A triangular scheme for congruence distributivity

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Dedicated to Professor Béla Csákány on his seventieth birthday

Abstract. We introduce a triangular scheme for congruences which is satisfied in any congruence distributive algebra $\mathcal{A}$. A condition called Weak Triangular Principle is studied, which is equivalent to the distributivity of $\mathbf{Con}\,\mathcal{A}$ for an arbitrary algebra $\mathcal{A}$. It follows that if $\mathcal{A}$ is congruence permutable then the Triangular Scheme is equivalent to the distributivity of $\mathbf{Con}\,\mathcal{A}$. We define the Triangular Principle as well, which is shown to hold in congruence distributive varieties.

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H.-P. Gumm [1] defined a certain rectangular scheme for relations of an algebra $\mathcal{A}$. He shows that if $\mathbf{V}$ is a congruence modular variety then this scheme is satisfied by suitable relations of members of $\mathbf{V}$ (the so called *Shifting Lemma* and *Shifting Principle*) and, conversely, if the scheme is satisfied by some congruences of an appropriate free algebra in $\mathbf{V}$ then $\mathbf{V}$ is congruence modular. We will show that a similar reasoning can be useful in the case of congruence distributivity.

Definition 1. An algebra $\mathcal{A} = (A, F)$ satisfies the *Triangular Scheme* if for any $x, y, z \in \mathcal{A}$ and every $\alpha, \beta, \gamma \in \mathbf{Con}\,\mathcal{A}$ with $\alpha \cap \beta \subseteq \gamma$ the following implication holds:

$$\text{if } \langle x, y\rangle \in \gamma,\ \langle x, z\rangle \in \alpha, \langle z, y\rangle \in \beta \text{ then } \langle y, z\rangle \in \gamma.$$

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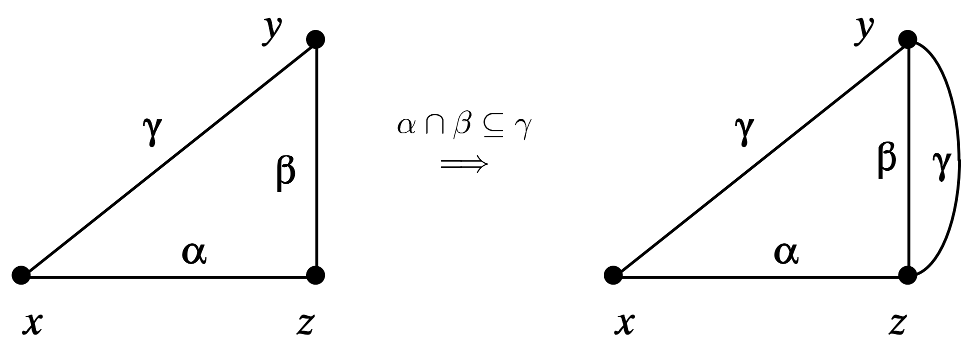

Figure 1

Remark. The Triangular Scheme can be visualized as shown in Figure 1.

Triangular Lemma. *Every congruence distributive algebra satisfies the Triangular Scheme.*

P r o o f. Suppose that $\mathcal{A} = (A, F)$ is congruence distributive, $x, y, z \in \mathcal{A}$, $\alpha, \beta, \gamma \in \mathbf{Con}\,\mathcal{A}$ with $\alpha \cap \beta \subseteq \gamma$ and $\langle x, y\rangle \in \gamma$, $\langle x, z\rangle \in \alpha$, $\langle z, y\rangle \in \beta$. Then $\langle z, y\rangle \in \beta \cap (\alpha \circ \gamma) \subseteq \beta \cap (\alpha \vee \gamma)$ and, due to congruence distributivity, also $\langle z, y\rangle \in (\beta \cap \alpha) \vee (\beta \cap \gamma) \subseteq \gamma \vee (\beta \cap \gamma) = \gamma$. $\diamond$

For $\mathcal{A}$ congruence permutable the converse assertion also holds, cf. Corollary 2 later.

Now let us introduce the following concept:

Definition 2. Given $n \in \mathbf{N}$ and an algebra $\mathcal{A} = (A, F)$, we say that $\mathcal{A}$ satisfies the *Weak Triangular Principle for n* if for any $x, y, z \in \mathcal{A}$ and every $\alpha, \beta, \gamma \in \mathbf{Con}\,\mathcal{A}$ with $\alpha \cap \beta \subseteq \gamma$ and $\Lambda_n = \gamma \circ \alpha \circ \gamma \circ \ldots$ (n factors) the following implication holds:

$$\text{if } \langle x, z\rangle \in \alpha,\ \langle z, y\rangle \in \beta,\ \langle x, y\rangle \in \Lambda_n \text{ then } \langle z, y\rangle \in \gamma.$$

If $\mathcal{A}$ satisfies the Weak Triangular Principle for all $n \in \mathbf{N}$ then we simply say that $\mathcal{A}$ satisfies the *Weak Triangular Principle.*

Remark. The Weak Triangular Principle can be visualized as shown in Figure 2.

Theorem 1. *An algebra $\mathcal{A}$ satisfies the Weak Triangular Principle if and only if* $\mathbf{Con}\,\mathcal{A}$ *is distributive.*

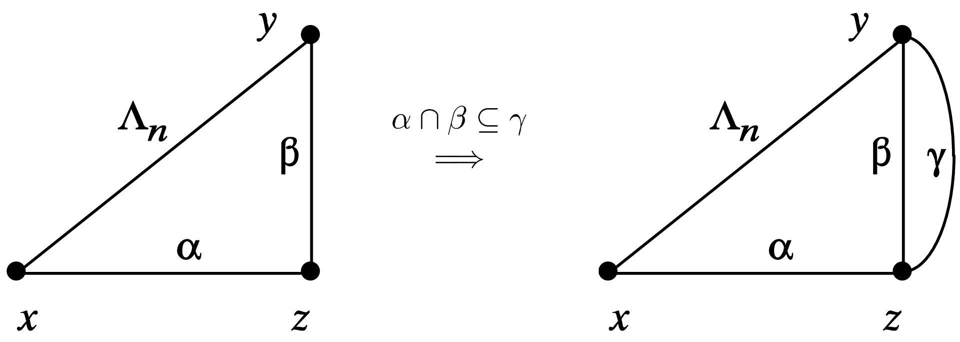

Figure 2

P r o o f. (a) Let $\mathbf{Con}\,\mathcal{A}$ be distributive. Let $\alpha, \beta, \gamma \in \mathbf{Con}\,\mathcal{A}$ and $\alpha \cap \beta \subseteq \gamma$. Suppose $\langle x, z\rangle \in \alpha, \langle y, z\rangle \in \beta$ and $\langle x, y\rangle \in \Lambda_n$ for some $n \in N$. Then $\langle z, y\rangle \in \beta \cap (\alpha \circ \Lambda_n) \subseteq \beta \cap (\alpha \vee \gamma) = (\beta \cap \alpha) \vee (\beta \cap \gamma) \subseteq \gamma \vee (\beta \cap \gamma) = \gamma$, thus $\mathcal{A}$ satisfies the Weak Triangular Principle.

(b) Suppose now that $\mathcal{A}$ satisfies the Weak Triangular Principle but $\mathbf{Con}\,\mathcal{A}$ contains a sublattice isomorphic to M_3 or N_5, i.e. there exist distinct $\alpha, \beta, \gamma \in \mathbf{Con}\,\mathcal{A}$ such that the situation of Figure 3 holds.

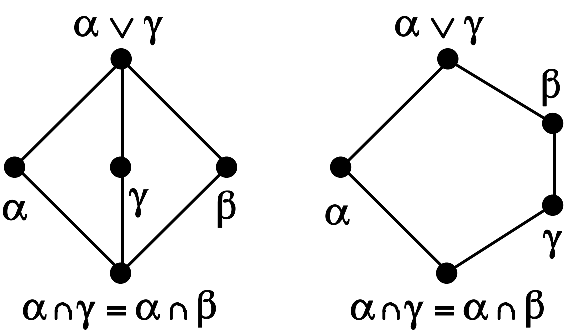

Figure 3

In both the cases, we have $\alpha \cap \beta \subseteq \gamma$. Let $\langle z, y\rangle \in \beta = \beta \cap (\alpha \vee \gamma)$. Then there exists $n \in \mathbf{N}$, such that $\langle z, y\rangle \in \beta \cap (\alpha \circ \Lambda_n)$ for $\Lambda_n = \gamma \circ \alpha \circ \gamma \circ \ldots$ (n factors) and, by the Weak Triangular Principle, we obtain $\langle z, y\rangle \in \beta \cap \gamma$. I.e., $\beta \subseteq \beta \cap \gamma$, a contradiction. $\diamond$

Remark. The distributivity of $\mathbf{Con}\,\mathcal{A}$ is (by Theorem 1) equivalent to the implication depicted in Figure 4.

In the case of a k-permutable algebra $\mathcal{A}$ we need not require the satisfaction of the Weak Triangular Principle for each $n \in \mathbf{N}$, for Theorem 2 yields almost

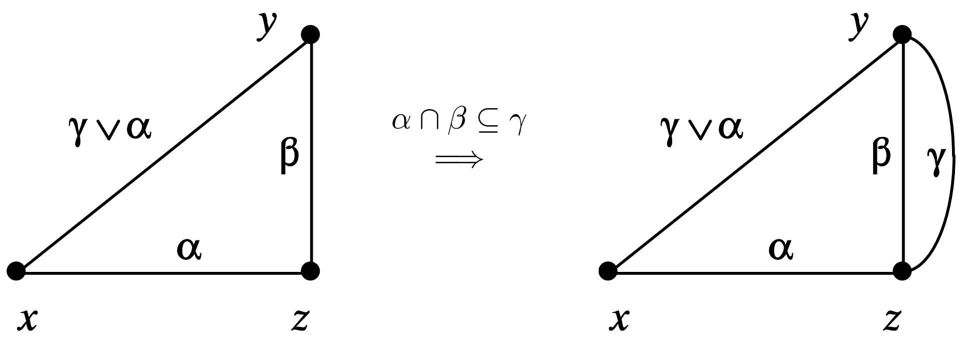

Figure 4

immediately the following

Corollary 1. *Let $\mathcal{A}$ be a k-permutable algebra. Then* $\mathbf{Con}\,\mathcal{A}$ *is distributive if and only if $\mathcal{A}$ satisfies the Weak Triangular Principle for $n = k - 1$.*

When $k = 2$, Corollary 1 yields the following assertion.

Corollary 2. *Let $\mathcal{A}$ be a congruence permutable algebra. Then $\mathcal{A}$ is congruence distributive if and only if $\mathcal{A}$ satisfies the Triangular Scheme.*

One can ask for an example of an algebra $\mathcal{A}$ satisfying the Triangular Scheme but not the Weak Triangular Principle, i.e. whose congruence lattice is not distributive. A suitable one is given below.

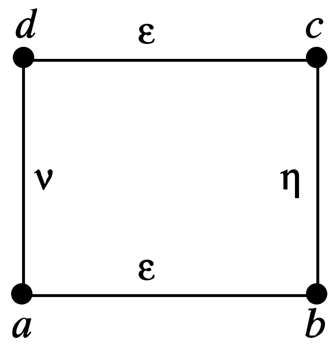

Figure 5

Example. Let $A = \{a, b, c, d\}$ depicted as a rectangle in Figure 5. Let f denote the projection to the lower side of the rectangle, i.e., $f : A \to A$, $a \mapsto a$, $b \mapsto b$, $c \mapsto b$ and $d \mapsto a$. Similarly, let g denote the projection to the upper side of the rectangle, i.e., $g : A \to A$, $a \mapsto d$, $b \mapsto c$, $c \mapsto c$ and $d \mapsto d$. We claim that the algebra $\mathcal{A} = (A, \{f, g\})$ satisfies the Triangular Scheme but not the Weak Triangular Principle.

P r o o f. The crucial step in the proof is the following observation:

each congruence collapsing at least one diagonal of the rectangle (i.e., $\{a, c\}$ or $\{b, d\}$) equals $\iota = A \times A$.

Now let ε, η, and ν be the equivalences on A with the respective partitions $\{\{a, b\}, \{c, d\}\}$, $\{\{b, c\}, \{a\}, \{d\}\}$ and $\{\{a, d\}, \{b\}, \{c\}\}$, cf. Figure 5. It is easy to see that $\iota = A \times A$, ω, ε, η and $\eta \vee \nu$ are congruences of $\mathcal{A}$ and they form a five-element nonmodular lattice. (In fact, the observation above implies easily that **Con** $\mathcal{A}$ is the lattice depicted in Figure 6, but we do not need the full description of the congruence lattice.) Hence $\mathcal{A}$ is not congruence distributive and therefore the Weak Triangular Principle fails in virtue of Theorem 1.

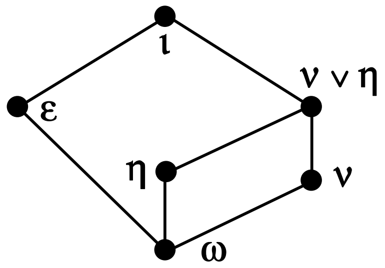

Figure 6

Yet, the Triangular Scheme holds. This is evident if x, y, z from Figure 1 are not pairwise distinct. On the other hand, if $\{x, y, z\}$ is a three element subset of A then one side of the triangle of Figure 1 is a diagonal of the rectangle of Figure 5. Hence, still keeping the notations of Figure 1, our observation implies $\iota \in \{\alpha, \beta, \gamma\}$, which easily makes the Triangular Scheme valid. ◇

Under the name *Shifting Principle* H.-P. Gumm [1] considers a condition in which not only congruences but tolerances also occur. Now the "congruence distributive counterpart" of this condition is introduced.

Definition 3. An algebra $\mathcal{A} = (A, F)$ satisfies the *Triangular Principle* if for each tolerance Φ and congruences β, γ the implication depicted in Figure 7 holds.

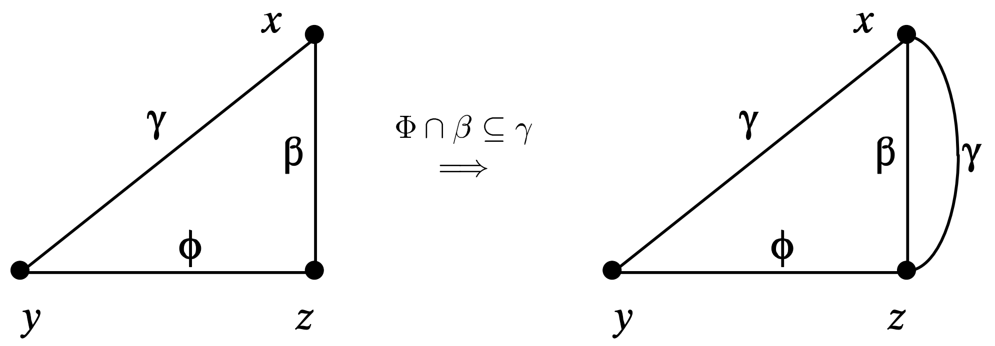

Figure 7

Theorem 2. *In congruence distributive varieties (i. e. in the algebras of such varieties) the Triangular Principle holds.*

P r o o f. Let **V** be a congruence distributive variety. Then we have Jónsson terms $t_0(x,y,z)\dots t_n(x,y,z)$ such that

$$t_0(x,y,z)=x,\quad t_n(x,y,z)=z,$$
$$t_i(x,y,x)=x \text{ for all i},$$
$$t_i(x,x,y)=t_{i+1}(x,x,y) \text{ for } i \text{ even, and}$$
$$t_i(x,y,y)=t_{i+1}(x,y,y) \text{ for } i \text{ odd.}$$

Let $\beta,\gamma\in\mathbf{Con}\,\mathcal{A}$ and $\Phi\in\mathbf{Tol}\,\mathcal{A}$, $\mathcal{A}\in\mathbf{V}$, $a,b,c\in\mathcal{A}$, and suppose that $\Phi\cap\beta\subseteq\gamma$ and we have the situation according to Figure 8.

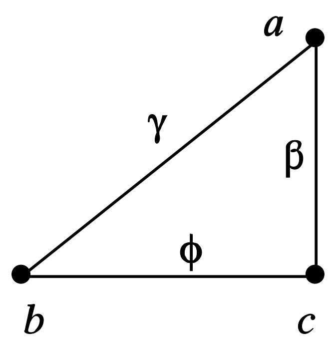

Figure 8

Consider the elements $d_i:=t_i(a,b,c)$, $i=0,1,\dots,n$. Now $d_0=a$, $d_n=c$. If i is even, then $d_i=t_i(a,b,c)\ \gamma\ t_i(a,a,c)=t_{i+1}(a,a,c)\ \gamma\ t_{i+1}(a,b,c)=d_{i+1}$. Consequently, $d_i\ \gamma\ d_{i+1}$ for i even. If i is odd, then we have to work a little bit more: first of all $d_i=t_i(a,b,c)\ \Phi\ t_i(a,c,c)$, and on the other hand, since $d_i=t_i(a,b,c)\ \beta$

$t_i(a,b,a) = a = t_i(a,a,a) \ \beta \ t_i(a,c,c)$, we have $(d_i, t_i(a,c,c)) \in \Phi \cap \beta \subseteq \gamma$. If we put $i+1$ instead of i, then in the same way we conclude $(d_{i+1}, t_{i+1}(a,c,c)) \in \Phi \cap \beta \subseteq \gamma$. But in this case $d_i \ \gamma \ t_i(a,c,c) = t_{i+1}(a,c,c) = \gamma \ d_{i+1}$ and, by the transitivity of γ, we get that $d_i \ \gamma \ d_{i+1}$ holds. Hence, for all i, $d_i \ \gamma \ d_{i+1}$, so $(a,c) = (d_0, d_n) \in \gamma$, i. e. the Triangular Principle holds. $\diamond$

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