

# Szigetek és függetlenségi fogalmak

Eszter K. Horváth, Szeged

Co-authors: Stephan Foldes, Sándor Radeleczki, Tamás Waldhauser

Szeged, 2012, Október 10.

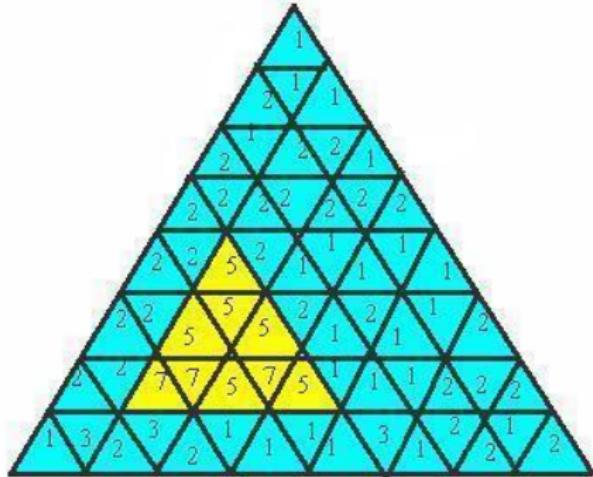
# Islands?



# Rectangular and triangular islands

We call a rectangle/triangle a *rectangular/triangular island*, if for the cell  $t$ , if we denote its height by  $a_t$ , then for each cell  $\hat{t}$  neighbouring with a cell of the rectangle/triangle  $T$ , the inequality  $a_{\hat{t}} < \min\{a_t : t \in T\}$  holds.

1	2	1	2	1
1	5	7	2	2
1	7	5	1	1
2	5	7	2	2
1	2	1	1	2
1	1	1	1	1



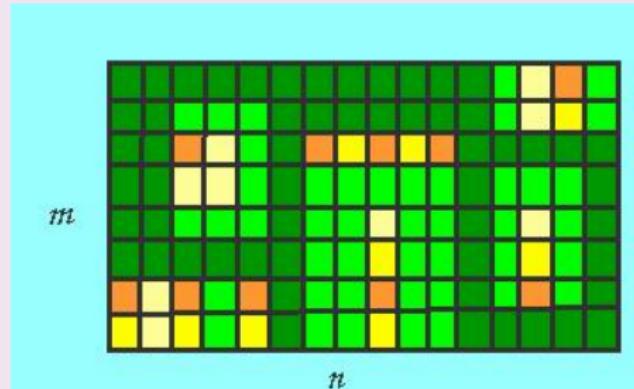
# Rectangular islands

## Rectangular islands

G. Czédli: The number of rectangular islands by means of distributive lattices, European Journal of Combinatorics 30 (2009), 208-215.

The maximum number of rectangular islands in a  $m \times n$  rectangular board on square grid:

$$f(m, n) = \left\lceil \frac{mn + m + n - 1}{2} \right\rceil.$$



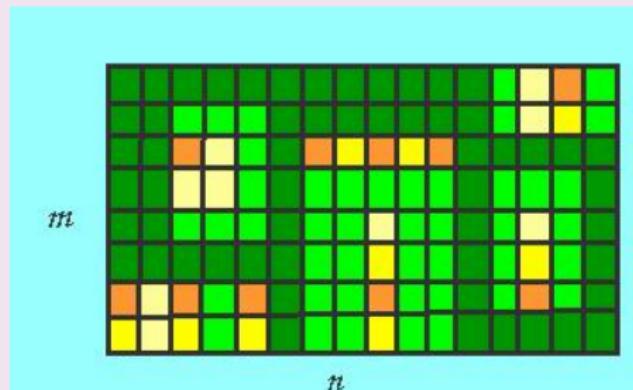
# Rectangular islands

## Rectangular islands

G. Czédli: The number of rectangular islands by means of distributive lattices, European Journal of Combinatorics 30 (2009), 208-215.

The maximum number of rectangular islands in a  $m \times n$  rectangular board on square grid:

$$f(m, n) = \left\lceil \frac{mn + m + n - 1}{2} \right\rceil.$$



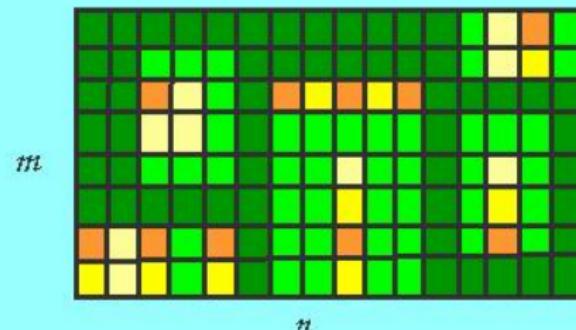
# Rectangular islands

## Rectangular islands

G. Czédli: The number of rectangular islands by means of distributive lattices, European Journal of Combinatorics 30 (2009), 208-215.

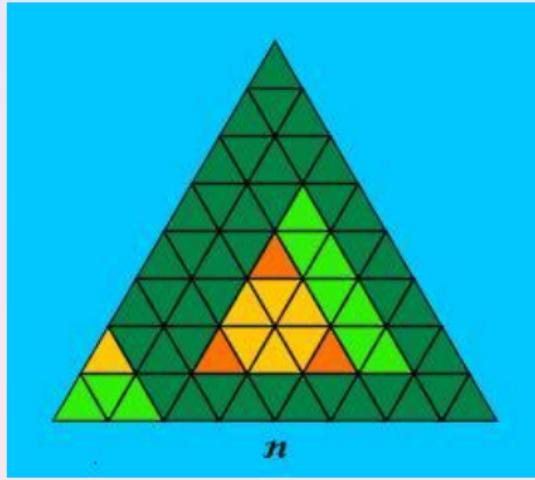
The maximum number of rectangular islands in a  $m \times n$  rectangular board on square grid:

$$f(m, n) = \left\lceil \frac{mn + m + n - 1}{2} \right\rceil.$$



# Triangular islands

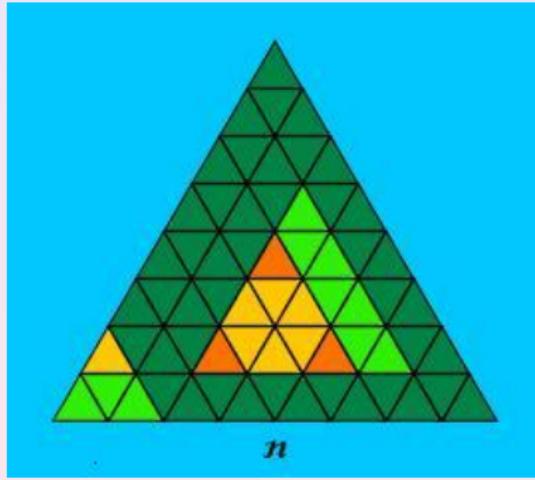
For the maximum number of triangular islands in an equilateral triangle of side length  $n$ ,  $\frac{n^2+3n}{5} \leq f(n) \leq \frac{3n^2+9n+2}{14}$  holds.



E. K. Horváth, Z. Németh and G. Pálházi: The number of triangular islands in an equilateral triangle. Periodica Mathematica Hungarica, 63 (2011), 229–238.

# Triangular islands

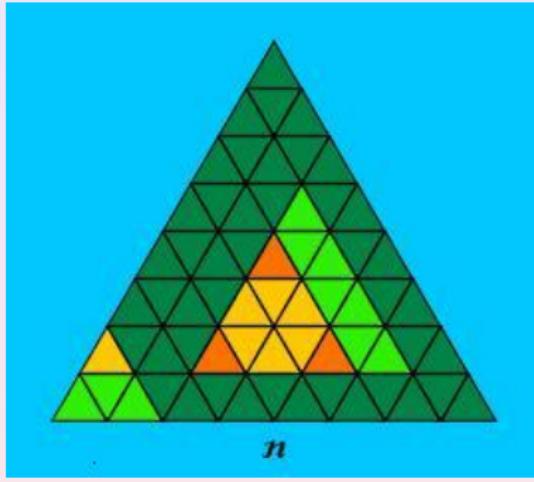
For the maximum number of triangular islands in an equilateral triangle of side length  $n$ ,  $\frac{n^2+3n}{5} \leq f(n) \leq \frac{3n^2+9n+2}{14}$  holds.



E. K. Horváth, Z. Németh and G. Pluhár: The number of triangular islands on a triangular grid, Periodica Mathematica Hungarica, 58 (2009), 25–34.

# Triangular islands

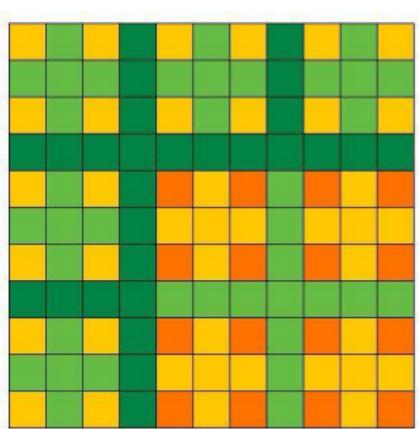
For the maximum number of triangular islands in an equilateral triangle of side length  $n$ ,  $\frac{n^2+3n}{5} \leq f(n) \leq \frac{3n^2+9n+2}{14}$  holds.



E. K. Horváth, Z. Németh and G. Pluhár: The number of triangular islands on a triangular grid, Periodica Mathematica Hungarica, 58 (2009), 25–34.

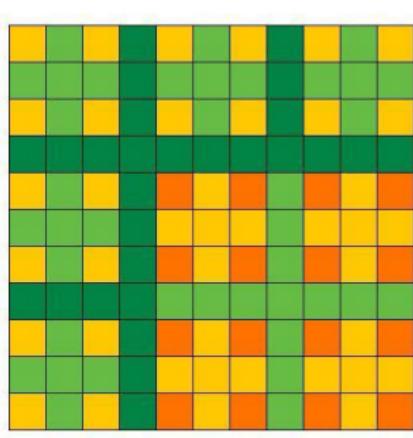
# Square islands (also in higher dimensions)

$$\frac{1}{3}(rs - 2r - 2s) \leq f(r, s) \leq \frac{1}{3}(rs - 1)$$



# Square islands (also in higher dimensions)

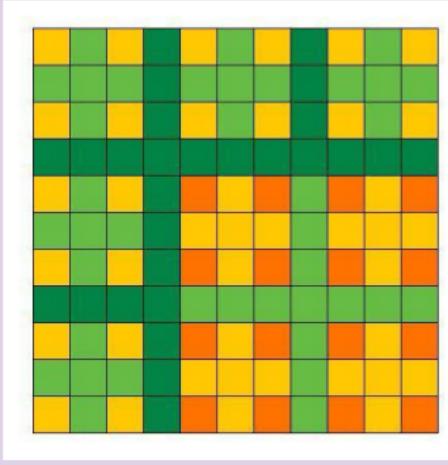
$$\frac{1}{3}(rs - 2r - 2s) \leq f(r, s) \leq \frac{1}{3}(rs - 1)$$



E. R. Berlekamp, G. Guy, and J. Selfridge, On Sets: The number of square islands in a rectangular sea, *American Mathematical Monthly* 70 (1963) 313–320.

# Square islands (also in higher dimensions)

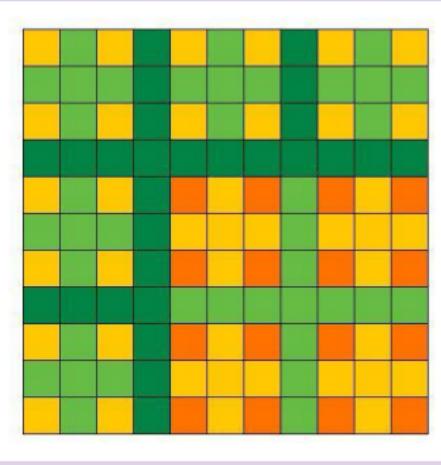
$$\frac{1}{3}(rs - 2r - 2s) \leq f(r, s) \leq \frac{1}{3}(rs - 1)$$



E. K. Horváth, G. Horváth, Z. Németh, Cs. Szabó: The number of square islands on a rectangular sea, Acta Sci. Math.(Szeged) **76** (2010) 35-48.

# Square islands (also in higher dimensions)

$$\frac{1}{3}(rs - 2r - 2s) \leq f(r, s) \leq \frac{1}{3}(rs - 1)$$



E. K. Horváth, G. Horváth, Z. Németh, Cs. Szabó: The number of square islands on a rectangular sea, Acta Sci. Math.(Szeged) **76** (2010) 35-48.

# CD-independent subsets in posets

## Definitions

Let  $\mathbb{P} = (P, \leq)$  be a partially ordered set and  $a, b \in P$ . The elements  $a$  and  $b$  are called *disjoint* and we write  $a \perp b$  if

either  $\mathbb{P}$  has least element  $0 \in P$  and  $\inf\{a, b\} = 0$ ,

A nonempty set  $X \subseteq P$  is called *CD-independent* if for any  $x, y \in X$ ,  $x \leq y$  or  $y \leq x$  or  $x \perp y$  holds.

Maximal CD-independent sets (with respect to  $\subseteq$ ) are called *CD-bases* in  $\mathbb{P}$ .

# CD-independent subsets in posets

## Definitions

Let  $\mathbb{P} = (P, \leq)$  be a partially ordered set and  $a, b \in P$ . The elements  $a$  and  $b$  are called *disjoint* and we write  $a \perp b$  if

either  $\mathbb{P}$  has least element  $0 \in P$  and  $\inf\{a, b\} = 0$ ,

or if  $\mathbb{P}$  is without 0, then  $a$  and  $b$  have no common lowerbound.

A nonempty set  $X \subseteq P$  is called *CD-independent* if for any  $x, y \in X$ ,  $x \leq y$  or  $y \leq x$  or  $x \perp y$  holds.

Maximal CD-independent sets (with respect to  $\subseteq$ ) are called *CD-bases* in  $\mathbb{P}$ .

# CD-independent subsets in posets

## Definitions

Let  $\mathbb{P} = (P, \leq)$  be a partially ordered set and  $a, b \in P$ . The elements  $a$  and  $b$  are called *disjoint* and we write  $a \perp b$  if

either  $\mathbb{P}$  has least element  $0 \in P$  and  $\inf\{a, b\} = 0$ ,

or if  $\mathbb{P}$  is without 0, then  $a$  and  $b$  have no common lowerbound.

A nonempty set  $X \subseteq P$  is called *CD-independent* if for any  $x, y \in X$ ,  $x \leq y$  or  $y \leq x$  or  $x \perp y$  holds.

Maximal CD-independent sets (with respect to  $\subseteq$ ) are called *CD-bases* in  $\mathbb{P}$ .

# CD-independent subsets in posets

## Definitions

Let  $\mathbb{P} = (P, \leq)$  be a partially ordered set and  $a, b \in P$ . The elements  $a$  and  $b$  are called *disjoint* and we write  $a \perp b$  if

either  $\mathbb{P}$  has least element  $0 \in P$  and  $\inf\{a, b\} = 0$ ,

or if  $\mathbb{P}$  is without 0, then  $a$  and  $b$  have no common lowerbound.

A nonempty set  $X \subseteq P$  is called *CD-independent* if for any  $x, y \in X$ ,  $x \leq y$  or  $y \leq x$  or  $x \perp y$  holds.

Maximal CD-independent sets (with respect to  $\subseteq$ ) are called *CD-bases* in  $\mathbb{P}$ .

# CD-independent subsets in posets

## Definitions

Let  $\mathbb{P} = (P, \leq)$  be a partially ordered set and  $a, b \in P$ . The elements  $a$  and  $b$  are called *disjoint* and we write  $a \perp b$  if

either  $\mathbb{P}$  has least element  $0 \in P$  and  $\inf\{a, b\} = 0$ ,

or if  $\mathbb{P}$  is without 0, then  $a$  and  $b$  have no common lowerbound.

A nonempty set  $X \subseteq P$  is called *CD-independent* if for any  $x, y \in X$ ,  $x \leq y$  or  $y \leq x$  or  $x \perp y$  holds.

Maximal CD-independent sets (with respect to  $\subseteq$ ) are called *CD-bases* in  $\mathbb{P}$ .

# CD-independent subsets in posets

## Definitions

Let  $\mathbb{P} = (P, \leq)$  be a partially ordered set and  $a, b \in P$ . The elements  $a$  and  $b$  are called *disjoint* and we write  $a \perp b$  if

either  $\mathbb{P}$  has least element  $0 \in P$  and  $\inf\{a, b\} = 0$ ,

or if  $\mathbb{P}$  is without 0, then  $a$  and  $b$  have no common lowerbound.

A nonempty set  $X \subseteq P$  is called *CD-independent* if for any  $x, y \in X$ ,  $x \leq y$  or  $y \leq x$  or  $x \perp y$  holds.

Maximal CD-independent sets (with respect to  $\subseteq$ ) are called *CD-bases* in  $\mathbb{P}$ .

# CD-independent subsets in distributive lattices

Any two CD-bases of a finite distributive lattice have the same number of elements.

If all finite lattices in a lattice variety have this property, then the variety must coincide with the variety of distributive lattices.

# CD-independent subsets in distributive lattices

Any two CD-bases of a finite distributive lattice have the same number of elements.

If all finite lattices in a lattice variety have this property, then the variety must coincide with the variety of distributive lattices.

G. Czédli, M. Hartmann and E. T. Schmidt: CD-independent subsets in distributive lattices, *Publicationes Mathematicae Debrecen*, 74/1-2 (2009)

# CD-independent subsets in distributive lattices

Any two CD-bases of a finite distributive lattice have the same number of elements.

If all finite lattices in a lattice variety have this property, then the variety must coincide with the variety of distributive lattices.

G. Czédli, M. Hartmann and E. T. Schmidt: CD-independent subsets in distributive lattices, *Publicationes Mathematicae Debrecen*, 74/1-2 (2009).

# CD-independent subsets in distributive lattices

Any two CD-bases of a finite distributive lattice have the same number of elements.

If all finite lattices in a lattice variety have this property, then the variety must coincide with the variety of distributive lattices.

G. Czédli, M. Hartmann and E. T. Schmidt: CD-independent subsets in distributive lattices, *Publicationes Mathematicae Debrecen*, 74/1-2 (2009).

# Definitions

$$U \in \mathcal{C} \subseteq \mathcal{K} \subseteq \mathcal{P}(U)$$

Let  $h: U \rightarrow \mathbb{R}$  be a height function and let  $S \in \mathcal{C}$  be a nonempty set.

We say that  $S$  is an *island* with respect to the triple  $(\mathcal{C}, \mathcal{K}, h)$ , if every  $K \in \mathcal{K}$  with  $S \prec K$  satisfies

$$\min h(K) < \min h(S).$$

We say that  $S$  is a *strict island* with respect to the triple  $(\mathcal{C}, \mathcal{K}, h)$ , if every  $K \in \mathcal{K}$  with  $S \prec K$  satisfies

$$h(u) < \min h(S) \text{ for all } u \in K \setminus S.$$

# Definitions

$$U \in \mathcal{C} \subseteq \mathcal{K} \subseteq \mathcal{P}(U)$$

Let  $h: U \rightarrow \mathbb{R}$  be a height function and let  $S \in \mathcal{C}$  be a nonempty set.

We say that  $S$  is an *island* with respect to the triple  $(\mathcal{C}, \mathcal{K}, h)$ , if every  $K \in \mathcal{K}$  with  $S \prec K$  satisfies

$$\min h(K) < \min h(S).$$

We say that  $S$  is a *strict island* with respect to the triple  $(\mathcal{C}, \mathcal{K}, h)$ , if every  $K \in \mathcal{K}$  with  $S \prec K$  satisfies

$$h(u) < \min h(S) \text{ for all } u \in K \setminus S.$$

## Example

Let  $G = (U, E)$  be a connected simple graph with vertex set  $U$  and edge set  $E$ ; let  $\mathcal{K}$  consist of the connected subsets of  $U$ , and let  $\mathcal{C} \subseteq \mathcal{K}$  such that  $U \in \mathcal{C}$ .

## Example

Let  $A_1, \dots, A_n$  be nonempty sets, and let  $\mathcal{I} \subseteq A_1 \times \dots \times A_n$ . Let us define

$$U = A_1 \times \dots \times A_n,$$

$$\mathcal{K} = \{B_1 \times \dots \times B_n : \emptyset \neq B_i \subseteq A_i, 1 \leq i \leq n\}$$

$$\mathcal{C} = \{C \in \mathcal{K} : C \subseteq \mathcal{I}\} \cup \{U\},$$

and let  $h: U \longrightarrow \{0, 1\}$  be the height function given by

$$h(a_1, \dots, a_n) := \begin{cases} 1, & \text{if } (a_1, \dots, a_n) \in \mathcal{I}; \\ 0, & \text{if } (a_1, \dots, a_n) \in U \setminus \mathcal{I}; \end{cases} \quad \text{for all } (a_1, \dots, a_n) \in U.$$

It is easy to see that the islands corresponding to the triple  $(\mathcal{C}, \mathcal{K}, h)$  are exactly  $U$  and the maximal elements of the poset  $(\mathcal{C} \setminus \{U\}, \subseteq)$ .

formal concepts

prime implicants of a Boolean function

## Example

Let  $A_1, \dots, A_n$  be nonempty sets, and let  $\mathcal{I} \subseteq A_1 \times \dots \times A_n$ . Let us define

$$U = A_1 \times \dots \times A_n,$$

$$\mathcal{K} = \{B_1 \times \dots \times B_n : \emptyset \neq B_i \subseteq A_i, 1 \leq i \leq n\}$$

$$\mathcal{C} = \{C \in \mathcal{K} : C \subseteq \mathcal{I}\} \cup \{U\},$$

and let  $h: U \longrightarrow \{0, 1\}$  be the height function given by

$$h(a_1, \dots, a_n) := \begin{cases} 1, & \text{if } (a_1, \dots, a_n) \in \mathcal{I}; \\ 0, & \text{if } (a_1, \dots, a_n) \in U \setminus \mathcal{I}; \end{cases} \quad \text{for all } (a_1, \dots, a_n) \in U.$$

It is easy to see that the islands corresponding to the triple  $(\mathcal{C}, \mathcal{K}, h)$  are exactly  $U$  and the maximal elements of the poset  $(\mathcal{C} \setminus \{U\}, \subseteq)$ .

formal concepts

prime implicants of a Boolean function

# Islands and admissible systems

## Definition

Let  $\mathcal{H} \subseteq \mathcal{C} \setminus \{\emptyset\}$  be a family of sets such that  $U \in \mathcal{H}$ . We say that  $\mathcal{H}$  is *admissible*, if for every nonempty antichain  $\mathcal{A} \subseteq \mathcal{H}$

$$\exists H \in \mathcal{A} \forall K \in \mathcal{K} : H \subset K \implies K \not\subseteq \bigcup \mathcal{A}. \quad (1)$$

## Proposition

Every system of islands is admissible.

# Islands and admissible systems

## Definition

Let  $\mathcal{H} \subseteq \mathcal{C} \setminus \{\emptyset\}$  be a family of sets such that  $U \in \mathcal{H}$ . We say that  $\mathcal{H}$  is *admissible*, if for every nonempty antichain  $\mathcal{A} \subseteq \mathcal{H}$

$$\exists H \in \mathcal{A} \forall K \in \mathcal{K} : H \subset K \implies K \not\subseteq \bigcup \mathcal{A}. \quad (1)$$

## Proposition

Every system of islands is admissible.

# Every system of islands is admissible.

## Proof

Let  $h: U \rightarrow \mathbb{R}$  be a height function and let  $\mathcal{S}$  be the system of islands corresponding to  $(\mathcal{C}, \mathcal{K}, h)$ . Clearly, we have  $\emptyset \notin \mathcal{S}$  and  $U \in \mathcal{S}$ . Let us assume for contradiction that there exists an antichain

$\mathcal{A} = \{S_i : i \in I\} \subseteq \mathcal{S}$  such that (1) does not hold. Then for every  $i \in I$  there exists  $K_i \in \mathcal{K}$  such that  $S_i \subset K_i$  and  $K_i \subseteq \bigcup_{i \in I} S_i$ . Since  $S_i$  is an island, we have

$$\min h(S_i) > \min h(K_i) \geq \min h\left(\bigcup_{i \in I} S_i\right)$$

for all  $i \in I$ . Taking the minimum of these inequalities we arrive at the contradiction

$$\min \{\min h(S_i) \mid i \in I\} > \min h\left(\bigcup_{i \in I} S_i\right).$$

# Canonical height functions

Let  $\mathcal{H} \subseteq \mathcal{C}$  be an admissible family of sets.

We define subfamilies  $\mathcal{H}^{(i)} \subseteq \mathcal{H}$  ( $i = 0, 1, 2, \dots$ ) recursively as follows.

Let  $\mathcal{H}^{(0)} = \{U\}$ .

For  $i > 0$ , if  $\mathcal{H} \neq \mathcal{H}^{(0)} \cup \dots \cup \mathcal{H}^{(i-1)}$ , then let  $\mathcal{H}^{(i)}$  consist of all those sets  $H \in \mathcal{H} \setminus (\mathcal{H}^{(0)} \cup \dots \cup \mathcal{H}^{(i-1)})$  that have the following property:

$$\forall K \in \mathcal{K} : H \subset K \implies K \not\subseteq (\mathcal{H} \setminus (\mathcal{H}^{(0)} \cup \dots \cup \mathcal{H}^{(i-1)})). \quad (2)$$

Since  $\mathcal{H}$  is finite and admissible, after finitely many steps we obtain a partition  $\mathcal{H} = \mathcal{H}^{(0)} \cup \dots \cup \mathcal{H}^{(r)}$ .

The *canonical height function corresponding to  $\mathcal{H}$*  is the function  $h_{\mathcal{H}}: U \rightarrow \mathbb{N}$  defined by

$$h_{\mathcal{H}}(x) := \max \left\{ i \in \{1, \dots, r\} : x \in \bigcup \mathcal{H}^{(i)} \right\} \text{ for all } x \in U. \quad (3)$$

# Canonical height functions

Let  $\mathcal{H} \subseteq \mathcal{C}$  be an admissible family of sets.

We define subfamilies  $\mathcal{H}^{(i)} \subseteq \mathcal{H}$  ( $i = 0, 1, 2, \dots$ ) recursively as follows.

Let  $\mathcal{H}^{(0)} = \{U\}$ .

For  $i > 0$ , if  $\mathcal{H} \neq \mathcal{H}^{(0)} \cup \dots \cup \mathcal{H}^{(i-1)}$ , then let  $\mathcal{H}^{(i)}$  consist of all those sets  $H \in \mathcal{H} \setminus (\mathcal{H}^{(0)} \cup \dots \cup \mathcal{H}^{(i-1)})$  that have the following property:

$$\forall K \in \mathcal{K} : H \subset K \implies K \not\subseteq (\mathcal{H} \setminus (\mathcal{H}^{(0)} \cup \dots \cup \mathcal{H}^{(i-1)})). \quad (2)$$

Since  $\mathcal{H}$  is finite and admissible, after finitely many steps we obtain a partition  $\mathcal{H} = \mathcal{H}^{(0)} \cup \dots \cup \mathcal{H}^{(r)}$ .

The *canonical height function corresponding to  $\mathcal{H}$*  is the function  $h_{\mathcal{H}}: U \rightarrow \mathbb{N}$  defined by

$$h_{\mathcal{H}}(x) := \max \left\{ i \in \{1, \dots, r\} : x \in \bigcup \mathcal{H}^{(i)} \right\} \text{ for all } x \in U. \quad (3)$$

# Canonical height functions

Let  $\mathcal{H} \subseteq \mathcal{C}$  be an admissible family of sets.

We define subfamilies  $\mathcal{H}^{(i)} \subseteq \mathcal{H}$  ( $i = 0, 1, 2, \dots$ ) recursively as follows.

Let  $\mathcal{H}^{(0)} = \{U\}$ .

For  $i > 0$ , if  $\mathcal{H} \neq \mathcal{H}^{(0)} \cup \dots \cup \mathcal{H}^{(i-1)}$ , then let  $\mathcal{H}^{(i)}$  consist of all those sets  $H \in \mathcal{H} \setminus (\mathcal{H}^{(0)} \cup \dots \cup \mathcal{H}^{(i-1)})$  that have the following property:

$$\forall K \in \mathcal{K} : H \subset K \implies K \not\subseteq (\mathcal{H} \setminus (\mathcal{H}^{(0)} \cup \dots \cup \mathcal{H}^{(i-1)})). \quad (2)$$

Since  $\mathcal{H}$  is finite and admissible, after finitely many steps we obtain a partition  $\mathcal{H} = \mathcal{H}^{(0)} \cup \dots \cup \mathcal{H}^{(r)}$ .

The *canonical height function corresponding to  $\mathcal{H}$*  is the function  $h_{\mathcal{H}}: U \rightarrow \mathbb{N}$  defined by

$$h_{\mathcal{H}}(x) := \max \left\{ i \in \{1, \dots, r\} : x \in \bigcup \mathcal{H}^{(i)} \right\} \text{ for all } x \in U. \quad (3)$$

# Canonical height functions

Let  $\mathcal{H} \subseteq \mathcal{C}$  be an admissible family of sets.

We define subfamilies  $\mathcal{H}^{(i)} \subseteq \mathcal{H}$  ( $i = 0, 1, 2, \dots$ ) recursively as follows.

Let  $\mathcal{H}^{(0)} = \{U\}$ .

For  $i > 0$ , if  $\mathcal{H} \neq \mathcal{H}^{(0)} \cup \dots \cup \mathcal{H}^{(i-1)}$ , then let  $\mathcal{H}^{(i)}$  consist of all those sets  $H \in \mathcal{H} \setminus (\mathcal{H}^{(0)} \cup \dots \cup \mathcal{H}^{(i-1)})$  that have the following property:

$$\forall K \in \mathcal{K} : H \subset K \implies K \not\subseteq (\mathcal{H} \setminus (\mathcal{H}^{(0)} \cup \dots \cup \mathcal{H}^{(i-1)})). \quad (2)$$

Since  $\mathcal{H}$  is finite and admissible, after finitely many steps we obtain a partition  $\mathcal{H} = \mathcal{H}^{(0)} \cup \dots \cup \mathcal{H}^{(r)}$ .

The *canonical height function corresponding to  $\mathcal{H}$*  is the function  $h_{\mathcal{H}}: U \rightarrow \mathbb{N}$  defined by

$$h_{\mathcal{H}}(x) := \max \left\{ i \in \{1, \dots, r\} : x \in \bigcup \mathcal{H}^{(i)} \right\} \text{ for all } x \in U. \quad (3)$$

# Canonical height functions

Let  $\mathcal{H} \subseteq \mathcal{C}$  be an admissible family of sets.

We define subfamilies  $\mathcal{H}^{(i)} \subseteq \mathcal{H}$  ( $i = 0, 1, 2, \dots$ ) recursively as follows.

Let  $\mathcal{H}^{(0)} = \{U\}$ .

For  $i > 0$ , if  $\mathcal{H} \neq \mathcal{H}^{(0)} \cup \dots \cup \mathcal{H}^{(i-1)}$ , then let  $\mathcal{H}^{(i)}$  consist of all those sets  $H \in \mathcal{H} \setminus (\mathcal{H}^{(0)} \cup \dots \cup \mathcal{H}^{(i-1)})$  that have the following property:

$$\forall K \in \mathcal{K} : H \subset K \implies K \not\subseteq (\mathcal{H} \setminus (\mathcal{H}^{(0)} \cup \dots \cup \mathcal{H}^{(i-1)})). \quad (2)$$

Since  $\mathcal{H}$  is finite and admissible, after finitely many steps we obtain a partition  $\mathcal{H} = \mathcal{H}^{(0)} \cup \dots \cup \mathcal{H}^{(r)}$ .

The *canonical height function corresponding to  $\mathcal{H}$*  is the function  $h_{\mathcal{H}} : U \rightarrow \mathbb{N}$  defined by

$$h_{\mathcal{H}}(x) := \max \left\{ i \in \{1, \dots, r\} : x \in \bigcup \mathcal{H}^{(i)} \right\} \text{ for all } x \in U. \quad (3)$$

# Canonical height functions

Let  $\mathcal{H} \subseteq \mathcal{C}$  be an admissible family of sets.

We define subfamilies  $\mathcal{H}^{(i)} \subseteq \mathcal{H}$  ( $i = 0, 1, 2, \dots$ ) recursively as follows.

Let  $\mathcal{H}^{(0)} = \{U\}$ .

For  $i > 0$ , if  $\mathcal{H} \neq \mathcal{H}^{(0)} \cup \dots \cup \mathcal{H}^{(i-1)}$ , then let  $\mathcal{H}^{(i)}$  consist of all those sets  $H \in \mathcal{H} \setminus (\mathcal{H}^{(0)} \cup \dots \cup \mathcal{H}^{(i-1)})$  that have the following property:

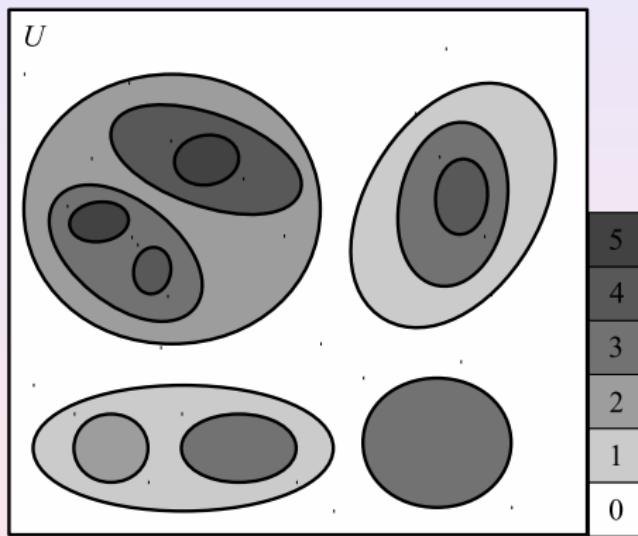
$$\forall K \in \mathcal{K} : H \subset K \implies K \not\subseteq (\mathcal{H} \setminus (\mathcal{H}^{(0)} \cup \dots \cup \mathcal{H}^{(i-1)})). \quad (2)$$

Since  $\mathcal{H}$  is finite and admissible, after finitely many steps we obtain a partition  $\mathcal{H} = \mathcal{H}^{(0)} \cup \dots \cup \mathcal{H}^{(r)}$ .

The *canonical height function corresponding to  $\mathcal{H}$*  is the function  $h_{\mathcal{H}}: U \rightarrow \mathbb{N}$  defined by

$$h_{\mathcal{H}}(x) := \max \left\{ i \in \{1, \dots, r\} : x \in \bigcup \mathcal{H}^{(i)} \right\} \text{ for all } x \in U. \quad (3)$$

# Canonical height functions



# Islands and admissible systems

## Proposition

If  $\mathcal{H} \subseteq \mathcal{C}$  is an admissible family of sets and  $h_{\mathcal{H}}$  is the corresponding canonical height function, then every member of  $\mathcal{H}$  is an island with respect to  $(\mathcal{C}, \mathcal{K}, h_{\mathcal{H}})$ .

## Theorem

A subfamily of  $\mathcal{C}$  is a maximal system of islands if and only if it is a maximal admissible family.

# Islands and admissible systems

## Proposition

If  $\mathcal{H} \subseteq \mathcal{C}$  is an admissible family of sets and  $h_{\mathcal{H}}$  is the corresponding canonical height function, then every member of  $\mathcal{H}$  is an island with respect to  $(\mathcal{C}, \mathcal{K}, h_{\mathcal{H}})$ .

## Theorem

A subfamily of  $\mathcal{C}$  is a maximal system of islands if and only if it is a maximal admissible family.

# Subsets of island systems

The following two conditions are equivalent for any pair  $(\mathcal{C}, \mathcal{K})$ :

Any subset of a system of islands corresponding to  $(\mathcal{C}, \mathcal{K})$  is also a system of islands.

The systems of islands corresponding to  $(\mathcal{C}, \mathcal{K})$  are exactly the admissible families.

# Subsets of island systems

The following two conditions are equivalent for any pair  $(\mathcal{C}, \mathcal{K})$ :

Any subset of a system of islands corresponding to  $(\mathcal{C}, \mathcal{K})$  is also a system of islands.

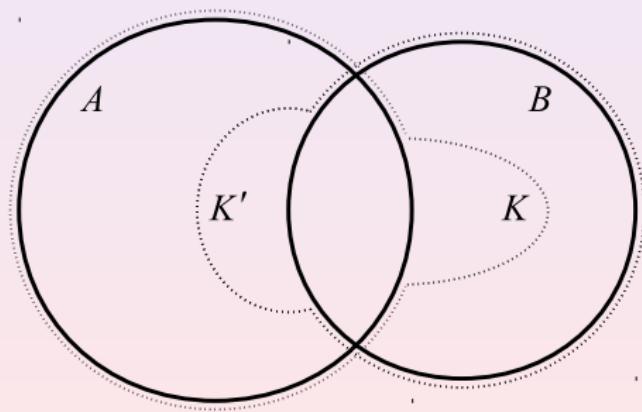
The systems of islands corresponding to  $(\mathcal{C}, \mathcal{K})$  are exactly the admissible families.

# Island domains

## Definition

A pair  $(\mathcal{C}, \mathcal{K})$  is an *island domain* if

$$\forall A, B \in \mathcal{C} : (A \cap B \neq \emptyset \text{ and } B \not\subseteq A) \implies \exists K \in \mathcal{K} : A \subset K \subseteq A \cup B.$$



**Definition** A family  $\mathcal{H} \subseteq \mathcal{P}(U)$  is *weakly independent* if

$$H \subseteq \bigcup_{i \in I} H_i \implies \exists i \in I : H \subseteq H_i \quad (4)$$

holds for all  $H \in \mathcal{H}, H_i \in \mathcal{H} (i \in I)$ . If  $\mathcal{H}$  is both CD-independent and weakly independent, then we say that  $\mathcal{H}$  is *CDW-independent*.

## Lemma

If  $(\mathcal{C}, \mathcal{K})$  is an island domain, then every admissible subfamily of  $\mathcal{C}$  is CDW-independent. [[But not conversely.]]

# Island domains

## Theorem

The following three conditions are equivalent for any pair  $(\mathcal{C}, \mathcal{K})$ :

(i)  $(\mathcal{C}, \mathcal{K})$  is an island domain.

(ii) Every system of islands corresponding to  $(\mathcal{C}, \mathcal{K})$  is CD-independent.

(iii) Every system of islands corresponding to  $(\mathcal{C}, \mathcal{K})$  is CDW-independent.

# Island domains

## Theorem

The following three conditions are equivalent for any pair  $(\mathcal{C}, \mathcal{K})$ :

- (i)  $(\mathcal{C}, \mathcal{K})$  is an island domain.
- (ii) Every system of islands corresponding to  $(\mathcal{C}, \mathcal{K})$  is CD-independent.
- (iii) Every system of islands corresponding to  $(\mathcal{C}, \mathcal{K})$  is CDW-independent.

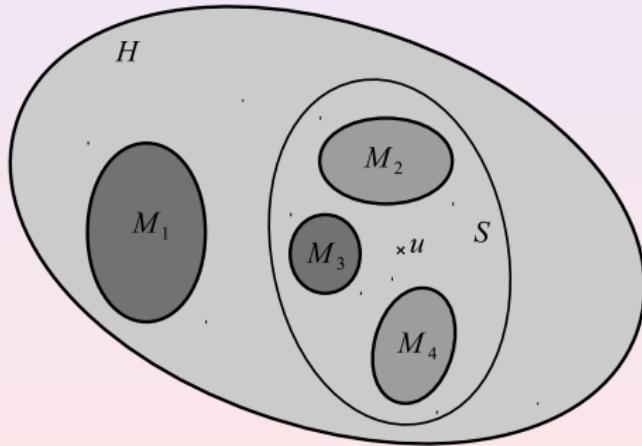
## Theorem

The following three conditions are equivalent for any pair  $(\mathcal{C}, \mathcal{K})$ :

- (i)  $(\mathcal{C}, \mathcal{K})$  is an island domain.
- (ii) Every system of islands corresponding to  $(\mathcal{C}, \mathcal{K})$  is CD-independent.
- (iii) Every system of islands corresponding to  $(\mathcal{C}, \mathcal{K})$  is CDW-independent.

## Theorem

If  $(\mathcal{C}, \mathcal{K})$  is an island domain, then a subfamily of  $\mathcal{C}$  is a system of islands if and only if it is admissible.



## Theorem

If  $(\mathcal{C}, \mathcal{K})$  is an island domain and  $\mathcal{S}$  is a system of islands corresponding to  $(\mathcal{C}, \mathcal{K})$ , then  $|\mathcal{S}| \leq |U|$ .

## Proof.

Let  $(\mathcal{C}, \mathcal{K})$  be an island domain and let  $\mathcal{S} \subseteq \mathcal{C} \setminus \{\emptyset\}$  be a system of islands corresponding to  $(\mathcal{C}, \mathcal{K})$ .  $\mathcal{S}$  is CDW-independent, and hence  $\mathcal{S} \cup \{\emptyset\}$  is also CDW-independent. From the results of G. Czédli and E. T. Schmidt it follows that every maximal CDW-independent subset of  $\mathcal{P}(U)$  has  $|U| + 1$  elements. Thus we have  $|\mathcal{S}| + 1 \leq |U| + 1$ .

## Theorem

If  $(\mathcal{C}, \mathcal{K})$  is an island domain and  $\mathcal{S}$  is a system of islands corresponding to  $(\mathcal{C}, \mathcal{K})$ , then  $|\mathcal{S}| \leq |U|$ .

## Proof.

Let  $(\mathcal{C}, \mathcal{K})$  be an island domain and let  $\mathcal{S} \subseteq \mathcal{C} \setminus \{\emptyset\}$  be a system of islands corresponding to  $(\mathcal{C}, \mathcal{K})$ .  $\mathcal{S}$  is CDW-independent, and hence  $\mathcal{S} \cup \{\emptyset\}$  is also CDW-independent. From the results of G. Czédli and E. T. Schmidt it follows that every maximal CDW-independent subset of  $\mathcal{P}(U)$  has  $|U| + 1$  elements. Thus we have  $|\mathcal{S}| + 1 \leq |U| + 1$ .

# Strict islands and proximity domains

$(\mathcal{C}, \mathcal{K})$

$\delta \subseteq \mathcal{C} \times \mathcal{C}$

$$A\delta B \Leftrightarrow \exists K \in \mathcal{K} : A \preceq K \text{ and } K \cap B \neq \emptyset. \quad (5)$$

It is easy to verify that relation  $\delta$  satisfies the following properties for all  $A, B, C \in \mathcal{C}$ :

$$A\delta B \Rightarrow B \neq \emptyset;$$

$$A \cap B \neq \emptyset \Rightarrow A\delta B;$$

$$A\delta(B \cup C) \Leftrightarrow (A\delta B \text{ or } A\delta C).$$

# Strict islands and proximity domains

$(\mathcal{C}, \mathcal{K})$

$\delta \subseteq \mathcal{C} \times \mathcal{C}$

$$A\delta B \Leftrightarrow \exists K \in \mathcal{K} : A \preceq K \text{ and } K \cap B \neq \emptyset. \quad (5)$$

It is easy to verify that relation  $\delta$  satisfies the following properties for all  $A, B, C \in \mathcal{C}$ :

$$A\delta B \Rightarrow B \neq \emptyset;$$

$$A \cap B \neq \emptyset \Rightarrow A\delta B;$$

$$A\delta(B \cup C) \Leftrightarrow (A\delta B \text{ or } A\delta C).$$

# Strict islands and proximity domains

$(\mathcal{C}, \mathcal{K})$

$\delta \subseteq \mathcal{C} \times \mathcal{C}$

$$A\delta B \Leftrightarrow \exists K \in \mathcal{K} : A \preceq K \text{ and } K \cap B \neq \emptyset. \quad (5)$$

It is easy to verify that relation  $\delta$  satisfies the following properties for all  $A, B, C \in \mathcal{C}$ :

$$A\delta B \Rightarrow B \neq \emptyset;$$

$$A \cap B \neq \emptyset \Rightarrow A\delta B;$$

$$A\delta(B \cup C) \Leftrightarrow (A\delta B \text{ or } A\delta C).$$

## Strict islands and proximity domains

$(\mathcal{C}, \mathcal{K})$

$\delta \subseteq \mathcal{C} \times \mathcal{C}$

$$A\delta B \Leftrightarrow \exists K \in \mathcal{K} : A \preceq K \text{ and } K \cap B \neq \emptyset. \quad (5)$$

It is easy to verify that relation  $\delta$  satisfies the following properties for all  $A, B, C \in \mathcal{C}$ :

$$A\delta B \Rightarrow B \neq \emptyset;$$

$$A \cap B \neq \emptyset \Rightarrow A\delta B;$$

$$A\delta(B \cup C) \Leftrightarrow (A\delta B \text{ or } A\delta C).$$

## Strict islands and proximity domains

We say that  $A, B \in \mathcal{C}$  are *distant* if neither  $A\delta B$  nor  $B\delta A$  holds.

It is easy to see that in this case  $A$  and  $B$  are also incomparable (in fact, disjoint), whenever  $A, B \neq \emptyset$ .

A nonempty family  $\mathcal{H} \subseteq \mathcal{C}$  will be called a *distant family*, if any two incomparable members of  $\mathcal{H}$  are distant.

**Lemma** If  $\mathcal{H} \subseteq \mathcal{C}$  is a distant family, then  $\mathcal{H}$  is CDW-independent. Moreover, if  $U \in \mathcal{H}$ , then  $U$  is admissible.

## Strict islands and proximity domains

We say that  $A, B \in \mathcal{C}$  are *distant* if neither  $A\delta B$  nor  $B\delta A$  holds.

It is easy to see that in this case  $A$  and  $B$  are also incomparable (in fact, disjoint), whenever  $A, B \neq \emptyset$ .

A nonempty family  $\mathcal{H} \subseteq \mathcal{C}$  will be called a *distant family*, if any two incomparable members of  $\mathcal{H}$  are distant.

**Lemma** If  $\mathcal{H} \subseteq \mathcal{C}$  is a distant family, then  $\mathcal{H}$  is CDW-independent. Moreover, if  $U \in \mathcal{H}$ , then  $U$  is admissible.

## Strict islands and proximity domains

We say that  $A, B \in \mathcal{C}$  are *distant* if neither  $A\delta B$  nor  $B\delta A$  holds.

It is easy to see that in this case  $A$  and  $B$  are also incomparable (in fact, disjoint), whenever  $A, B \neq \emptyset$ .

A nonempty family  $\mathcal{H} \subseteq \mathcal{C}$  will be called a *distant family*, if any two incomparable members of  $\mathcal{H}$  are distant.

**Lemma** If  $\mathcal{H} \subseteq \mathcal{C}$  is a distant family, then  $\mathcal{H}$  is CDW-independent. Moreover, if  $U \in \mathcal{H}$ , then  $U$  is admissible.

## Strict islands and proximity domains

We say that  $A, B \in \mathcal{C}$  are *distant* if neither  $A\delta B$  nor  $B\delta A$  holds.

It is easy to see that in this case  $A$  and  $B$  are also incomparable (in fact, disjoint), whenever  $A, B \neq \emptyset$ .

A nonempty family  $\mathcal{H} \subseteq \mathcal{C}$  will be called a *distant family*, if any two incomparable members of  $\mathcal{H}$  are distant.

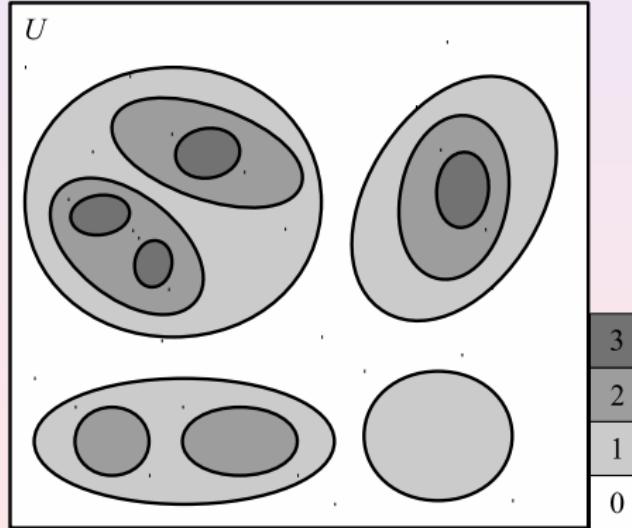
**Lemma** If  $\mathcal{H} \subseteq \mathcal{C}$  is a distant family, then  $\mathcal{H}$  is CDW-independent. Moreover, if  $U \in \mathcal{H}$ , then  $U$  is admissible.

# Standard height function

Let us consider a CD-independent family  $\mathcal{H}$ .

Clearly, for every  $u \in U$ , the set of members of  $\mathcal{H}$  containing  $u$  is a finite chain.

The *standard height function* of  $\mathcal{H}$  assigns to each element  $u$  the length of this chain, i.e., one less than the number of members of  $\mathcal{H}$  that contain  $u$ .

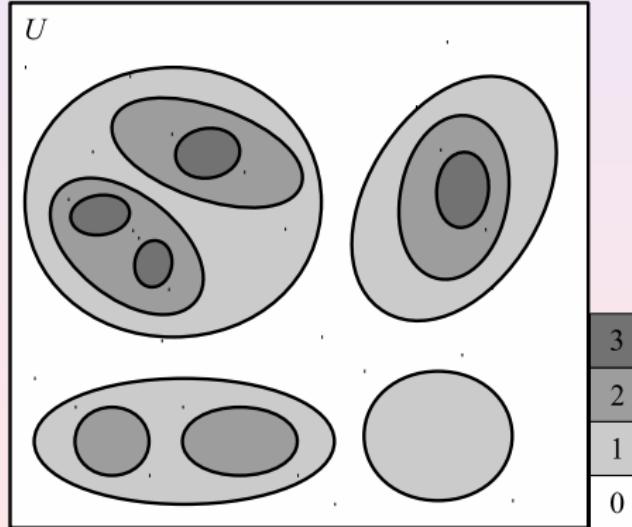


# Standard height function

Let us consider a CD-independent family  $\mathcal{H}$ .

Clearly, for every  $u \in U$ , the set of members of  $\mathcal{H}$  containing  $u$  is a finite chain.

The *standard height function* of  $\mathcal{H}$  assigns to each element  $u$  the length of this chain, i.e., one less than the number of members of  $\mathcal{H}$  that contain  $u$ .

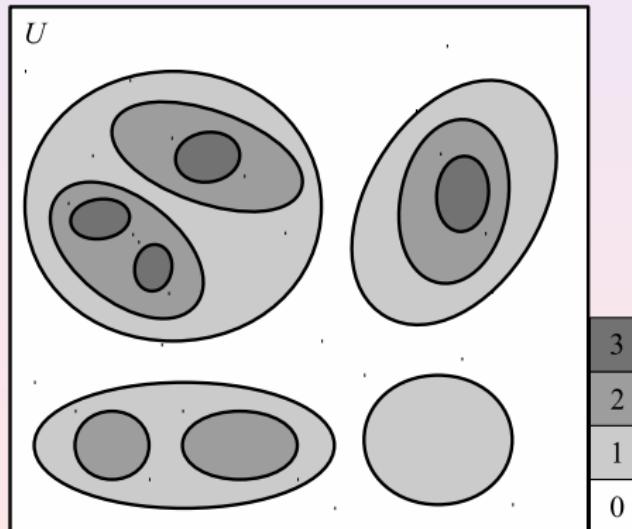


# Standard height function

Let us consider a CD-independent family  $\mathcal{H}$ .

Clearly, for every  $u \in U$ , the set of members of  $\mathcal{H}$  containing  $u$  is a finite chain.

The *standard height function* of  $\mathcal{H}$  assigns to each element  $u$  the length of this chain, i.e., one less than the number of members of  $\mathcal{H}$  that contain  $u$ .

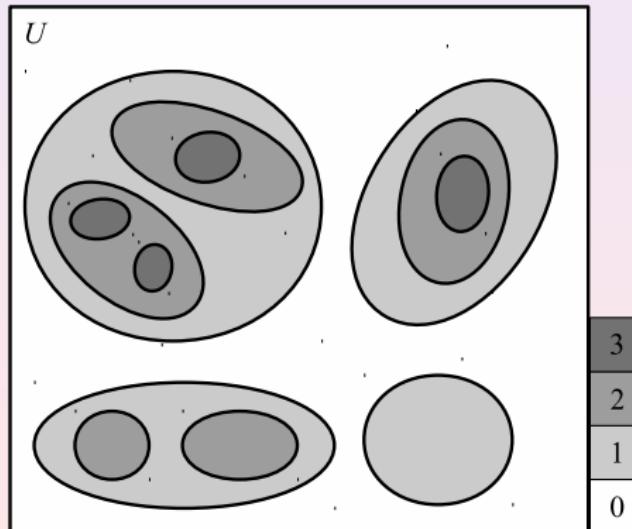


# Standard height function

Let us consider a CD-independent family  $\mathcal{H}$ .

Clearly, for every  $u \in U$ , the set of members of  $\mathcal{H}$  containing  $u$  is a finite chain.

The *standard height function* of  $\mathcal{H}$  assigns to each element  $u$  the length of this chain, i.e., one less than the number of members of  $\mathcal{H}$  that contain  $u$ .



# Strict islands and proximity domains

## Theorem

Let  $(\mathcal{C}, \mathcal{K})$  be an island domain and let  $\mathcal{H} \subseteq \mathcal{C} \setminus \{\emptyset\}$  with  $U \in \mathcal{H}$ . If  $\mathcal{H}$  is a distant family, then  $\mathcal{H}$  is a system of strict islands; moreover,  $\mathcal{H}$  is the system of strict islands corresponding to its standard height function.

## Strict islands and proximity domains

The pair  $(\mathcal{C}, \mathcal{K})$  is called a *proximity domain*, if it is an island domain and the relation  $\delta$  is symmetric for nonempty sets, that is

$$\forall A, B \in \mathcal{C} \setminus \{\emptyset\} : A\delta B \Leftrightarrow B\delta A. \quad (6)$$

If a relation  $\delta$  defined on  $\mathcal{P}(U)$  satisfies the mentioned three properties and  $\delta$  is symmetric for nonempty sets, then  $(U, \delta)$  is called a *proximity space*.

The notion goes back to Frigyes Riesz (1908), however this axiomatization is due to Vadim A. Efremovich.

## Strict islands and proximity domains

The pair  $(\mathcal{C}, \mathcal{K})$  is called a *proximity domain*, if it is an island domain and the relation  $\delta$  is symmetric for nonempty sets, that is

$$\forall A, B \in \mathcal{C} \setminus \{\emptyset\} : A\delta B \Leftrightarrow B\delta A. \quad (6)$$

If a relation  $\delta$  defined on  $\mathcal{P}(U)$  satisfies the mentioned three properties and  $\delta$  is symmetric for nonempty sets, then  $(U, \delta)$  is called a *proximity space*.

The notion goes back to Frigyes Riesz (1908), however this axiomatization is due to Vadim A. Efremovich.

## Strict islands and proximity domains

The pair  $(\mathcal{C}, \mathcal{K})$  is called a *proximity domain*, if it is an island domain and the relation  $\delta$  is symmetric for nonempty sets, that is

$$\forall A, B \in \mathcal{C} \setminus \{\emptyset\} : A\delta B \Leftrightarrow B\delta A. \quad (6)$$

If a relation  $\delta$  defined on  $\mathcal{P}(U)$  satisfies the mentioned three properties and  $\delta$  is symmetric for nonempty sets, then  $(U, \delta)$  is called a *proximity space*.

The notion goes back to Frigyes Riesz (1908), however this axiomatization is due to Vadim A. Efremovich.

# Strict islands and proximity domains

## Proposition

If  $(\mathcal{C}, \mathcal{K})$  is a proximity domain, then any system of strict islands corresponding to  $(\mathcal{C}, \mathcal{K})$  is a distant system.

## Corollary

If  $(\mathcal{C}, \mathcal{K})$  is a proximity domain, and  $\mathcal{H} \subseteq \mathcal{C} \setminus \{\emptyset\}$  with  $U \in \mathcal{H}$ , then  $\mathcal{H}$  is a system of strict islands if and only if  $\mathcal{H}$  is a distant family. Moreover, in this case  $\mathcal{H}$  is the system of strict islands corresponding to its standard height function.

# Strict islands and proximity domains

## Proposition

If  $(\mathcal{C}, \mathcal{K})$  is a proximity domain, then any system of strict islands corresponding to  $(\mathcal{C}, \mathcal{K})$  is a distant system.

## Corollary

If  $(\mathcal{C}, \mathcal{K})$  is a proximity domain, and  $\mathcal{H} \subseteq \mathcal{C} \setminus \{\emptyset\}$  with  $U \in \mathcal{H}$ , then  $\mathcal{H}$  is a system of strict islands if and only if  $\mathcal{H}$  is a distant family. Moreover, in this case  $\mathcal{H}$  is the system of strict islands corresponding to its standard height function.

# Strict islands and proximity domains

Finally, let us consider the following condition on  $(\mathcal{C}, \mathcal{K})$ , which is stronger than that of being an island domain:

$$\forall K_1, K_2 \in \mathcal{K} : K_1 \cap K_2 \neq \emptyset \implies K_1 \cup K_2 \in \mathcal{K}. \quad (7)$$

## Theorem

Suppose that  $(\mathcal{C}, \mathcal{K})$  satisfies condition (7), and assume that for all  $C \in \mathcal{C}$ ,  $K \in \mathcal{K}$  with  $C \prec K$  we have  $|K \setminus C| = 1$ . Then  $(\mathcal{C}, \mathcal{K})$  is a proximity domain, and islands and strict islands corresponding to  $(\mathcal{C}, \mathcal{K})$  coincide. Therefore, if  $\mathcal{H} \subseteq \mathcal{C} \setminus \{\emptyset\}$  and  $U \in \mathcal{H}$ , then  $\mathcal{H}$  is a system of (strict) islands if and only if  $\mathcal{H}$  is a distant family. Moreover, in this case  $\mathcal{H}$  is the system of (strict) islands corresponding to its standard height function.

# Strict islands and proximity domains

## Corollary

Let  $G$  be a graph with vertex set  $U$ ; let  $(\mathcal{C}, \mathcal{K})$  be an island domain corresponding to  $(\mathcal{C}, \mathcal{K})$ , and let  $\mathcal{H} \subseteq \mathcal{C} \setminus \{\emptyset\}$  with  $U \in \mathcal{H}$ . Then  $\mathcal{H}$  is a system of (strict) islands if and only if  $\mathcal{H}$  is distant; moreover, in this case  $\mathcal{H}$  is the system of (strict) islands corresponding to its standard height function.