A scheme for congruence semidistributivity

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Abstract

A diagrammatic statement is developed for the generalized semidistributive law in case of single algebras assuming that their congruences are permutable. Without permutable congruences, a diagrammatic statement is developed for the $\wedge$-semidistributive law.

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Some attempts show that instead of identities in congruence lattices, certain diagrammatic statements are reasonable to consider, see [1] and Gumm [4]. The aim of the present paper is to show that this phenomenon can be extended to lattice Horn sentences as well.

Definition 1. A lattice L is $\wedge$-*semidistributive* if it satisfies the following implication for all $\alpha, \beta, \gamma \in L$:

$$\alpha \wedge \beta = \alpha \wedge \gamma \quad \Rightarrow \quad \alpha \wedge (\beta \vee \gamma) = \alpha \wedge \beta.$$

The $\wedge$-semidistributive law above is often denoted by $SD_\wedge$. More general (in fact, weaker) Horn sentences have been investigated in Geyer [3] and Czédli [2]. For $n \geq 2$ put $\mathbf{n} = \{0, 1, \ldots, n-1\}$ and let $P_2(\mathbf{n})$ denote the set $\{S : S \subseteq \mathbf{n} \text{ and } |S| \geq 2\}$.

Definition 2. For $\emptyset \neq H \subseteq P_2(\mathbf{n})$ we define the *generalized meet semidistributive law* $SD_\wedge(n, H)$ for lattices as follows: for all $\alpha, \beta_0, \ldots, \beta_{n-1}$

$$\alpha \wedge \beta_0 = \alpha \wedge \beta_1 = \ldots = \alpha \wedge \beta_{n-1} \quad \Rightarrow \quad \alpha \wedge \beta_0 = \alpha \wedge \bigwedge_{I \in H} \bigvee_{i \in I} \beta_i.$$

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As a particular case, when $H = \{S : S \subseteq \mathbf{n} \text{ and } |S| = 2\}$, $SD_\wedge(n, H)$ is denoted by $SD_\wedge(n, 2)$. Notice that $SD_\wedge(n, 2)$ is the following lattice Horn sentence:

$$\alpha \wedge \beta_0 = \alpha \wedge \beta_1 = \cdots = \alpha \wedge \beta_{n-1} \quad \Rightarrow \quad \alpha \wedge \bigwedge_{0 \le i < j < n} (\beta_i \vee \beta_j) = \alpha \wedge \beta_0,$$

which was originally studied by Geyer [3], and $SD_\wedge(2, 2)$ is the $\wedge$-semidistributivity law defined in Definition 1. Czédli [2] has noticed that $SD_\wedge(n, 2)$ is strictly weakening in n, i. e. $SD_\wedge(n, 2)$ implies $SD_\wedge(n + 1, 2)$ but not conversely.

Our goal is to study $SD_\wedge(n, H)$ in congruence lattices of *single* algebras. Although it is usual to consider lattice identities and Horn sentences in congruence lattices of all algebras of a variety, this is not our case. The reason is that, for an arbitrary variety $\mathcal{V}$, if $SD_\wedge(n, H)$ holds in $\{\mathrm{Con}\, A : A \in \mathcal{V}\}$ then so does $SD_\wedge$. (This was proved by Czédli [2] and an anonymous referee of [2] who pointed out that both Kearnes and Szendrei [5] and Lipparini [6] contain implicitly the statement that if a lattice Horn sentence λ can be characterized by a weak Mal'cev condition and, for each nontrivial module variety $\mathcal{M}$, λ fails in $\mathrm{Con}\, M$ for some $M \in \mathcal{M}$ then for an arbitrary variety $\mathcal{V}$, if λ holds in $\{\mathrm{Con}\, A : A \in \mathcal{V}\}$ then so does $SD_\wedge$, cf. the last paragraph in [2].) In particular, for any variety $\mathcal{V}$ and any $n \ge 2$, $SD_\wedge(n, 2)$ and $SD_\wedge$ are equivalent for the class $\{\mathrm{Con}\, A : A \in \mathcal{V}\}$. Hence $SD_\wedge(n, 2)$ does not deserve a separate study for *varieties*.

First, we consider congruence permutable algebras.

Theorem 1. *Let A be a congruence permutable algebra. Then $\mathrm{Con}\, A$ satisfies $SD_\wedge(n, 2)$ if and only if A satisfies the scheme depicted in Figure 1 for $\alpha, \beta_0, \ldots, \beta_{n-1} \in \mathrm{Con}\, A$ and $x_0, \ldots, x_k, y, z \in A$ where $k = \frac{n(n-1)}{2} - 1$ and δ stands for $\beta_0 \cap \beta_1 \cap \cdots \cap \beta_{n-1}$.*

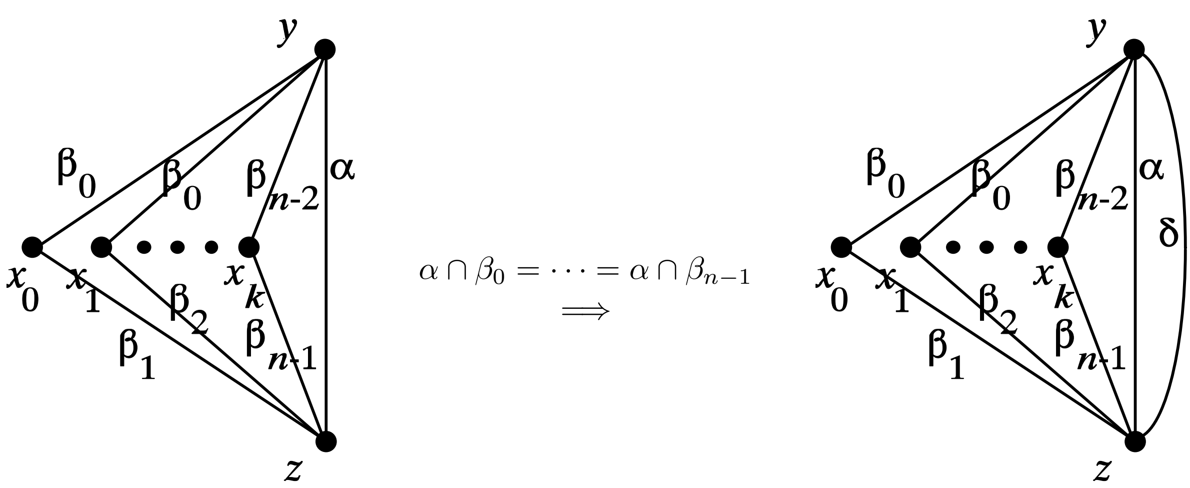

Figure 1

Proof. Suppose $SD_\wedge(n, 2)$ holds. Using the premise of $SD_\wedge(n, 2)$ we obtain

$$\alpha \cap \beta_0 = (\alpha \cap \beta_0) \cap \cdots \cap (\alpha \cap \beta_{n-1}) = \alpha \cap (\beta_0 \cap \cdots \cap \beta_{n-1}) \subseteq \delta,$$

whence Con A satisfies the Horn sentence

$$\alpha \cap \beta_0 = \cdots = \alpha \cap \beta_{n-1} \quad \Rightarrow \quad \alpha \cap \bigcap_{0 \le i < j < n} (\beta_i \vee \beta_j) \le \delta.$$

This implies the scheme, for the situation on the left hand side in Figure 1 then gives

$$(y, z) \in \alpha \cap \bigcap_{0 \le i < j < n} (\beta_i \circ \beta_j) \subseteq \alpha \cap \bigcap_{0 \le i < j < n} (\beta_i \vee \beta_j) \subseteq \delta.$$

To show the converse suppose that the scheme given by Figure 1 holds, $\alpha, \beta_0, \ldots, \beta_{n-1} \in \mathrm{Con}\, A$ with $\alpha \cap \beta_0 = \cdots = \alpha \cap \beta_{n-1}$, and suppose that $(y, z) \in \alpha \cap \bigcap_{0 \le i < j < n} (\beta_i \vee \beta_j)$. Since $\beta_i \vee \beta_j = \beta_i \circ \beta_j$ by congruence permutability, there exist $x_0, x_1, \ldots, x_k$ of A such that for each j $(1 \le j \le k)$ there exist u, v such that $(z, x_j) \in \beta_u$ and $(x_j, y) \in \beta_v$ (according to the left hand side of Figure 1). Then the scheme applies and we conclude $(y, z) \in \delta$. Since $\delta \subseteq \beta_0$, $(y, z) \in \beta_0$. Hence $(y, z) \in \alpha \cap \beta_0$. This proves the "$\le$" part of $SD_\wedge(n, 2)$. The reverse part is simpler and does not need the scheme: $\alpha \supseteq \alpha \cap \beta_0$ and $\beta_i \vee \beta_j \supseteq \beta_i \supseteq \alpha \cap \beta_i = \alpha \cap \beta_0$ clearly give

$$\alpha \cap \bigcap_{0 \le i < j < n} (\beta_i \vee \beta_j) \supseteq \alpha \cap \beta_0,$$

proving the theorem.

In the particular case when $n = 2$ we trivially conclude the following assertion:

Theorem 2. *Let A be a congruence permutable algebra. Then* Con A *is $\wedge$-semidistributive if and only if A satisfies the so-called triangular scheme in Figure 2 for any $\alpha, \beta, \gamma \in$* Con A *and $x, y, z \in A$.*

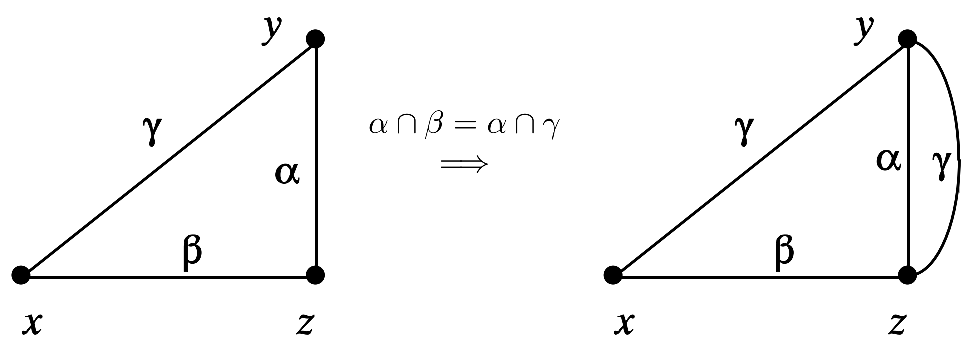

Figure 2

Proof. If Con A is $\wedge$-semidistributive, then the premise of the Triangular Scheme gives $(y, z) \in \beta \cap \gamma \subseteq \gamma$ by Theorem 1. Conversely, if the Triangular Scheme holds for A then its premise, after interchanging the role of β and γ, implies $(y, z) \in \beta \cap \gamma$, so $SD_\wedge(2, 2)$, which is the usual $\wedge$-semidistributivity, holds in Con A by Theorem 1.

One may observe that this scheme in Theorem 2 is the same as that in [1] characterising congruence distributivity in the congruence permutable case. This implies that: *in presence of congruence permutability, congruence $\wedge$-semidistributivity is equivalent to congruence distributivity.*

This follows also from another direction. Let A be congruence permutable and satisfying $SD_\wedge$. In this case A is congruence distributive since otherwise its congruence lattice, being modular due to congruence permutability, contains M_3 but with the choice α, β, γ on Figure 3 we see that $SD_\wedge$ fails.

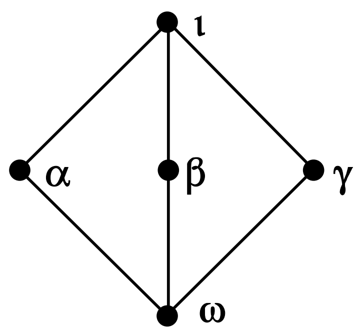

Figure 3

Remark. For $SD_\wedge(n, H)$, a similar scheme can be derived as in Theorem 1.

Without congruence permutability, for the case $SD_\wedge(2, 2) = SD_\wedge$, the following theorem can be stated:

Theorem 3. *Let A be an algebra. The congruence lattice* Con A *is $\wedge$-semidistributive if and only if for each n, A satisfies the scheme in Figure 4 for $\alpha, \beta, \gamma \in$* Con A *and $x, y, z \in A$, where $\Lambda_0 = \beta$ and $\Lambda_{m+1} = \Lambda_m \circ \gamma \circ \beta$.*

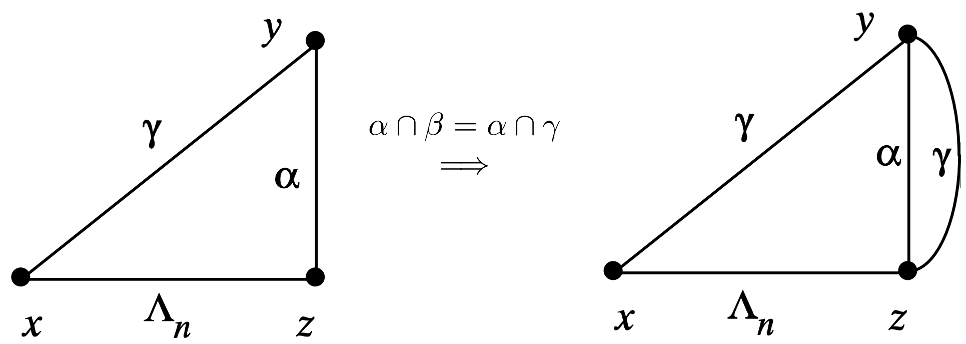

Figure 4

Proof. Suppose that Con A is $\wedge$- semidistributive and $\alpha, \beta, \gamma \in$ Con A with $\alpha \cap \beta = \alpha \cap \gamma$. Let $x, y, z \in A$ and let $(x, y) \in \gamma$, $(y, z) \in \alpha$ and $(x, z) \in \Lambda_n$. Then

$$(y, z) \in \alpha \cap (\Lambda_n \circ \gamma) \subseteq \alpha \cap (\beta \vee \gamma) = \alpha \cap \beta = \alpha \cap \gamma$$

due to the $\wedge$-semidistributivity. Thus $(y, z) \in \gamma$, proving the validity of the scheme.

Conversely, let A satisfy the scheme for each $n \in \mathbf{N}_0$, let $\alpha, \beta, \gamma \in \mathrm{Con}\, A$ with $\alpha \cap \beta = \alpha \cap \gamma$. Suppose $(z, y) \in \alpha \cap (\beta \vee \gamma)$. Then there exists $n \in \mathbf{N}_0$ such that $(z, y) \in \alpha \cap (\Lambda_n \circ \gamma)$ and hence $(x, y) \in \gamma$ and $(y, z) \in \alpha$ and $(x, z) \in \Lambda_n$ for some $x \in A$. Due to the scheme, we conclude $(x, y) \in \alpha \cap \gamma$, i.e. $\alpha \cap (\beta \vee \gamma) \subseteq \alpha \cap \gamma \subseteq \alpha \cap \beta$. The converse inclusion is trivial, thus $\mathrm{Con}\, A$ is $\wedge$-semidistibutive.

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