STUDENT RESEARCH PROJECT

EDITOR

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Student Research Projects are articles that describe how to give students the experience of independent, research-style work in a specific mathematical setting. For example, a Student Research Project might present a subject in the form of a series of open-ended questions. Student Research Projects should be submitted to Brigitte Servatius at bservat@math.wpi.edu.

One-dimensional Czédli-type Islands

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Here is a research topic in which elementary mathematics plays crucial role. It is suitable for nearly any college-level mathematics student. The exercises include interesting examples of induction arguments, particularly strong induction. Moreover, this is a *living* research topic (see the references), so *any* good ideas may influence *present or future* investigations (or not).

Introducing islands

Let n consecutive square cells be given; this is our "board." We write nonnegative integers into each cell and call these *heights*. Consecutive cells constitute an *island* if the integers in them are all greater than the integers in the neighboring cells. Note that this is different from the islands we are used to. Islands in our experience are surrounded by water (which could be height 0 here). Our Czédli-type islands can occur at any height. Moreover, islands can be included in other islands (See Figure 1.)

The following exercises will help the reader become familiar with islands. Note: we always use n for the number of cells and h for the maximum height on a board. We further assume that every board is bordered with cells of height 0. Then, if there are no other 0s, the whole board is an island.

Exercise 1. Show how to put heights into the cells of any one-dimensional board in order to obtain 0 islands. Repeat with 1, 2, and 3 islands.

Exercise 2. Find a formula for the number of possible boards with n cells and maximum height h. (Check: for $n = 20$ and $h = 2$, there are approximately 3.48×10^9 boards.)

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MSC: 05A99, 05C05, 06A07, 06B99

0	0	1	1	1	0	0	0

one island

0	1	2	2	1	0	1	2	1	2

5 islands

0	1	2	2	1	0	1	2	1	2	3	3

6 islands

Figure 1. Example boards (darker shading = higher height).

Exercise 3. Given n and h, discover how to create boards with exactly the following numbers of islands, or show that it is not possible: $\lfloor \frac{n}{2} \rfloor - 1$ islands, $\lfloor \frac{n}{2} \rfloor$ islands, $\lfloor \frac{n}{2} \rfloor + 1$, $n - 2$ islands, $n - 1$ islands, n islands, $n + 1$ islands, h islands, $2h$ islands, and h^2 islands. (Here $\lfloor x \rfloor$ is the greatest integer in x.)

Exercise 4. Prove using induction on n that the maximum number of islands on a board of length n is n.

The main problem What is the maximum number of islands, given n and h?

At first glance this does not look easy. As Exercise 2 shows, brute force will not solve it. On the other hand, Exercise 4 completely solves the problem when $h > n$. Trial and error with small boards leads eventually to the conjecture that the maximum number of islands is

$$I(n, h) = n - \left\lfloor \frac{n}{2^h} \right\rfloor.$$

(More detail is available in [**3**].)

A proof can be accomplished in two steps: first show that it is possible to create $I(n, h)$ islands on a board with n cells and maximum height h; then prove that no more than $I(n, h)$ islands are possible.

Boards exist with $I(n, h)$ islands

In fact one can prove that $I(n, h)$ islands are possible on a board in which every other cell has height h. The next exercises outline how to complete these steps using strong induction on h.

Exercise 5. Base case. Show that if $h = 1$, then boards exist with $I(n, h) = n - \lfloor \frac{n}{2} \rfloor$ islands with every other cell of maximum height.

The induction step proceeds simultaneously in four subcases depending on the remainder when n is divided by 4.

Exercise 6. In case n is of form $4k$, then by the induction hypothesis on $2k$ cells with height $h - 1$, we can realize at least

$$2k - \left\lfloor \frac{2k}{2^{h-1}} \right\rfloor$$

islands and every other cell has height $h-1$ starting with the first. (See Figure 2(a).)

2	1	2	0	2	1

(a) $n = 6,\ h = 2.$

3	2	3	1	3	2	3	0	3	2	3	1

(b) $n = 12,\ h = 3.$

Figure 2. Example of induction on h.

Now replace each cell of maximum height $h-1$ by three cells with the heights h, $h-1$, h respectively. Then we obtain $n = 4k$ cells with

$$2k - \left\lfloor \frac{2k}{2^{h-1}} \right\rfloor + 2k = 4k - \left\lfloor \frac{4k}{2^{h}} \right\rfloor$$

islands. (See Figure 2(b).)

Exercise 7. Complete the induction on h in the cases when n is congruent to 1, 2, or 3 (mod 4).

No more than $I(n, h)$ islands are possible

One can prove this by induction on n. The base case ($n = 1$) is trivial.

Exercise 8. Let $n > 1$. The induction hypothesis is that for all $n' < n$, no more than

$$I(n', h) = n' - \left\lfloor \frac{n'}{2^h} \right\rfloor$$

islands are possible.

First, suppose that there is a height 0 on the board. The cell 0 divides the board into two parts. We denote the lengths of these parts by k and l, and obviously $k + l + 1 = n$, where $k, l \geq 0$. Apply the induction hypothesis and use the following inequality,

$$\left\lfloor \frac{k}{2^h} \right\rfloor + \left\lfloor \frac{l}{2^h} \right\rfloor + 1 \geq \left\lfloor \frac{k+l+1}{2^h} \right\rfloor$$

to complete the induction in this case.

Exercise 9. Denote by m the minimum height on the board. If $m > 0$, then the whole board is an island. The other islands on this board are the same as those on the new board obtained by subtracting m from each cell. Apply the result of exercise 8 to this board to complete the induction proof.

Conclusion

In two dimensions, Gábor Czédli [**2**] has determined the maximum number of rectangular islands on an $m \times n$ size rectangular board:

$$f(m, n) = \left\lfloor \frac{mn + m + n - 1}{2} \right\rfloor.$$

His proof is based on a result in lattice theory, but now by [**1**] two elementary ways are also known to prove the same result. We mention, that the exact result, i.e., the exact formula for the maximum number of *triangular* islands on the *triangular* grid is not yet known, only upper and lower bounds [**4**]. The situation is the same with the square islands on a square grid, see [**5**]. The topic of islands is still developing, already many branches of mathematics are involved. We hope that creative students will find more interesting problems and their solutions in this field.

Finally, here are some further exercises:

Exercise 10. Prove that if $n \leq 2^h - 1$, then if we put elements of the set $\{0, 1, 2, \ldots, h\}$ in the cells, then we can have n islands, but if $n \geq 2^h$, then the maximum number of islands is *less* than n.

Exercise 11. Prove that if the numbers $\{1, 2, \ldots, n\}$ are put into the cells of a one dimensional board of length n in *any* order, then we will always have *exactly* n islands.

Exercise 12. Show, that for arbitrary h the number of islands cannot be maximal if we have height h in (at least two) neighbouring cells.

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Summary. The notion of an island has surfaced in recent algebra and coding theory research. Discrete versions provide interesting combinatorial problems. This paper presents the one-dimensional case with finitely many heights, a topic convenient for student research.

References

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The Perfect Ploy?

After Audrey, protagonist of the novel *The Perfect Play* by Louise Wener, recounts to her tutee Ryan how Hippasus was sentenced to drowning by Pythagoras for demonstrating the irrationality of the square root of two, they have the following exchange.

Audrey: "Anyway, the thing is, numbers were so important to the philosophers of ancient Greece that people sometimes killed each other because of them. The early Christians even had the first libraries burnt to the ground and all the maths books destroyed because they were so afraid of what numbers might tell them."

Ryan: "So, it wasn't just blokes in togas sitting around on their arses, plotting ways to ruin my life with homework?"

Audrey: "No. It was violence and murder and secret sects and drowning and war and lots of stabbing."

Ryan: "A bit like the Mafia?"

Audrey: "In a manner of speaking, yes."

Ryan: "Cool."

Audrey: "So you'll try and have some of this algebra homework done for me when I come round again next week?"

Ryan: "Yeah. I'll have a think about it."

—extracted from *The Perfect Play* by Louise Wener
(Harper Paperbacks, 2005) p. 50–51.
—suggested by Michael A. Jones