THE NUMBER OF SQUARE ISLANDS ON A RECTANGULAR SEA

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Abstract. The aim of the present paper is to carry on the research of Czédli in determining the maximum number of rectangular islands on a rectangular grid. We estimate the maximum of the number of square islands on a rectangular grid.

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1. Introduction

Recently, the notion of islands on a grid caught the attention of several researchers. The original combinatorial problem was raised in connection with instantaneous (prefix-free) codes, see [3]. For every square (cell) of a rectangular grid a positive real number a_{ij} is given, its height. The height of the bottom (above sea level) is constant 0 on each cell. Now a rectangle R is called a rectangular island iff there is a possible water level such that R is an island of the lake in the usual sense. There are other examples requiring only $m \times n$ cells; for example, a_{ij} may mean a colour on a gray-scale (before we convert the picture to black and white), transparency (against X-rays), or melting temperature can be modelled with the three dimensional version of the problem.

Czédli [2] has considered a rectangular lake whose bottom is divided into $(m + 2) \times (n + 2)$ cells. In other words, we identify the bottom of the lake with the table $\{0, 1, \ldots, m + 1\} \times \{0, 1, \ldots, n + 1\}$. In his paper, he shows that the maximum number of rectangular islands is $\lfloor (mn+m+n-1)/2 \rfloor$. Pluhár [6], generalizing the earlier methods gave upper and lower bounds in higher dimensions. In [5] the dual problem of Czédli is investigated: the minimal size of a maximal system of islands is presented. Czédli's work is continued on different surfaces (torus, cylinder) in [1].

In [4], the maximum number of triangular islands on the triangular grid is investigated. There, the methods for finding the proper lower and upper bounds are highly non-trivial and different from the rectangular cases. This gave us the motivation to investigate the number of *square* islands on a *rectangular* grid.

For $n \in \mathbf{N}$ let $\mathbf{n} = [1, n] = \{1, \ldots, n\}$. For $m, n \in \mathbf{N}$ the set $\mathbf{m} \times \mathbf{n}$ will be called a *table* of size $m \times n$. We will consider $\mathbf{m} \times \mathbf{n}$ as a collection of *cells*: (i, j) will mean the j-th cell in the i-th row. We identify $\mathbf{m} \times \mathbf{n}$ by the rectangular grid $\{ 0, 1, \ldots, m \} \times \{ 0, 1, \ldots, n \}$, and the cell (i, j) is identified by the set $\{ (i - 1, j - 1), (i, j - 1), (i - 1, j), (i, j) \}$ of four gridpoints. Geometrically $\mathbf{m} \times \mathbf{n}$ is identified with a rectangle of sidelengths m and n, and the cell (i, j) is identified with the unit subsquare whose upper-right vertex is gridpoint (i, j).

Two cells are called *neighbouring cells* if the distance of their centers is at most $\sqrt{2}$. That is, two cells are neighbouring if their corresponding unit subsquares intersect by a side or by a vertex, i.e. if they have a common gridpoint.

By a height function of $m \times n$ we mean a mapping $h\colon \mathbf{m} \times \mathbf{n} \to \mathbf{R}$, $(i, j) \mapsto a_{ij}$. Let R be an arbitrary subsquare of $\mathbf{m} \times \mathbf{n}$. We say that R

is a *square island* of the height function h if $h((i,j)) < \min(h|_R)$ holds for each $(i,j) \in (\mathbf{m} \times \mathbf{n}) \setminus R$ such that (i,j) is neighbouring with some cell of R. The set of square islands of h will be denoted by $\mathcal{I}_{\rm sqr}(h)$. We are looking for $\max |\mathcal{I}_{\rm sqr}(h)|$, where h runs through all possible height functions.

In Section 2 we prove that for the rectangular grid $(\mathbf{m}-\mathbf{1}) \times (\mathbf{n}-\mathbf{1})$ the maximum number of square islands $\max |\mathcal{I}_{\rm sqr}(h)|$ satisfies the following inequalities:

$$\frac{1}{3} \cdot (mn - 2m - 2n) \leq \max |\mathcal{I}_{\rm sqr}(h)| \leq \frac{1}{3} \cdot (mn - 1)\,.$$

Then in Section 3 we generalize our two dimensional results to higher dimensions.

The key of all proofs for the upper estimates is an ingenious observation of Czédli: in the rectangular case the number of islands can be measured by the area of the rectangle, and this was the case for triangular islands, as well.

Let $\mathbf{Sq}_{m,n}$ denote the set of subsquares in $\mathbf{m} \times \mathbf{n}$ and $T_1, T_2 \in \mathbf{Sq}_{m,n}$ be two squares of $\mathbf{m} \times \mathbf{n}$. We say that T_1 and T_2 are *far from each other*, if they are disjoint and no cell of T_1 is neighbouring with any cell of T_2. Obviously, the role of T_1 and T_2 is symmetric in the definition. The following statement is from Section 2 of [2].

Lemma 1. *Let $\mathcal{H}$ be a subset of $\mathbf{Sq}_{m,n}$. The following two conditions are equivalent:*

(i) *There exists a height function h such that $\mathcal{H} = \mathcal{I}_{sqr}(h)$.*
(ii) *For any $T_1, T_2 \in \mathcal{H}$ either $T_1 \subseteq T_2$ or $T_2 \subseteq T_1$ or T_1 and T_2 are far from each other. Moreover, if $m = n$, then $\mathbf{m} \times \mathbf{n} \in \mathcal{H}$.*

In the sequel, subsets $\mathcal{H}$ of $\mathbf{Sq}_{m,n}$ satisfying the (equivalent) conditions of Lemma 1 will be called *systems of square islands.* Let $\mathbf{L}_{m \times n}$ denote the set of gridpoints contained in the rectangle $(\mathbf{m}-\mathbf{1}) \times (\mathbf{n}-\mathbf{1})$. Indeed, these gridpoints form an m by n grid. For a rectangle (or square) T let $L(T)$ denote the set of gridpoints contained in T. Lemma 1 can be reformulated on the following way:

Lemma 2. *Let $\mathcal{H}$ be a subset of $\mathbf{Sq}_{m,n}$. The following two conditions are equivalent:*

(i) *There exists a height function h such that $\mathcal{H} = \mathcal{I}_{sqr}(h)$.*
(ii) *For any $T_1, T_2 \in \mathcal{H}$ either $L(T_1) \subseteq L(T_2)$ or $L(T_2) \subseteq L(T_1)$ or $L(T_1)$ and $L(T_2)$ are disjoint. Moreover, if $m = n$, then $\mathbf{m} \times \mathbf{n} \in \mathcal{H}$.*

Proof. We only have to observe, that two squares T_1 and T_2 are far if and only if $L(T_1) \cap L(T_2) = \emptyset$. $\square$

2. Estimating the number of islands

Let $f(m,n)$ denote the maximum number of square islands on the rectangular grid $(\mathbf{m-1})\times(\mathbf{n-1})$, or equivalently on gridpoints $\mathbf{L}_{m\times n}$. In this Section we prove the following inequalities:

$$\frac{1}{3}\cdot(mn-2m-2n)\leq f(m,n)\leq\frac{1}{3}\cdot(mn-1).$$

First in Subsection 2.1 we inductively construct a set $\mathcal{H}=\mathcal{S}_{(\mathbf{m-1})\times(\mathbf{n-1})}$ of squares on the rectangular grid $(\mathbf{m-1})\times(\mathbf{n-1})$ (or equivalently on gridpoints $\mathbf{L}_{m\times n}$) satisfying condition (ii) of Lemma 1 (or equivalently Lemma 2). Hence $\mathcal{S}_{(\mathbf{m-1})\times(\mathbf{n-1})}$ is a system of square islands. Let $s(m,n)=|\mathcal{S}_{(\mathbf{m-1})\times(\mathbf{n-1})}|$. Clearly, $s(m,n)$ is a lower bound for $f(m,n)$, that is $s(m,n)\leq f(m,n)$ holds. We prove that $\frac{1}{3}\cdot(mn-2m-2n)\leq s(m,n)$ in Subsection 2.2, which finishes the proof of the lower bound.

Secondly, we prove the upper bound in Subsection 2.3 by calculating the number of gridpoints covered by the maximal elements of an arbitrary set of rectangular islands. We shall see that our upper bound is sharp, when $m=n=2^k$ for some positive integer k.

2.1. Constructing the set $\mathcal{S}_{(\mathbf{m-1})\times(\mathbf{n-1})}$.

Let m, n be arbitrary positive integers. For the table $T=(\mathbf{m-1})\times(\mathbf{n-1})$ (i.e. where $\mathbf{L}_{m\times n}$ is an $m\times n$ grid) we define a system of square islands, $\mathcal{S}_T$. Then we calculate the number $s(m,n)=\left|\mathcal{S}_{(\mathbf{m-1})\times(\mathbf{n-1})}\right|$.

We define $\mathcal{S}_T$ by induction on m and n. For technical reasons we define $\mathcal{S}_{(\mathbf{m-1})\times(\mathbf{0})}=\mathcal{S}_{(\mathbf{0})\times(\mathbf{n-1})}$ as the emptyset and hence we define $s(m,0)=s(0,n)=0$. First we define our system of islands when $m=n=2^i$ (for some positive integer i), then for $m=2^i$, $n=2^j$ (for some positive integers $i\neq j$) and finally for arbitrary positive integers m, n of not the above form.

If $m=1$ or $n=1$, then let $\mathcal{S}_{(\mathbf{m-1})\times(\mathbf{n-1})}=\emptyset$. Now, let T be a square of $2^i\times 2^i$ gridpoints, i.e. $m=n=2^i$. Into this square we draw four subsquares of $2^{i-1}\times 2^{i-1}$ gridpoints as shown in Figure 1. Let us denote these squares by T_1, T_2, T_3 and T_4. Now, let

$$\mathcal{S}_{(\mathbf{m-1})\times(\mathbf{n-1})}=\{\,T\,\}\cup\mathcal{S}_{T_1}\cup\mathcal{S}_{T_2}\cup\mathcal{S}_{T_3}\cup\mathcal{S}_{T_4}.$$

Let $T=(\mathbf{m-1})\times(\mathbf{n-1})$ such that $m=2^i$, $n=2^j$, for some positive integers $i\neq j$. Let us assume that $i<j$, the other case can be handled similarly. Into T we draw 2^{j-i}-many equilateral subsquares of $2^i\times 2^i$ gridpoints as shown in Figure 2. Let these squares be $T_1,T_2,\ldots,T_{2^{j-i}}$. Now, let

$$\mathcal{S}_{(\mathbf{m-1})\times(\mathbf{n-1})}=\mathcal{S}_{T_1}\cup\mathcal{S}_{T_2}\cup\cdots\cup\mathcal{S}_{T_{2^{j-i}}}.$$

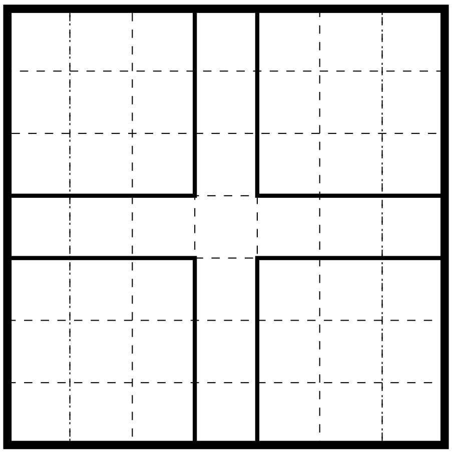

FIGURE 1. Construction of $\mathcal{S}_{(\mathbf{m-1})\times(\mathbf{n-1})}$ for $m = n = 8$.

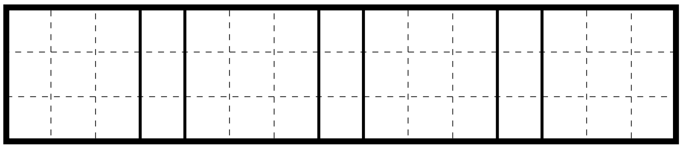

FIGURE 2. Construction of $\mathcal{S}_{(\mathbf{m-1})\times(\mathbf{n-1})}$ for $m = 4$, $n = 16$.

Finally, let m and n be arbitrary positive integers not of the above form, i.e. at least one of m and n is not a power of 2. Let the binary form of m be $\overline{a_k a_{k-1} \dots a_1 a_0}$, i.e. $m = \sum_{i=0}^{k} a_i \cdot 2^i$. Similarly, let the binary form of n be $\overline{b_l b_{l-1} \dots b_1 b_0}$, i.e. $n = \sum_{j=0}^{l} b_j \cdot 2^j$. Let us divide the sides of the grid $\mathbf{L}_{m\times n}$ into segments of legths $a_i \cdot 2^i$ and $b_j \cdot 2^j$ (for $0 \le i \le k$ and $0 \le j \le l$). This defines a (rectangular) tiling on $\mathbf{L}_{m\times n}$ (and thus on $T = (\mathbf{m-1}) \times (\mathbf{n-1})$) by taking the direct products of the appropriate intervals. A general rectangle is of the form $(a_i \cdot 2^i) \times (b_j \cdot 2^j)$. Figure 3 shows this tiling for $m = 6$, $n = 14$. Let us denote by $T_{i,j}$ the subrectangle of gridpoints $2^i \times 2^j$, if $a_i = b_j = 1$. Now, if $m \ne n$ then let

$$\mathcal{S}_{(\mathbf{m-1})\times(\mathbf{n-1})} = \bigcup_{\substack{0\le i\le k\\ a_i=1}} \bigcup_{\substack{0\le j\le l\\ b_j=1}} \mathcal{S}_{T_{i,j}},$$

if $m = n$, then let

$$\mathcal{S}_{(\mathbf{m-1})\times(\mathbf{n-1})} = \{T\} \cup \bigcup_{\substack{0\le i\le k\\ a_i=1}} \bigcup_{\substack{0\le j\le l\\ b_j=1}} \mathcal{S}_{T_{i,j}}.$$

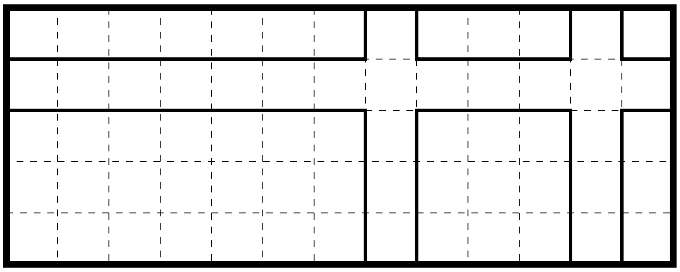

FIGURE 3. Construction of $\mathcal{S}_{(\mathbf{m-1})\times(\mathbf{n-1})}$ for $m = 6$, $n = 14$.

2.2. **Lower bound.** Let

$$\varepsilon(m,n) = \begin{cases} 1, & \text{if } m = n \text{ and is not a power of 2,}\\ 0, & \text{otherwise,}\end{cases}$$

We remind the reader that $s(m,n) = \left|\mathcal{S}_{(\mathbf{m-1})\times(\mathbf{n-1})}\right|$. From our construction it is easy to see that s satisfies the following properties:

$$s(m,1) = s(1,n) = 0, \tag{1}$$

$$s\left(2^i,2^i\right) = 4\cdot s\left(2^{i-1},2^{i-1}\right) + 1, \tag{2}$$

$$s\left(2^i,2^j\right) = 2^{|j-i|}\cdot s\left(2^{\min\{i,j\}},2^{\min\{i,j\}}\right), \tag{3}$$

$$s(m,n) = \sum_{i=0}^{k}\sum_{j=0}^{l} a_i b_j \cdot s\left(2^i,2^j\right) + \varepsilon(m,n), \tag{4}$$

where the binary form of m is $\overline{a_k a_{k-1}\dots a_1 a_0}$ and the binary form of n is $\overline{b_l b_{l-1}\dots b_1 b_0}$.

Theorem 3. *Let the binary form of m be $\overline{a_k a_{k-1}\dots a_1 a_0}$ and let the binary form of n be $\overline{b_l b_{l-1}\dots b_1 b_0}$. Then*

$$s(m,n) = \frac{1}{3}\cdot mn - \frac{1}{3}\cdot\sum_{i=0}^{k}\sum_{j=0}^{l} a_i b_j\cdot 2^{|i-j|} + \varepsilon(m,n).$$

Proof. By induction on i we can easily deduce from (2) that

$$s\left(2^i,2^i\right)=\frac{1}{3}\cdot\left(4^i-1\right).$$

By applying (3) we immediately obtain for $i<j$ that

$$s\left(2^i,2^j\right)=\frac{1}{3}\cdot\left(2^{i+j}-2^{j-i}\right),$$

similarly for $j<i$ we have

$$s\left(2^i,2^j\right)=\frac{1}{3}\cdot\left(2^{i+j}-2^{i-j}\right).$$

The formula $s\left(2^i,2^j\right)=\frac{1}{3}\cdot\left(2^{i+j}-2^{|i-j|}\right)$ summarizes our previous two remarks.

Now let the binary form of m be $\overline{a_k a_{k-1}\dots a_1 a_0}$ and let the binary form of n be $\overline{b_l b_{l-1}\dots b_1 b_0}$. Then

$$\begin{aligned}
s\left(m,n\right)&=\sum_{i=0}^{k}\sum_{j=0}^{l}a_ib_j\cdot s\left(2^i,2^j\right)+\varepsilon\left(m,n\right)\\
&=\frac{1}{3}\cdot\sum_{i=0}^{k}\sum_{j=0}^{l}a_ib_j\cdot\left(2^{i+j}-2^{|i-j|}\right)+\varepsilon\left(m,n\right),\\
&=\frac{1}{3}\cdot mn-\frac{1}{3}\cdot\sum_{i=0}^{k}\sum_{j=0}^{l}a_ib_j\cdot 2^{|i-j|}+\varepsilon\left(m,n\right).
\end{aligned}$$

□

Lemma 4. *Let the binary form of m be $\overline{a_k a_{k-1}\dots a_1 a_0}$ and let the binary form of n be $\overline{b_l b_{l-1}\dots b_1 b_0}$. Then*

$$\sum_{i=0}^{k}\sum_{j=0}^{l}a_ib_j\cdot 2^{|i-j|}\le 2\cdot\left(m+n\right).$$

Proof. We have

$$\begin{aligned}
\sum_{i=0}^{k}\sum_{j=0}^{l}a_ib_j\cdot 2^{|i-j|}&\le\sum_{i=0}^{k}a_i\cdot 2^{-i}\cdot\sum_{j=i}^{l}b_j\cdot 2^j+\sum_{j=0}^{l}b_j\cdot 2^{-j}\cdot\sum_{i=j}^{k}a_i\cdot 2^i\\
&\le\sum_{i=0}^{k}a_i\cdot 2^{-i}\cdot n+\sum_{j=0}^{l}b_j\cdot 2^{-j}\cdot m\\
&\le 2\cdot\left(m+n\right).
\end{aligned}$$

□

Corollary 5.

$$f\left(m,n\right) \geq \frac{1}{3} \cdot \left(mn - 2m - 2n\right).$$

Proof. Let the binary form of m be $\overline{a_k a_{k-1} \dots a_1 a_0}$ and let the binary form of n be $\overline{b_l b_{l-1} \dots b_1 b_0}$. Then we have

$$\begin{aligned} f\left(m,n\right) &\geq s\left(m,n\right) \\ &= \frac{1}{3} \cdot mn - \frac{1}{3} \cdot \sum_{i=0}^{k} \sum_{j=0}^{l} a_i b_j \cdot 2^{|i-j|} + \varepsilon\left(m,n\right) \\ &\geq \frac{1}{3} \cdot \left(mn - 2m - 2n\right). \end{aligned}$$

□

2.3. **Upper bound.**

Theorem 6. *The following inequality holds:*

$$f(m,n) \leq \frac{1}{3} \cdot \left(mn - 1\right).$$

Proof. We prove the statement by induction on m and n. For $m \leq 2$ or for $n \leq 2$ the statement is clear. The induction hypothesis is, that for any rectangle T with uv-many gridpoints, such that $u \leq m$, $v \leq n$, $(u,v) \neq (m,n)$, we have $f(u,v) \leq \frac{1}{3} \cdot (uv - 1)$. We define $\mu(u,v) := uv$, which is exactly the number of gridpoints of the rectangle of sidelengths $u-1, v-1$. Let the two sides of the rectangle T contain u_T and v_T-many gridpoints. Let

$$\varepsilon^{*}\left(m,n\right) = \begin{cases} 1, & \text{if } m = n, \\ 0, & \text{if } m \neq n, \end{cases}$$

Let $\mathcal{H}^{*}$ be a system of square islands of $(\mathbf{m-1}) \times (\mathbf{n-1})$, such that $|\mathcal{H}^{*}| = f\left(m,n\right)$. Let $\max \mathcal{H}^{*}$ be the set of maximal islands in $\mathcal{H}^{*} \setminus$

$\{(\mathbf{m-1}) \times (\mathbf{n-1})\}$. Now,

$$\begin{aligned}
f(m,n) &= \varepsilon^*(m,n) + \sum_{T \in \max \mathcal{H}^*} f(u_T, v_T) \\
&\leq \varepsilon^*(m,n) + \sum_{T \in \max \mathcal{H}^*} \frac{1}{3} \cdot (u_T v_T - 1) \\
&= \varepsilon^*(m,n) + \sum_{T \in max \mathcal{H}^*} \frac{1}{3} \cdot (\mu(u_T, v_T) - 1) \\
&= \varepsilon^*(m,n) - \frac{1}{3} \cdot |\max(\mathcal{H}^*)| + \sum_{T \in \max \mathcal{H}^*} \frac{\mu(u_T, v_T)}{3} \\
&\leq \varepsilon^*(m,n) - \frac{1}{3} \cdot |\max(\mathcal{H}^*)| + \frac{\mu(m,n)}{3} \\
&= \varepsilon^*(m,n) - \frac{1}{3} \cdot |\max(\mathcal{H}^*)| + \frac{mn}{3}.
\end{aligned}$$

Recall that $\varepsilon^*(m,n) = 0$, whenever $m \neq n$ and $\varepsilon^*(m,n) = 1$ if $m = n$. Thus, if $m \neq n$ or $|max(\mathcal{H}^*)| \geq 4$ then we obtained that

$$f(m,n) \leq \frac{1}{3} \cdot (mn - 1).$$

For the remaining cases we assume that $m = n$. We use induction on m.

If $|\max(\mathcal{H}^*)| = 1$, then the maximum number of the square islands is $f(m-1, m-1) + 1$. In this case by the induction hypothesis

$$\begin{aligned}
f(m,m) = f(m-1, m-1) + 1 &= \frac{1}{3} \cdot \left((m-1)^2 - 1\right) + 1 \\
= \frac{1}{3} \cdot \left(m^2 - 2m + 3\right) &\leq \frac{1}{3} \cdot \left(m^2 - 1\right).
\end{aligned}$$

If $|\max(\mathcal{H}^*)| = 2$, then the maximum number of the square islands is $f(k,k) + f(m-k, m-k) + 1$ for some k where $2 \leq k \leq m-2$. In this case by the induction hypothesis

$$\begin{aligned}
f(m,m) &= f(k,k) + f(m-k, m-k) + 1 \\
&= \frac{1}{3} \cdot \left(k^2 - 1 + (m-k)^2 - 1\right) + 1 \\
&= \frac{1}{3} \cdot \left(m^2 - 2mk + 2k^2 + 1\right) \\
&\leq \frac{1}{3} \cdot \left(m^2 - 1\right).
\end{aligned}$$

If $|max(\mathcal{H}^*)| = 3$, then at least one gridpoint does not belong to any of the maximal islands, consequently,

$$\begin{aligned} f(m,m) &= 1 + \sum_{T \in \max \mathcal{H}^*} f(u_T, v_T) \\ &\leq 1 + \sum_{T \in \max \mathcal{H}^*} \frac{1}{3} \cdot (u_T v_T - 1) \\ &= 1 + \sum_{T \in max \mathcal{H}^*} \frac{1}{3} \cdot (\mu(u_T, v_T) - 1) \\ &= 1 - \frac{1}{3} \cdot |\max(\mathcal{H}^*)| + \sum_{T \in \max \mathcal{H}^*} \frac{\mu(u_T, v_T)}{3} \\ &\leq 1 - \frac{1}{3} \cdot 3 + \frac{\mu(m,n) - 1}{3} \\ &= \frac{1}{3} \cdot (mn - 1)\,. \end{aligned}$$

□

Remark 7. We note that our upper bound is sharp, whenever $m = n = 2^k$ for some k.

3. Higher dimensions

Now, let us consider dimension d, for $d \geq 3$. We use the definitions from [6]. For $n \in \mathbf{N}$ let $\mathbf{n} = [1, n] = \{1, \ldots, n\}$. For $m_1, m_2, \ldots m_d \in \mathbf{N}$ the set $\mathbf{m_1} \times \mathbf{m_2} \times \cdots \times \mathbf{m_d}$ will be called a (d dimensional) *table* of size $m_1 \times m_2 \times \cdots \times m_d$. We consider $\mathbf{m_1} \times \mathbf{m_2} \times \cdots \times \mathbf{m_d}$ as a collection of *cells*.

As with the two-dimensional case, we identify $\mathbf{m_1} \times \mathbf{m_2} \times \cdots \times \mathbf{m_d}$ by the rectangular grid $\{\,0, 1, \ldots, m_1\,\} \times \cdots \times \{\,0, 1, \ldots, m_d\,\}$. Geometrically $\mathbf{m_1} \times \mathbf{m_2} \times \cdots \times \mathbf{m_d}$ is identified with a d-dimensional rectangular cuboid of sidelengths $m_1, \ldots, m_d$. A cell (i, j) is identified with the corresponding unit subcube, or with the set of its 2^d-many vertices.

Two cells are called *neighbouring cells* if the distance of their centers is at most $\sqrt{d}$. That is, two cells are neighbouring if their corresponding unit subcubes intersect by an at most $(d-1)$-dimensional face, i.e. if they have a common gridpoint.

By an array of size $m_1 \times m_2 \times \cdots \times m_d$ we mean a mapping $A \colon \mathbf{m_1} \times \mathbf{m_2} \times \cdots \times \mathbf{m_d} \to \mathbf{R}$, $(h_1, h_2, \ldots, h_d) \mapsto a_{h_1,h_2,\ldots h_d}$. Given A, for a cube R of the table $\mathbf{m_1} \times \mathbf{m_2} \times \cdots \times \mathbf{m_d}$ let $\min(A|_R)$ denote the minimum of $\{a_{h_1 h_2 \ldots, h_d} \colon (h_1, h_2, \ldots, h_d) \in R\}$. We say that R is a *cube island* of the array A if R is a d-dimensional subcube of the table and $a_{h_1 h_2 \ldots h_d} <$

$\min(A|_R)$ holds for each $(h_1, h_2, \ldots, h_d) \in (\mathbf{m_1} \times \mathbf{m_2} \times \cdots \times \mathbf{m_d}) \setminus R$ such that $(h_1, h_2, \ldots, h_d)$ is neighbouring with some cell of R. Let $\mathbf{L}_{m_1 \times \cdots \times m_d}$ denote the set of gridpoints contained in the table $(\mathbf{m_1} - \mathbf{1}) \times \cdots \times (\mathbf{m_d} - \mathbf{1})$. These gridpoints form an $m_1 \times \cdots \times m_d$ grid. The set of cube islands of A will be denoted by $\mathcal{I}_{\text{sqr}}(A)$. For a given table of size $(m_1 - 1) \times (m_2 - 1) \times \cdots \times (m_d - 1)$ (or equivalently on gridpoints $\mathbf{L}_{m_1 \times \cdots \times m_d}$). Notice that Lemmas 1 and 2 can be reformulated easily for higher dimensions.

Let $f(m_1, m_2, \ldots, m_d)$ denote the maximum number of cube islands. Using the ideas and proofs we presented in Section 2, we prove the following upper and lower bounds:

$$f\left(m_1, \ldots, m_d\right) \leq \frac{1}{2^d - 1} \cdot \left(m_1 \cdots m_d - 1\right),$$
$$f\left(m_1, \ldots, m_d\right) \geq \frac{1}{2^d - 1} \cdot \left(m_1 \cdots m_d - 2 \cdot \sum m_{j_1} m_{j_2} \cdots m_{j_{d-1}}\right),$$

where the sum is over the $d-1$ element subsets of $\{1, 2, \ldots, d\}$.

We prove the lower bound in Theorem 8, the upper bound in Theorem 9. Considering that the proof for higher dimensional cases is pretty similar to the proof for the two dimensional case, we only sketch the proofs of these theorems. We note, that our upper bound is sharp, whenever $m_1 = \cdots = m_d = 2^k$ for some positive integer k.

Theorem 8.

$$f\left(m_1, \ldots, m_d\right) \geq \frac{1}{2^d - 1} \cdot \left(m_1 \cdots m_d - 2 \cdot \sum m_{j_1} m_{j_2} \cdots m_{j_{d-1}}\right),$$

where the sum is over the $d-1$ element subsets of $\{1, 2, \ldots, d\}$.

Proof. Since the proof is quite similar to the two dimensional case, we just state the key observations without further explanations. Let $t\left(j_1, \ldots, j_d\right) = \sum_{i=1}^{d} j_i - d \cdot \min\{j_1, \ldots, j_d\}$ and let

$$\varepsilon\left(m_1, \ldots, m_d\right) = \begin{cases} 1, & \text{if } m_1 = \cdots = m_d \text{ and is not a power of } 2, \\ 0, & \text{otherwise.} \end{cases}$$

The extension of the two dimensional construction in Section 2.1 to higher dimension is straightforward. The number of cubes used is denoted by s. It is a lower bound of f and its recursive description is as follows:

$$\begin{aligned}
0 &= s\left(m_1,\dots,m_d\right), \text{ if } m_i = 1 \text{ for some } 1 \le i \le d,\\
s\left(2^j,\dots,2^j\right) &= 2^d \cdot s\left(2^{j-1},\dots,2^{j-1}\right) + 1,\\
s\left(2^{j_1},\dots 2^{j_d}\right) &= 2^{t(j_1,\dots,j_d)} \cdot s\left(2^{\min\{j_1,\dots,j_d\}}, 2^{\min\{j_1,\dots,j_d\}}\right),\\
s\left(m_1,\dots m_d\right) &= \sum_{0\le j_1\le k_1} \cdots \sum_{0\le j_d\le k_d} a^{(1)}_{j_1}\cdots a^{(d)}_{j_d} \cdot s\left(2^{j_1},\dots,2^{j_d}\right)\\
&\quad + \varepsilon\left(m_1,\dots,m_d\right),
\end{aligned}$$

where the binary form of m_i is $\overline{a^{(i)}_{k_i}\dots a^{(i)}_0}$ (i.e. $m_i = \sum_{j_i=0}^{k_i} a^{(i)}_{j_i}\cdot 2^{j_i}$) for every $1 \le i \le d$.

Similarly to the two dimensional case, by induction on j we have

$$s\left(2^j,\dots,2^j\right) = \frac{1}{2^d-1}\cdot\left(2^{d\cdot j}-1\right).$$

Similarly as in Subsection 2.2 we have for $j_1 \le \dots \le j_d$ that

$$\begin{aligned}
s\left(2^{j_1},\dots,2^{j_d}\right) &= \frac{1}{2^d-1}\cdot\left(2^{j_1+\dots+j_d} - 2^{j_2+\dots+j_d-(d-1)\cdot j_1}\right)\\
&= \frac{1}{2^d-1}\cdot\left(2^{j_1+\dots+j_d} - 2^{t(j_1,\dots,j_d)}\right).
\end{aligned}$$

Let the binary form of m_i be $\overline{a^{(i)}_{k_i}\dots a^{(i)}_0}$ for every $1 \le i \le d$. Now,

$$\begin{aligned}
s\left(m_1,\dots,m_d\right) &= \sum_{0\le j_1\le k_1} \cdots \sum_{0\le j_d\le k_d} a^{(1)}_{j_1}\cdots a^{(d)}_{j_d} \cdot s\left(2^{j_1},\dots,2^{j_d}\right)\\
&\quad + \varepsilon\left(m_1,\dots,m_d\right)\\
&= \frac{1}{2^d-1}\cdot \sum_{0\le j_1\le k_1} \cdots \sum_{0\le j_d\le k_d} a^{(1)}_{j_1}\cdots a^{(d)}_{j_d} \cdot \left(2^{j_1+\dots+j_d} - 2^{t(j_1,\dots,j_d)}\right)\\
&\quad + \varepsilon\left(m_1,\dots,m_d\right)\\
&= \frac{1}{2^d-1}\cdot m_1\cdots m_d\\
&\quad - \frac{1}{2^d-1}\cdot \sum_{0\le j_1\le k_1} \cdots \sum_{0\le j_d\le k_d} a^{(1)}_{j_1}\cdots a^{(d)}_{j_d} \cdot 2^{t(j_1,\dots,j_d)}\\
&\quad + \varepsilon\left(m_1,\dots,m_d\right).
\end{aligned}$$

As we proved in Lemma 4, we have

$$\sum_{0\le j_1\le k_1}\cdots\sum_{0\le j_d\le k_d} a^{(1)}_{j_1}\cdots a^{(d)}_{j_d}\cdot 2^{t(j_1,\dots,j_d)} \le$$
$$\sum_{i=1}^{d}\sum_{j_1=0}^{k} a^{(i)}_{j_i}\cdot 2^{-(d-1)\cdot j_i}\cdot \sum_{j_1,\dots j_{i-1},j_{i+1},\dots,j_d=j_i}^{k} \frac{a^{(1)}_{j_1}\cdots a^{(d)}_{j_d}}{a^{(i)}_{j_i}}\cdot 2^{j_1+\dots+j_d-j_i}$$
$$\le 2\cdot\sum m_{j_1}m_{j_2}\cdots m_{j_{d-1}},$$

where the last sum is over the $d-1$ element subsets of $\{1,2,\dots,d\}$. This finishes the proof of Theorem 8. □

Theorem 9.

$$f(m_1,\dots,m_d)\le \frac{1}{2^d-1}\cdot(m_1\cdots m_d-1).$$

Proof. The proof is quite similar to the two dimensional case, hence we just state the key observations without further explanations.

We prove the statement by induction on $m_1,\dots,m_d$. If $m_i\le 2$ for all but one $1\le i\le d$, then the statement is clear. The induction hypothesis is, that for any table T with $u_1\dots u_d$-many gridpoints, such that $u_i\le m_i$ for every $1\le i\le d$, $(u_1,\dots,u_d)\ne(m_1,\dots m_d)$, we have $f(u_1,\dots,u_d)\le\frac{1}{2^d-1}\cdot(u_1\cdots u_d-1)$. We define $\mu(u_1,\dots,u_d):=u_1\cdots u_d$, which is exactly the number of grid points of the table of sidelengths $u_1-1,\dots,u_d-1$. Let the sides of the table T contain $u_{T,1},u_{T,2},\dots,u_{T,d}$-many gridpoints. Let

$$\varepsilon^*(m_1,\dots,m_d)=\begin{cases}1, & \text{if } m_1=\cdots=m_d,\\ 0, & \text{otherwise.}\end{cases}$$

Let $\mathcal{H}^*$ be a system of cube islands of $(\mathbf{m_1}-\mathbf{1})\times\cdots\times(\mathbf{m_d}-\mathbf{1})$, such that $|\mathcal{H}^*|=f(m_1,\dots,m_d)$. Let $\max\mathcal{H}^*$ be the set of maximal islands in $\mathcal{H}^* \setminus \{ (\mathbf{m_1 - 1}) \times \cdots \times (\mathbf{m_d - 1}) \}$. Now,

$$\begin{aligned}
f(m_1,\dots,m_d) &= \varepsilon^*(m_1,\dots,m_d) + \sum_{T\in\max\mathcal{H}^*} f(u_{T,1},\dots,u_{T,d}) \\
&\le \varepsilon^*(m_1,\dots,m_d) + \sum_{T\in\max\mathcal{H}^*} \frac{1}{2^d-1}\cdot(u_{T,1}\cdots u_{T,d}-1) \\
&= \varepsilon^*(m_1,\dots,m_d) + \sum_{T\in max\mathcal{H}^*} \frac{1}{2^d-1}\cdot(\mu(u_{T,1},\dots,u_{T,d})-1) \\
&= \varepsilon^*(m_1,\dots,m_d) - \frac{1}{2^d-1}\cdot|\max(\mathcal{H}^*)| \\
&\qquad + \sum_{T\in\max\mathcal{H}^*} \frac{\mu(u_{T,1},\dots,u_{T,d})}{2^d-1} \\
&\le \varepsilon^*(m_1,\dots,m_d) - \frac{1}{2^d-1}\cdot|\max(\mathcal{H}^*)| + \frac{\mu(m_1,\dots,m_d)}{2^d-1} \\
&= \varepsilon^*(m_1,\dots,m_d) - \frac{1}{2^d-1}\cdot|\max(\mathcal{H}^*)| + \frac{m_1\cdots m_d}{2^d-1}.
\end{aligned}$$

Recall that $\varepsilon^*(m_1,\dots,m_d) = 1$, whenever $m_1 = m_2 = \cdots = m_d$, otherwise $\varepsilon^*(m_1,\dots,m_d) = 0$. Thus if $m_i \neq m_j$ for some $1 \le i < j \le d$ or $|max(\mathcal{H}^*)| \ge 2^d$ then we obtained that

$$f(m_1,\dots,m_d) \le \frac{1}{2^d-1}\cdot(m_1\cdots m_d - 1).$$

For the remaining cases we assume that $m_1 = m_2 = \cdots = m_d \ge 3$. Let $m = m_1$ and let $t = |max(\mathcal{H}^*)|$. We only need to consider the case $t \le 2^d - 1$. We use induction on m. The maximum side length of the cubes in $max(\mathcal{H}^*)$ is denoted by $m-k$. All the other side lengths are at most k, we denote them by $k_1, k_2, \dots, k_{t-1}$. Then the maximum number of cube islands is $1 + f(m-k,\dots,m-k) + f(k_1,\dots,k_1) + \cdots + f(k_{t-1},\dots,k_{t-1})$. Now,

$$\begin{aligned}
f(m,\dots,m) &= 1 + \sum_{T\in\max\mathcal{H}^*} f(u_{T,1},\dots,u_{T,d}) \\
&\le 1 + \sum_{T\in\max\mathcal{H}^*} \frac{1}{2^d-1}\cdot(u_{T,1}\cdots u_{T,d}-1) \\
&= 1 - \frac{1}{2^d-1}\cdot|\max(\mathcal{H}^*)| + \frac{(m-k)^d + k_1^d + \cdots + k_{t-1}^d}{2^d-1} \\
&\le \frac{2^d - 1 - t + (m-k)^d + (t-1)\cdot k^d}{2^d-1}.
\end{aligned}$$

Let $g(x) = (m-x)^d + (t-1) \cdot x^d$. For $x \in [1, m/2]$ we have $g''(x) > 0$, i.e. g is convex in the interval $[1, m/2]$, thus $g(x) \leq \max\{g(1), g(m/2)\}$ for $x \in [1, m/2]$. Now,

$$\begin{aligned} g(1) &= (m-1)^d + (t-1) \\ &\leq m^d + t - 1 - d \cdot (m-1)^{d-1} \\ &\leq m^d + t - 1 - 3 \cdot 2^{d-1} \\ &\leq m^d + t - 2^d, \\ g(m/2) &= t \cdot (m/2)^d \\ &= m^d - \frac{m^d}{2^d} \cdot \left(2^d - t\right) \\ &\leq m^d + t - 2^d, \end{aligned}$$

hence

$$\begin{aligned} f(m, \ldots, m) &\leq \frac{2^d - 1 - t + (m-k)^d + (t-1) \cdot k^d}{2^d - 1} \\ &\leq \frac{2^d - 1 - t + m^d + t - 2^d}{2^d - 1} \\ &= \frac{1}{2^d - 1} \cdot \left(m^d - 1\right). \end{aligned}$$

□

Remark 10. Our upper bound is sharp, whenever $m_1 = \cdots = m_d = 2^k$ for some k. We conjecture that $f(m_1, \ldots, m_d) = s(m_1, \ldots, m_d)$ always holds, but we think that the verification would require a fair amount of technical estimations, including the distinction of several cases and subcases and might not be worth of the effort and space. This is also suggested by the fact that in most cases the arrengement of the squares is not unique.

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