Trapezoid Lemma and congruence distributivity

Version of December 4, 2002

Ivan Chajda, Gábor Czédli and Eszter K. Horváth

(I.Ch.:) Department of Algebra and Geometry, Palacký University Olomouc, Tomkova 40, 779 00 Olomouc, Czech Republic

(G.Cz. and E.K.H.:) University of Szeged, Bolyai Institute, Szeged, Aradi vértanúk tere 1, HUNGARY–6720.

E-mail: chajda@risc.upol.cz, czedli@math.u-szeged.hu, horeszt@math.u-szeged.hu

Key words: congruence distributivity, trapezoid lemma, shifting lemma, triangular lemma, Mal'cev condition = Mal'tsev condition.

Mathematics Subject Classification 2000: Primary 08B10, secondary 08B05.

Dedicated to Béla Csákány on his seventieth birthday

Abstract. Motivated by Gumm's (rectangular) Shifting Lemma, in our context a condition rather than a statement, and Shifting Principle, which play a key role in his treatment of congruence modularity and the theory of modular commutator, the present paper relates analogous triangular and trapezoid lemmas and principles to the distributivity of congruence lattices of single algebras and varieties. For varieties, the Trapezoid Lemma is equivalent to congruence distributivity. As a byproduct, congruence distributivity is characterized by a Mal'cev condition with a very clear connection with Day terms characterizing congruence modularity. Some results presented here were previously announced by J. Duda.

This research was partially supported by the project J98:MSM153100011 of the Czech Government, by the NFSR of Hungary (OTKA), grant no. T034137, T026243 and T037877, and also by the Hungarian Ministry of Education, grant no. FKFP 0169/2001.

H.-P. Gumm [11] defined a certain condition by a rectangular scheme for congruences resp. congruences and tolerances of an algebra under the name Shifting Lemma and Shifting Principle. In a variety, each of these two conditions is equivalent to congruence modularity. Keeping congruence distributivity rather than congruence modularity in mind our goal is to study some other schemes which can be defined by triangles or trapezes. Following Gumm's style of [11, Cor. 3.6], schemes for congruences will be called *lemmas* although they are just conditions, and we keep the word *principle* for schemes where tolerances also occur. We are going to study our conditions for congruences of single algebras and for congruences of varieties. As it will be detailed at the end of the paper, some of our results have previously been announced by Duda [7].

Now we give some definitions. An algebra A is said to satisfy the *Shifting Lemma* (in other words, *Rectangular Lemma*) if for any $\alpha, \beta, \gamma \in \operatorname{Con} A$ if $\alpha \cap \beta \subseteq \gamma$, $(x, u), (y, v) \in \alpha$, $(x, y), (u, v) \in \beta$ and $(u, v) \in \gamma$ then $(x, y) \in \gamma$, cf. Gumm [11]. Pictorially, the Rectangular Lemma is the condition given by Figure 1.

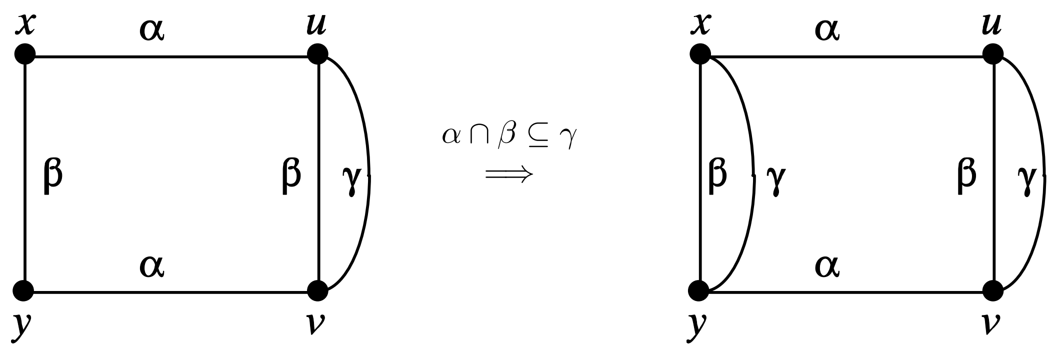

Figure 1

Similarly, A is said to satisfy the *Triangular Lemma* if for any $\alpha, \beta, \gamma \in \operatorname{Con} A$, $(x, y) \in \gamma$, $(x, z) \in \alpha$ and $(y, z) \in \beta$ if $\alpha \cap \beta \subseteq \gamma$ then $(y, z) \in \gamma$, cf. [1, 3] and Duda [8]. The Triangular Lemma is depicted in Figure 2.

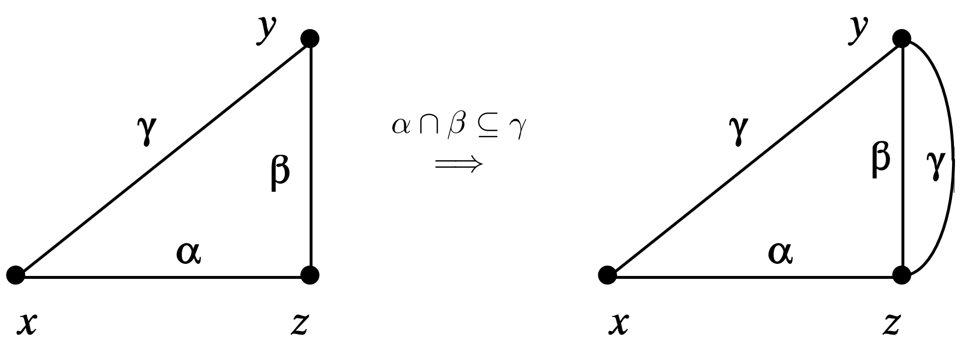

Figure 2

Now we introduce a new condition under the name *Trapezoid Lemma* as follows: for any $\alpha, \beta, \gamma \in \operatorname{Con} A$ if $\alpha \cap \beta \subseteq \gamma$, $(x, u), (y, v) \in \alpha$, $(x, y) \in \beta$ and $(u, v) \in \gamma$ then $(x, y) \in \gamma$. The Trapezoid Lemma is depicted in Figure 3.

Three corresponding conditions called *Shifting* (or *Rectangular*) *Principle* (cf. Gumm[11]), *Triangular Principle* (cf. [3]) and *Trapezoid Principle* are defined similarly, the only difference is that α should be replaced by Φ, which stands for an arbitrary *tolerance* (i.e., compatible, reflexive and symmetric binary relation) of A.

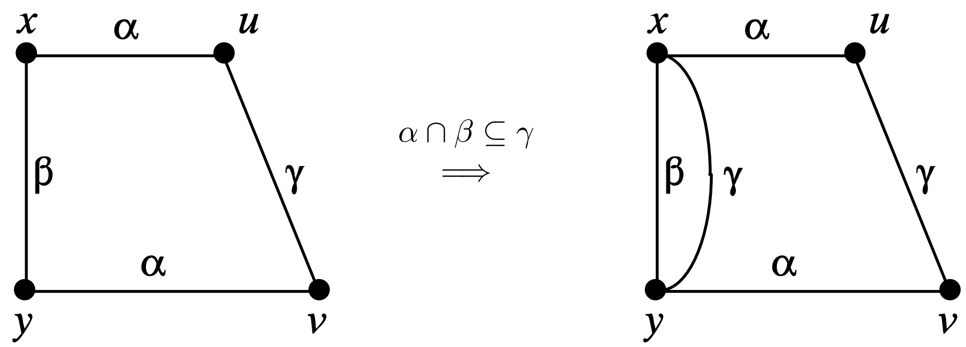

Figure 3

Our figures follow the tradition that parallel edges have the same label. Sometimes we do not require the above-defined conditions for all triplets (α, β, γ) just for a single triplet $(\alpha_0, \beta_0, \gamma_0)$; in this case we will say so.

Given a direct product $A = A_1 \times A_2$, a congruence $\gamma \in \operatorname{Con} A$ is called *directly decomposable* if $\gamma = \gamma_1 \times \gamma_2$ for appropriate $\gamma_1 \in \operatorname{Con} A_1$ and $\gamma_2 \in \operatorname{Con} A_2$. One of the motivations for introducing the Trapezoid Lemma is revealed by the following statement, which strengthens Assertion 1 in [1].

Proposition 1. *Let $\gamma \in \operatorname{Con}(A_1 \times A_2)$ and let π_i denote the kernel of the projection $A_1 \times A_2 \to A_i$, $(x_1, x_2) \mapsto x_i$, $i = 1, 2$. Then the following three conditions are equivalent:*

(a) γ is directly decomposable;

(b) the Trapezoid Lemma holds for (π_1, π_2, γ) and (π_2, π_1, γ);

(c) both the Rectangular Lemma and the Triangular Lemma holds for (π_1, π_2, γ) and (π_2, π_1, γ).

Proof. The equivalence of (a) and (b), in a slightly different formulation, is proved by Fraser and Horn [9, Thm. 1 (1,3)], cf. also the trapezes in [2, Figure 31, page 128]. The implication (b) $\Longrightarrow$ (c) is evident; this will also be clear from the forthcoming Proposition 2. Proving (c) $\Longrightarrow$ (b) is obvious, too: if $(x, u), (y, v) \in \pi_1$, $(x, y) \in \pi_2$ and $(u, v) \in \gamma$ then with $w := (y_1, u_2) = (v_1, u_2)$ the Triangular Lemma gives $(u, w) \in \gamma$, whence the Rectangular Lemma yields $(x, y) \in \gamma$. $\diamond$

The following statement presents some connections among our conditions in case of a single algebra; for varieties of algebras we will soon state more.

Proposition 2. *Let A be an algebra.*

(1) If A satisfies the Trapezoid Lemma resp. the Trapezoid Principle then it satisfies the Rectangular Lemma and the Triangular Lemma resp. the Rectangular Principle and the Triangular Principle. Moreover, each of the three principles implies the corresponding lemma.

(2) If $\operatorname{Con} A$ *is distributive then A satisfies the Trapezoid Lemma (and therefore the other two lemmas as well).*

(3) If A satisfies the Trapezoid Principle then $\operatorname{Con} A$ *is distributive.*

(4) If A satisfies the Rectangular Principle then $\operatorname{Con} A$ *is modular (cf. Gumm [11, Lemma 3.2].*

(5) If A is congruence permutable then $\operatorname{Con} A$ *is distributive if and only if A satisfies the Triangular Lemma (cf. [3, Cor. 2]).*

Proof. (1) is trivial. (2) comes easily from the fact that a lattice is distributive iff it satisfies the Horn sentence

$$\alpha \wedge \beta \leq \gamma \Longrightarrow \beta \wedge (\alpha \vee \gamma) \leq \gamma, \tag{$*$}$$

cf. [1]. Hence only (3) needs a proof. Suppose A is an algebra satisfying the Trapezoid Principle and $\alpha, \beta, \gamma \in \operatorname{Con} A$ with $\alpha \wedge \beta \leq \gamma$. According to $(*)$ it suffices to show $\beta \wedge (\alpha \vee \gamma) \leq \gamma$. Borrowing the idea from the proof of Lemma 3.2 in Gumm [11], define tolerances $\Phi_0 = \alpha$ and $\Phi_{n+1} = \Phi_n \circ \gamma \circ \alpha$, $n \in \mathbf{N}$. Via induction on n we want to show that $\beta \cap \Phi_n \subseteq \gamma$. For $n = 0$ this is clear. Now suppose $\beta \cap \Phi_n \subseteq \gamma$ and let (x, y) be an arbitrary pair in $\beta \cap \Phi_{n+1}$. Then $(x, y) \in \beta \cap \Phi_{n+1} = \beta \cap (\Phi_n \circ \gamma \circ \alpha) \subseteq \beta \cap (\Phi_n \circ \gamma \circ \Phi_n)$, so there are $u, v \in A$ such that $(x, u), (y, v) \in \Phi_n$, $(x, y) \in \beta$ and $(u, v) \in \gamma$. Hence the induction hypothesis $\beta \cap \Phi_n \subseteq \gamma$ and the Trapezoid Principle gives $(x, y) \in \gamma$. This shows $\beta \cap \Phi_{n+1} \subseteq \gamma$, completing the induction. Finally,

$$\beta \wedge (\alpha \vee \gamma) = \beta \cap \bigcup_{n=0}^{\infty} \Phi_n = \bigcup_{n=0}^{\infty} (\beta \cap \Phi_n) \subseteq \gamma,$$

proving $(*)$ and (3). ◇

We do not know if the implication in (1), (2), (3) and (4) of Proposition 2 can be reversed but we guess the answer is negative in each case. However, for varieties rather than single algebras much more can be said. Of course, a condition is said to hold in a variety if it holds in all algebras of the variety. Part (a) $\Longleftrightarrow$ (c) of the following theorem was announced by Duda [7].

Theorem 1. *Let $\mathcal{V}$ be a variety of algebras. Then the following five conditions are equivalent.*

(a) $\mathcal{V}$ is congruence distributive;

(b) the Trapezoid Principle holds in $\mathcal{V}$;

(c) the Trapezoid Lemma holds in $\mathcal{V}$;

(d) the Rectangular Lemma and the Triangular Lemma hold in $\mathcal{V}$;

(e) there is a positive integer n and there are quaternary terms $d_0, d_1, \ldots, d_n$ such that the identities

(e1) $d_0(x, y, u, v) = x$, $\quad d_n(x, y, u, v) = y$,

(e2) $d_i(x, y, x, y) = d_{i+1}(x, y, x, y)$ for i even,

(e3) $d_i(x, y, z, z) = d_{i+1}(x, y, z, z)$ for i odd, and

(e4) $d_i(x, x, y, z) = x$ for all i

hold in $\mathcal{V}$.

Remark 1. Congruence distributivity and congruence modularity of varieties are characterized by classical Mal'cev conditions, namely by the Jónsson terms, cf. Jónsson [13], and the Day terms, cf. Day [6]. Since distributivity implies modularity, one would expect that Jónsson terms trivially produce Day terms, but this is not the case. To fulfil this wish (and also to reduce the number of variables) Gumm [11, 12] characterizes congruence modularity with another Mal'cev condition, the Gumm terms, and he points out that Jónsson terms trivially produce Gumm terms. Now (e) of Theorem 1 gives an alternative way to meet the mentioned expectation. Namely, Day terms are quaternary terms satisfying (e1), (e2), (e3) and

(e4') $d_i(x,x,y,y) = x$ for all i,

so our terms in (e) clearly produce (and in fact, constitute) Day terms. Notice that (e) is a byproduct of studying the Trapezoid Lemma; indeed, the proof of Theorem 1 is easier with (e) than with Jónsson terms. To reveal the connection between (e) and Jónsson terms we mention that the $p_i(x,y,z) = d_i(x,z,y,z)$ are Jónsson terms provided the d_i are (e) terms.

Remark 2. Theorem 1 and Proposition 2 clearly imply Theorem 2 of [3], which says that congruence distributive varieties satisfy the Triangular Principle.

Proof of Theorem 1. (a) $\Longrightarrow$ (e) follows in the standard way of deriving Mal'cev conditions if we consider the the principal congruences $\beta = \mathrm{con}(u,v)$ and $\gamma = \mathrm{con}(x,y)$, and the congruence $\alpha = \mathrm{con}(x,u) \vee \mathrm{con}(y,v)$ of the free algebra $F_{\mathcal{V}}(x,y,u,v)$.

(e) $\Longrightarrow$ (b): Assuming that (e) holds in $\mathcal{V}$, let $A \in \mathcal{V}$, let Φ be a tolerance relation of A, let $\beta,\gamma \in \mathrm{Con}\,A$ with $\Phi \cap \beta \subseteq \gamma$, let $x,y,u,v \in A$ and suppose $(x,u),(y,v) \in \Phi$, $(x,y) \in \beta$ and $(u,v) \in \gamma$. We have to show that $(x,y) \in \gamma$. Consider the elements $h_i = d_i(x,y,u,v)$, $i = 0,\ldots,n$, where the terms d_i are provided by (e). Then for i odd, $h_i = d_i(x,y,u,v)\;\gamma\;d_i(x,y,u,u) = d_{i+1}(x,y,u,u)\;\gamma\;d_{i+1}(x,y,u,v) = h_{i+1}$, i.e., $(h_i,h_{i+1}) \in \gamma$ for i odd. For i even we have to work a bit more. We start with $h_i = d_i(x,y,u,v)\;\Phi\;d_i(x,y,x,y)$ and $h_i = d_i(x,y,u,v)\;\beta\;d_i(x,x,u,v) = x = d_i(x,x,x,x)\;\beta\;d_i(x,y,x,y)$. Hence $\big(h_i, d_i(x,y,x,y)\big) \in \Phi \cap \beta \subseteq \gamma$. We obtain $\big(h_{i+1}, d_{i+1}(x,y,x,y)\big) \in \gamma$ similarly. But $d_i(x,y,x,y) = d_{i+1}(x,y,x,y)$, whence the transitivity of γ gives $(h_i,h_{i+1}) \in \gamma$ for i even. Now $(h_i,h_{i+1}) \in \gamma$ for all i, and we conclude $(x,y) = (h_0,h_n) \in \gamma$. I.e., $\mathcal{V}$ satisfies (b).

Observe that (b) $\Longrightarrow$ (c) and (c) $\Longrightarrow$ (d) are evident (or follow from Proposition 2).

(d) $\Longrightarrow$ (a): Let $\mathcal{V}$ be a variety satisfying the Rectangular Lemma and the Triangular Lemma. The Rectangular Lemma in itself implies that $\mathcal{V}$ is congruence modular by Gumm [11, Cor. 3.6]. Now, by way of contradiction, assume that $\mathcal{V}$ is not congruence distributive. Then there is an algebra $A \in \mathcal{V}$ and there are congruences $\alpha,\beta,\gamma \in \mathrm{Con}\,A$ generating a five-element nondistributive sublattice $M_3 = \{\alpha,\beta,\gamma,\omega,\iota\}$ of $\mathrm{Con}\,A$ with $\omega < \alpha < \iota$, $\omega < \beta < \iota$ and $\omega < \gamma < \iota$. The theory of modular commutator says, cf. Gumm [11, Cor. 8.9] or Freese and McKenzie [10, Lemma 13.1], that any two elements of this M_3 permute. Since $\beta \not\subseteq \gamma$, we can pick a pair $(y,z) \in \beta \setminus \gamma$. Since $(y,z) \in \beta \subseteq \iota = \gamma \vee \alpha = \gamma \circ \alpha$, there is an element x with $(y, x) \in \gamma$ and $(x, z) \in \alpha$, cf. the left hand side of Figure 2. Now $\alpha \cap \beta = \omega \subseteq \gamma$, so the Triangular Lemma yields $(y, z) \in \gamma$, a contradiction. This proves that $\mathcal{V}$ is congruence distributive. $\diamond$

Comparison with subsequent and previous results. While the Trapezoid Principle seems to be just a technical condition here, later it proved to be an essential milestone towards [4, 5], which follow the present paper. Several parts of the present paper are in close connection with former results of J. Duda. He also introduced the Trapezoid Lemma (under the name Upright Principle) and announced that conditions (a) and (c) of Theorem 1 are equivalent, cf. [7], and they are equivalent to the conjunction of congruence modularity and the Triangular Lemma, cf. [8]. (In virtue of Gumm's classical result, this conjunction is clearly equivalent to (d) of Theorem 1.) Duda [7] also gave a Mal'cev condition to characterize the Trapezoid Lemma; his Mal'cev condition consists of 6-ary terms. Although we have never had access to his proofs, his highly sophisticated Mal'cev conditions for the Trapezoid Lemma and also for the Triangular Lemma, cf. [8], convinced us that our approach to his result is new and simpler than the original one.

References

[1] I. Chajda, G. Czédli and E. K. Horváth, Shifting lemma and shifting lattice identities, Algebra Universalis, submitted.

[2] I. Chajda and K. Głazek, A Basic Course on Algebra, Technical University Press, Zielona Góra, Poland, 2000, x+151 pp. ISBN 83-85911-81-2.

[3] I. Chajda and E. K. Horváth, A triangular scheme for congruence distributivity, Acta Sci. Math. (Szeged), to appear.

[4] G. Czédli and E. K. Horváth, Congruence distributivity and modularity permit tolerances, Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica, to appear.

[5] G. Czédli and E. K. Horváth, All congruence lattice identities implying modularity have Mal'tsev conditions, Algebra Universalis, to appear.

[6] A. Day, A characterization of modularity for congruence lattices of algebras, Canad. Math. Bull., 12, 1969, 167–173.

[7] J. Duda, The Upright Principle for congruence distributive varieties, Abstract of a seminar lecture presented in Brno, March, 2000.

[8] J. Duda, The Triangular Principle for congruence distributive varieties, Abstract of a seminar lecture presented in Brno, March, 2000.

[9] G. A. Fraser and A. Horn, Congruence relations in direct products, Proc. Amer. Math. Soc., 26, 1970, 390–394.

[10] R. Freese and R. McKenzie, Commutator theory for congruence modular varieties, Cambridge Univ. Press, Cambridge, 1987, iv+227 pp. ISBN 0-521-34832-3.

[11] H. P. Gumm, Geometrical methods in congruence modular algebras, Mem. Amer. Math. Soc., 45, 1983, no. 286, viii+79 pp.

[12] H. P. Gumm, Congruence modularity is permutability composed with distributivity, Arch. d. Math., 36, 1981, 569–576.

[13] B. Jónsson, Algebras whose congruence lattices are distributive, Math. Scand., 21, 1967, 110–121.