cs1] B. Csákány: Homogeneous algebras are functionally complete, *Algebra Universalis*, **11** (1980), 149–158.

[CzH1] G. Czédli, E. K. Horváth: Reflexive relations and Mal'tsev conditions for congruence lattice identities in modular varieties, *Acta Univ. Palacki Olomouc., Fac. rer. nat., Mathematica*, **41** (2002), 43–53.

[CzH2] G. Czédli, E. K. Horváth: Congruence distributivity and modularity permit tolerances, *Acta Univ. Palacki. Olomouc., Fac.rer.nat., Mathematica*, **41** (2002), 39–42.

[CzH3] G. Czedli, E. K. Horváth: All congruence lattice identities implying modularity have Malt'sev conditions, *Algebra Universalis*, **50** (2003), 69–74.

[CHR] G. Czédli, E. K. Horváth, S. Radeleczki: On tolerance lattices of algebras in congruence mudular varieties, *Acta Math. Hungar.*, **100 (1–2)** (2003), 9–17.

[CHL] G. Czédli, E. K. Horváth, P. Lipparini: Optimal Mal'tsev conditions for congruence modular varieties, *Algebra Universalis*, to appear.

[Da1] A. Day: A characterization of modularity for congruence lattices of algebras, *Canad. Math. Bull.*, **12** (1969), 167–173.

[Da2] A. Day: p-modularity implies modularity in equational classes, *Algebra Universalis*, **3** (1973), 398–399.

[F] A. L. Foster: An existence theorem for functionally complete universal algebras, *Math Z.*, **71** (1959), 69–82.

[FM] R. Freese, R. McKenzie: *Commutator theory for congruence modular varieties*, Cambridge Univ. Press, Cambridge, 1987.

[Ged1] E. Gedeonová: Jordan-Hölder theorem for lines, *Mat. Časopis Sloven. Akad. Vied.*, **22.** (1972), 177–198

[Ged2] E. Gedeonová: A characterization of p-modularity for congruence lattices of algebras, *Acta Fac. Rerum Natur. Univ Comenian. Math Publ.*, **28** (1972), 99–106.

[Gei] D. Geiger: Closed systems of functions and predicates, *Pacific J. Math.*, **27** (1968), 95–100.

[Gr1] G. Gratzer: Two Mal'cev-type theorems in universal algebra , *J. Combinatorial Theory* , **8** (1970), 334–342.

[Gu1] H.-P. Gumm: *Geometrical methods in congruence modular algebras*, Memoirs of the Amer. Math. Soc.. **45** (1983)

[Gu2] H.-P. Gumm: Congruence modularity is permutability composed with distributivity, *Arch. Math. (Basel)*, **36** (1981), 569–576.

[HH] C. Herrmann, A. P. Huhn: Zum Begriff der Charakteristik modulárer Verbände, *Math Z.*, **144** (1975), 185–194.

[Ho1] E. K. Horváth: Invariance groups of threshold functions, *Acta Cybernetica.*, **11** (1994), 325–332.

[Ho2] E. K. Horváth: The Słupecki criterion by duality, *Discussiones Mathematicae (General Algebra and Applications.*, **21** (2001), 5–10.

[Hu1] A. P. Huhn: Schwach distributive Verbände I, *Acta Sci. Math. (Szeged)*, **33** (1972), 297–305.

[Hu2] A. P. Huhn: On Grätzer's problem concerning automorphisms of a finitely presented lattice, *Algebra Universalis.*, **5** (1975), 65–71.

[J1] B. Jónsson: Algebras whose congruence lattices are distributive, *Math. Scand.*, Polygon. **21** (1967), 110–121

[J2] B. Jónsson: Congruence varieties, *Algebra Universalis*, **10** (1980), 355–394.

[Kr1] P. H. Krauss: On primal algebras, *Algebra Universalis*, **I.X.** (1999), 37–46.

[Kr2] P. H. Krauss: On quasi primal algebras, *Math Z.*, **134** (1973), 85–89.

[Mc] R. McKenzie: Equational bases and non-modular lattice varieties, *Trans. Amer. Math. Soc.*, **174** (1972), 1–43.

[Me] P. Mederly: Three Mal'cev type theorems and their application, *Mat. Casopis Sloven. Akad. Vied* , **25** (1975), 83–95.

[N] G. B. Nation: Varieties whose congruences satisfy certain lattice identities, *Algebra Universalis*, **4** (1974), 78–88.

[PSz] P. P. Pálfy, Cs. Szabó: An identity for subgroup lattices of abelian group, *Algebra Universalis*, **33.** (1995), 191–195

[PK] R. Pöschel, L. A. Kalushnin: *Funktionen- und Relationenalgebren*, Verlag der Wissenschaften, Berlin, 1979.

[Sh] Ching Lai Sheng: *Threshold logic (H. G. Booker and N. DeClaris, eds.*, Academic Press, London and New York, 1969..

[Sl] J. Słupecki: Completeness criterion for systems of many-valued propositional calculus (in Polish), *C. R. des Séances de la Societé des Sciences et des Lettres de Varsovie Cl. II*, **32** (1939), 102–109.

[St] M. Stern: *Semimodular lattices, Theory and applications*, Cambridge-New York-Melbourne, 1999.

[Sz] Á. Szendrei: *Clones in Universal Algebra*, Université de Montréal, 1986.

[T] W. Taylor: Characterizing Mal'cev conditions, *Algebra Universalis*, **3** (1973), 351–397.

[Inv1] http://en.wikipedia.org/wiki/Invariants