A NOTE ON LATTICE VARIANT OF THRESHOLDNESS OF BOOLEAN FUNCTIONS

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Abstract. Lattice induced threshold function is a Boolean function determined by a particular linear combination of lattice elements. We prove that every isotone Boolean function is a lattice induced threshold function and vice versa.

1. Introduction

,,Threshold functions provide a simple but fundamental model for many questions investigated in electrical engineering, artificial intelligence, game theory and many other areas." (Quotation from [3].) In [10], modeling neurons and political decisions are also mentioned, as application of classical threshold functions. A classical **threshold function** is a Boolean function $f : \{0,1\}^n \to \{0,1\}$ such that there exist real numbers $w_1, \ldots, w_n, t$, fulfilling

$$f(x_1, \ldots, x_n) = 1 \text{ if and only if } \sum_{i=1}^{n} w_i \cdot x_i \geq t, \tag{1.1}$$

where w_i is called the **weight** of x_i, for $i = 1, 2, \ldots, n$ and t is a constant called the **threshold value**. In this paper we define a new but related notion: the so called *lattice induced threshold function* and we investigate its properties.

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1.1. **Historical background.** For the historical as well as for the basic mathematical background we recommend the books [3, 10, 11]. For some algebraic aspects of classical threshold functions, we list relevant algebraic papers from several areas of algebra. In [1] the authors reveal the connection between classical threshold functions and fundamental ideals of group-rings. Paper [8] determines the invariance groups of threshold functions. The topic of monotone Boolean functions has constantly been investigated. For the recent results see e.g. [2, 6, 9]. Paper [7] proves that classical threshold functions cannot be characterized by a finite set of standard or generalized constraints.

1.2. **Motivation.** Isotone Boolean functions constitute a clone; threshold functions are not closed under superposition, see Theorem 9.2 in [3], so they do not constitute a clone. It is easy to see that threshold functions with positive weights and a threshold value are isotone. However, an isotone Boolean function is not necessarily threshold, e.g. $f = x \cdot y \vee w \cdot z$ is isotone, but not a threshold function. To see this, it is enough to consider its invariance group, which is the following: $D8 = \{(), (1324), (12)(34), (1423), (12), (34), (13)(24), (14)(23)\}$, however by [8] the invariance group of any threshold function is a direct product of symmetric groups.

Therefore, our aim is to generalize threshold functions in the framework of lattice valued functions, and then, in particular, to obtain a characterization of all isotone Boolean functions and to represent them by particular linear combinations.

1.3. **Outline.** In Preliminaries, we present properties of closure operations, lattice valued functions, Boolean functions and classical threshold functions.

In Section 3 we define threshold functions induced by complete lattices; these are Boolean functions determined by particular linear combination of lattice elements. We prove that every isotone Boolean function is a lattice induced threshold function and vice versa.

2. Preliminaries

2.1. **Order, lattices.** Our basic notion is a partially ordered set, **ordered set, poset**, denoted by $(P, \leq)$, where $\leq$ is an order on a set P. If a poset is a **lattice**, then it is denoted by $(L, \leq)$, with the meet and the join of $a, b \in L$ being $a \wedge b$ and $a \vee b$ respectively. The bottom and the top of a poset, if they exist, are 0 and 1, respectively. If $(P, \leq)$ is a poset, then we denote by $\bigwedge M$ and $\bigvee M$ the meet and the join of $M \subseteq P$ respectively, if they exist. We deal with **complete lattices** and **free distributive lattices with n generators**. We also use finite **Boolean lattices**, represented by all n-tuples of 0 and 1, ordered componentwise, and denoted by $(\{0, 1\}^n, \leq)$.

2.2. **Boolean functions; threshold functions.** A **Boolean function** is a mapping $f : \{0,1\}^n \to \{0,1\}$, $n \in \mathbb{N}$.

The domain $\{0,1\}^n$ of a Boolean function is usually ordered componentwise, with respect to the natural order $0 \leq 1$: $(x_1, x_2, \ldots, x_n) \leq (y_1, y_2, \ldots, y_n)$ if and only if for every $i \in \{1, \ldots, n\}$, $x_i \leq y_i$.

As it is known, the poset $(\{0,1\}^n, \leq)$ is a Boolean lattice. Moreover, every finite Boolean lattice with n atoms is isomorphic to this one.

A Boolean function $f : \{0,1\}^n \to \{0,1\}$ is **isotone** (order preserving, positive, as in [3]), if for every $x, y \in \{0,1\}^n$, from $x \leq y$, it follows that $f(x) \leq f(y)$.

The following is easy to check.

Lemma 1. *The set $F \subseteq \{0,1\}^n$ is an order semi-filter on $(\{0,1\}^n, \leq)$ if and only if a Boolean function f defined by*

$$f(x) = 1 \text{ if and only if } x \in F$$

is isotone.

As mentioned in Introduction, a **threshold function** is a Boolean function $f : \{0,1\}^n \to \{0,1\}$ such that there exist real numbers $w_1, \ldots, w_n, t$, fulfilling

$$f(x_1, \ldots, x_n) = 1 \text{ if and only if } \sum_{i=1}^{n} w_i \cdot x_i \geq t,$$

where w_i are called the **weights** of x_i, for $i = 1, 2, \ldots, n$ and t is a constant called the **threshold value**.

3. Threshold functions induced by complete lattices

In this section we introduce threshold functions induced by complete lattices, and we use them for investigating isotone Boolean functions and their representation.

We deal with functions over the Boolean lattice $(\{0,1\}^n, \leq)$, and we use the complete lattice L in which the bottom and the top are (also) denoted by 0 and 1 respectively; however, it will be clear from the context whether 0 (1) is a component in some $(x_1, \ldots, x_n) \in \{0,1\}^n$, or it is from L.

For $x \in \{0,1\}$, and $w \in L$, we define a mapping $L \times \{0,1\}$ into L denoted by "$\cdot$", as follows:

$$w \cdot x := \begin{cases} w, & \text{if } \quad x = 1 \\ 0, & \text{if } \quad x = 0. \end{cases} \tag{3.1}$$

A function $f : \{0,1\}^n \to \{0,1\}$ is a **lattice induced threshold function**, if there is a complete lattice L and $w_1, \ldots, w_n, t \in L$, such that

$$f(x_1,\dots,x_n)=1 \text{ if and only if } \bigvee_{i=1}^{n}(w_i\cdot x_i)\geq t. \tag{3.2}$$

Proposition 1. *Every lattice induced threshold function is isotone.*

Proof. Let L be a complete lattice and $w_1,\dots,w_n,t\in L$, and $f:\{0,1\}^n\to\{0,1\}$ a lattice induced threshold function.

Let $(x_1,x_2,\dots,x_n)\leq(y_1,y_2,\dots,y_n)$. Then, for every i, we have $w_i\cdot x_i\leq w_i\cdot y_i$, by the definition (3.1). Hence,

$$\bigvee_{i=1}^{n}(w_i\cdot x_i)\leq\bigvee_{i=1}^{n}(w_i\cdot y_i).$$

Therefore, if $f(x_1,\dots,x_n)=1$, then

$$\bigvee_{i=1}^{n}(w_i\cdot x_i)\geq t,\quad\text{and hence}\quad\bigvee_{i=1}^{n}(w_i\cdot y_i)\geq t.$$

This implies $f(y_1,\dots,y_n)=1$ and we obtain

$$f(x_1,\dots,x_n)\leq f(y_1,\dots,y_n),$$

which proves that f is an isotone function. □

Theorem 1. *Every isotone Boolean function is a lattice induced threshold function.*

Proof. We prove that for every $n\in\mathbb{N}$, there is a lattice L such that every isotone Boolean function is a lattice induced threshold function over L.

Let $n\in\mathbb{N}$. We take L to be a free distributive lattice with n generators w_1, $w_2,\dots,w_n$ (the join and meet of empty set of generators are also counted here, as the bottom and the top of L, respectively). Recall that every element in a free distributive lattice can be uniquely represented in a "conjunctive normal form" by means of generators (i.e., every element is a meet of elements of the type $\bigvee_{i\in J}w_j$, where $J\subseteq\{1,\dots,n\}$,) see e.g., [4].

Therefore, for $x,y\in L$, if $x=\bigwedge_{k=1}^{p}\bigvee_{j\in I_k}w_j$ and $y=\bigwedge_{k=1}^{l}\bigvee_{s\in J_k}w_s$, then $x\leq y$ if and only if for every $u\in\{1,\dots,l\}$ there is $k\in\{1,\dots,p\}$ such that $I_k\subseteq J_u$. (*)

Let $f:\{0,1\}^n\to\{0,1\}$ be an isotone Boolean function. Let F be the corresponding order semi-filter on $\{0,1\}^n$ (according to Lemma 1). Further, let $m_1,\dots,m_p$ be minimal elements of this semi-filter. Let $I_1,\dots,I_p$ be subsets of $\{1,2,\dots,n\}$, i.e., sets of indices, such that $i\in I_k$ if and only if $i-th$ coordinate of m_k is equal to 1.

For the threshold $t \in L$ associated to the given function f we take

$$t = \bigwedge_{k=1}^{p} \bigvee_{j \in I_k} w_j.$$

Now, we prove that

$$f(x_1, \dots, x_n) = 1 \text{ if and only if } \bigvee_{i=1}^{n} (w_i \cdot x_i) \geq t. \tag{3.3}$$

Indeed, from $f(x_1, \dots, x_n) = 1$, it follows that there is a minimal element m_l in the corresponding semi-filter, such that $(x_1, \dots, x_n) \geq m_l$. Hence,

$$\bigvee_{i=1}^{n} (w_i \cdot x_i) \geq \bigvee_{j \in I_l} w_j \geq \bigwedge_{k=1}^{p} \bigvee_{j \in I_k} w_j = t.$$

Now we suppose that

$$\bigvee_{i=1}^{n} (w_i \cdot x_i) \geq \bigwedge_{k=1}^{p} \bigvee_{j \in I_k} w_j$$

for an ordered n-tuple $(x_1, \dots, x_n)$. Let $I \subseteq \{1, \dots n\}$ be the set of indices such that $x_i = 1$ if and only if $i \in I$. We prove that there is $s \in \{1, \dots p\}$ such that $I_s \subseteq I$. This follows directly from the above mentioned (*) property of the free distributive lattice with n generators $w_1, \dots, w_n$.

Now, it follows that $(x_1, \dots, x_n) \geq (y_1, \dots, y_n)$, where $y_i = 1$ if and only if $i \in I_s$. Therefore,

$$\bigvee_{i=1}^{n} (w_i \cdot y_i) \geq t, \quad \text{and} \quad f(y_1, \dots, y_n) = 1.$$

This finally implies $f(x_1, \dots, x_n) = 1$. □

Remark 1. In the previous proposition it is proved not only that every isotone n-ary Boolean function is a lattice induced threshold function, but also that the corresponding lattice in each case can be the free distributive lattice with n generators. □

4. Conclusion

To sum up, we have introduced a lattice induced threshold function, as a generalization of the class of classical threshold functions. Our motivation was to capture isotonicity of Boolean functions.

The obtained results could be further generalized in several directions. First of all, one could consider lattice functions instead of Boolean ones. Also, one might look for a characterization of the classical treshold functions within the class of lattice induced threshold functions. Furthermore, with appropriately chosen lattices as co-domain, interesting classes of Boolean functions could be obtained.

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