Some enumerative and lattice theoretic aspects of islands (and lakes) and related investigations

Eszter K. Horváth, Szeged

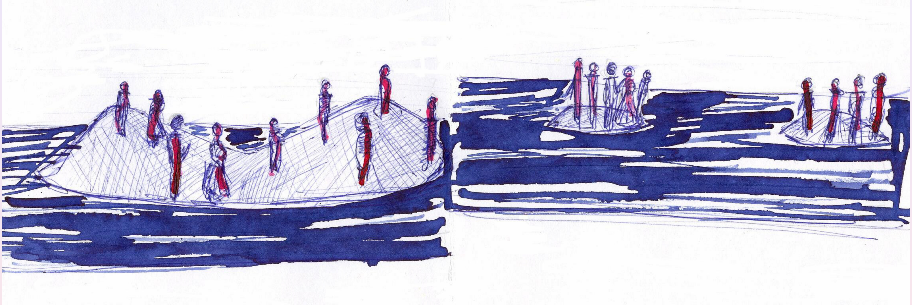
Co-authors: János Barát, Péter Hajnal, Branimir Šešelja, Andreja Tepavčević, Sándor Radeleczki

Novi Sad, 2015, June 5 .

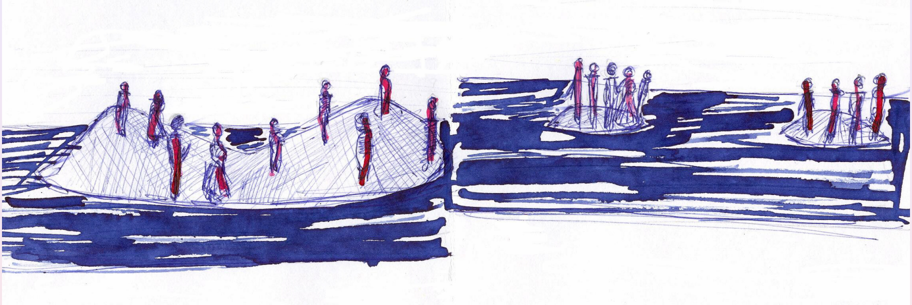
Islands? Alcatraz

Islands?

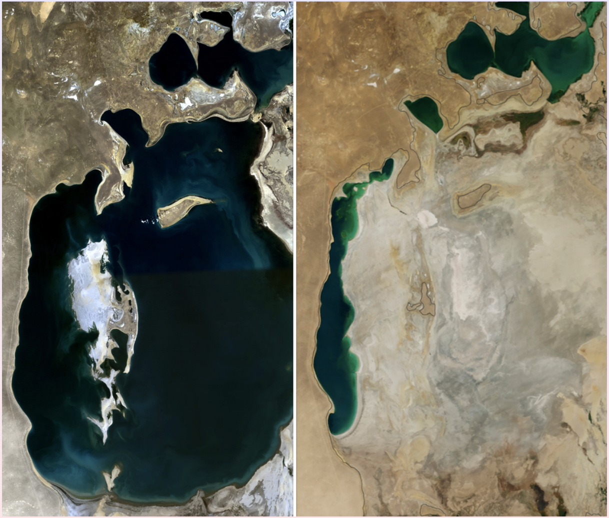
Islands?



Islands?



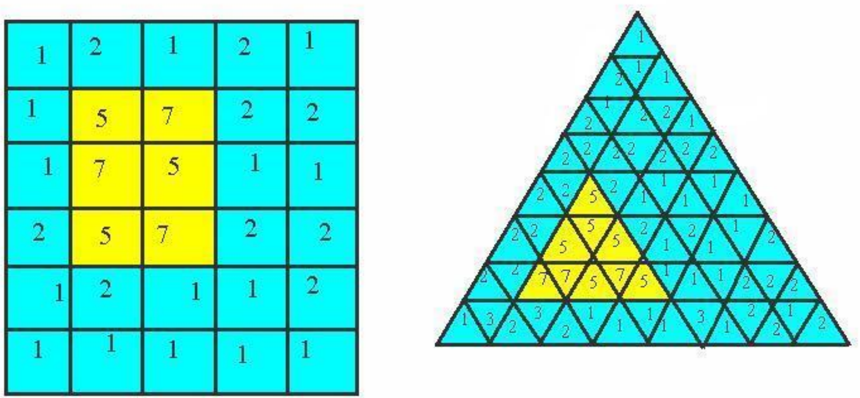
Lakes? (Aral sea, satellit photo)



Lakes?

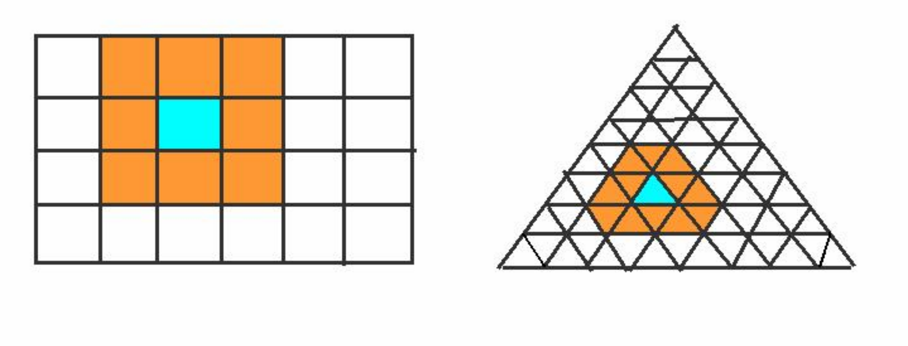
Definition

We call a rectangle/triangle a *rectangular/triangular island,* if for the cell t, if we denote its height by a_t, then for each cell $\hat{t}$ neighbouring with a cell of the rectange/triangle T, the inequality $a_{\hat{t}} < min\{a_t : t \in T\}$ holds.



Definition

Grid, neighbourhood



The number of rectangular islands

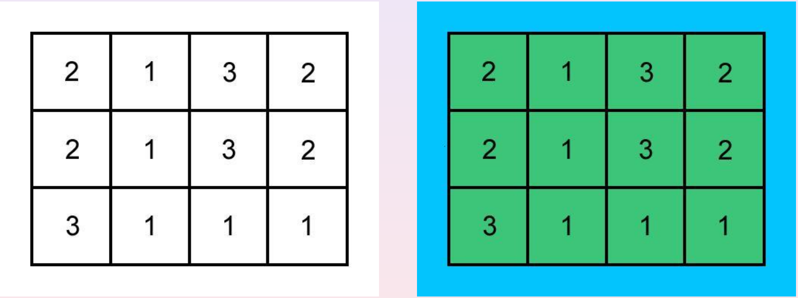
We put heights into the cells.
How many rectangular islands do we have?

2	1	3	2
2	1	3	2
3	1	1	1

The number of rectangular islands

Water level: 0,5

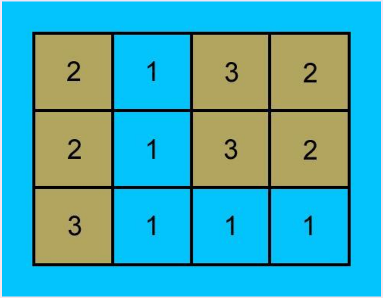
Number of rectangular islands: 1



The number of rectangular islands

Water level: 1,5
Number of rectangular islands: 2

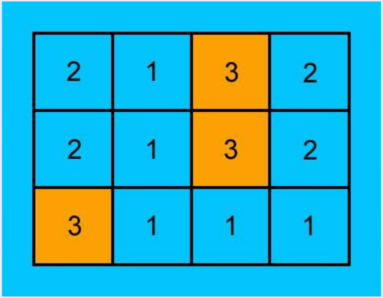
2	1	3	2
2	1	3	2
3	1	1	1



The number of rectangular islands

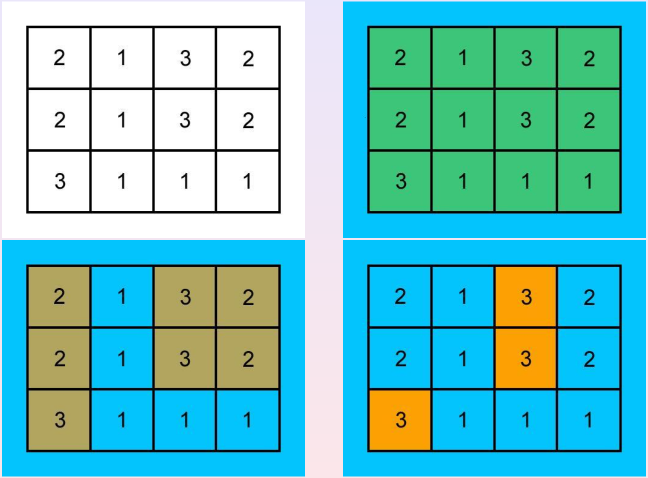
Water level: 2,5
Number of rectangular islands: 2

2	1	3	2
2	1	3	2
3	1	1	1



The number of rectangular islands

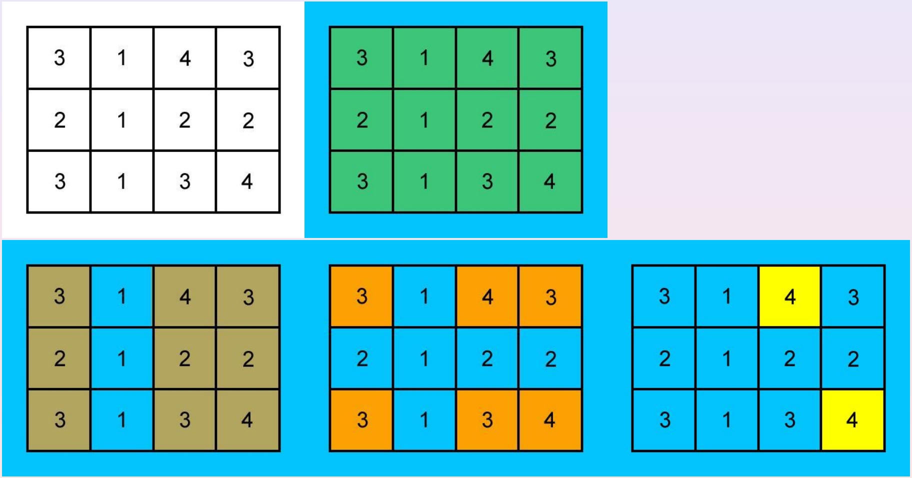
Altogether: $1 + 2 + 2 = 5$ rectangular islands.



Could we put more rectangular islands onto this grid? (With other heights?)

The number of rectangular islands

Yes, we could put more rectangular islands, here we have
$1 + 2 + 4 + 2 = 9$ rectangular islands.



HOWEWER, WE CANNOT PUT MORE RECTANGULAR ISLANDS!!!

The maximum number of rectangular islands on the $m \times n$ size grid (Gábor Czédli , Szeged, 2007. june 17.)

$$f(m,n) = \left[\frac{mn+m+n-1}{2}\right]$$

History/1

Coding theory

S. Földes and N. M. Singhi: On instantaneous codes, J. of Combinatorics, Information and System Sci., 31 (2006), 317-326.

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Rectangular islands

G. Czédli: The number of rectangular islands by means of distributive lattices, European Journal of Combinatorics 30 (2009), 208-215.

The maximum number of rectangular islands in a $m \times n$ rectangular board on square grid:

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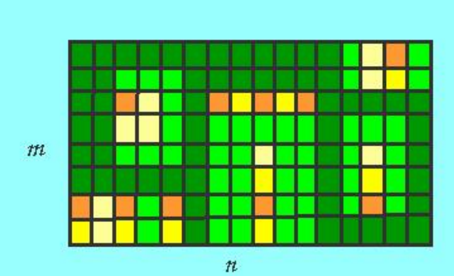

History/2

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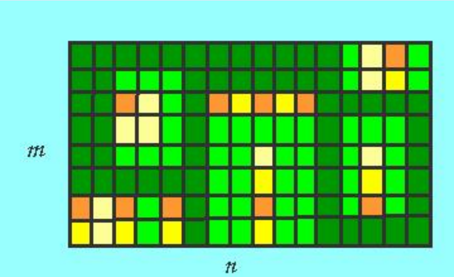

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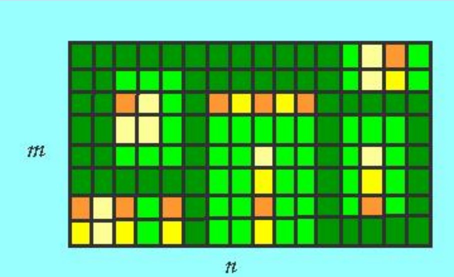


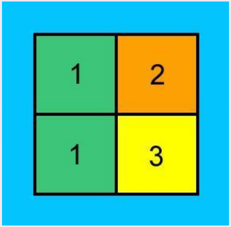
Proving $f(m,n) = \left[\frac{mn+m+n-1}{2}\right]$ THERE EXISTS:

By induction on the number of the cells: $f(m,n) \geq \left[\frac{mn+m+n-1}{2}\right]$.

If $m = 1$, then $\left[\frac{n+1+n-1}{2}\right] = n$, we put the numbers $1, 2, 3, \ldots, n$ in the cells and we will have exactly n islands.

If $n = 1$, then $\left[\frac{m+m+1-1}{2}\right] = m$.

If $m = n = 2$:



Proving $f(m,n) = \left[\frac{mn+m+n-1}{2}\right]$ THERE EXISTS:

Let $m, n > 2$.

$$f(m,n) \geq f(m-2,n)+f(1,n)+1 \geq \left[\frac{(m-2)n+(m-2)+n-1}{2}\right]+\left[\frac{n+1+n-1}{2}\right]+1 =$$
$$= \left[\frac{(m-2)n+(m-2)+n-1+2n}{2}\right] + 1 = \left[\frac{mn+m+n-1}{2}\right].$$

LATTICE METHOD

G. Czédli, A. P. Huhn and E. T. Schmidt: Weakly independent subsets in lattices, Algebra Universalis 20 (1985), 194-196.

Any two weak bases of a finite distributive lattice have the same number of elements.

Proving methods/1

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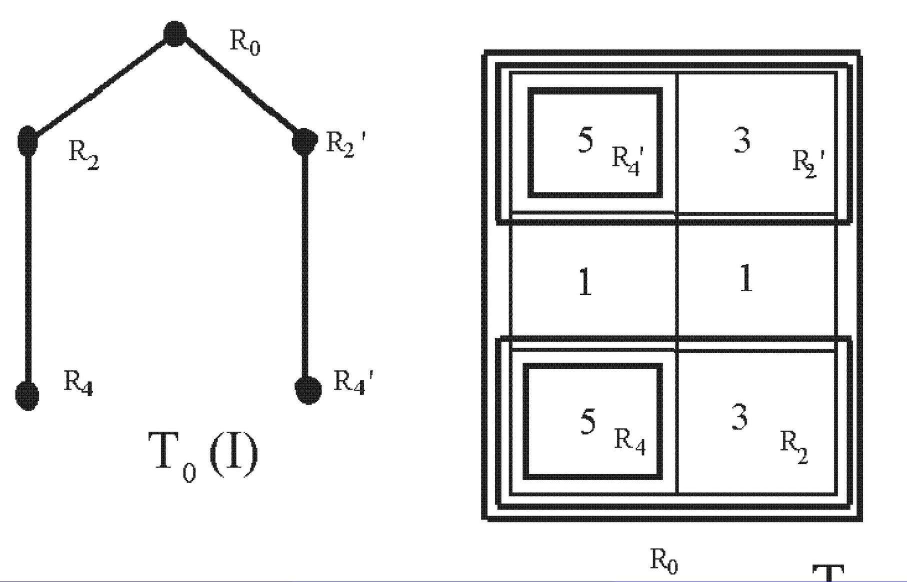
Proving methods/2

TREE-GRAPH METHOD



Proving methods/2

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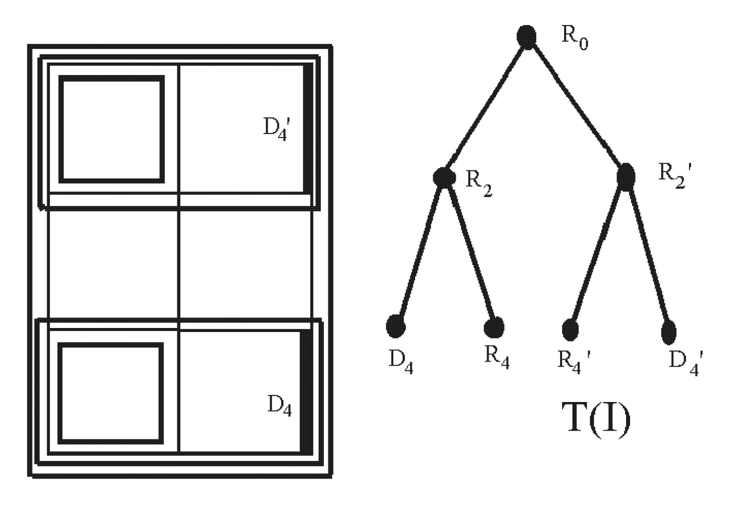

TREE-GRAPH METHOD

Lemma 2 (folklore)

Let T be a rooted tree such that any non-leaf node has at least 2 sons. Let ℓ be the number of leaves in T. Then $|V| \leq 2\ell - 1$.

We have $4s + 2d \leq (n+1)(m+1)$.
The number of leaves of $T(\mathcal{I})$ is $\ell = s + d$. Hence by Lemma 2 the number of islands is

$$|V| - d \leq (2\ell - 1) - d = 2s + d - 1 \leq \frac{1}{2}(n+1)(m+1) - 1.$$

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Proving methods/3

ELEMENTARY METHOD

We define

$$\mu(R) = \mu(u, v) := (u + 1)(v + 1).$$

Now

$$f(m, n) = 1 + \sum_{R \in max\mathcal{I}} f(R) = 1 + \sum_{R \in max\mathcal{I}} ([\frac{(u+1)(v+1)}{2}] - 1)$$

$$= 1 + \sum_{R \in max\mathcal{I}} ([\frac{\mu(u, v)}{2}] - 1) \leq 1 - |max\mathcal{I}| + [\frac{\mu(\mathrm{C})}{2}].$$

If $|max\mathcal{I}| \geq 2$, then the proof is ready. Case $|max\mathcal{I}| = 1$ is an easy exercise.

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Exact results

Cylindric board, rectangular islands (J. Barát, P. Hajnal, E.K. Horváth):
If $n \geq 2$, then $h_1(m,n) = [\frac{(m+1)n}{2}]$.

Cylindric board, cylindric and rectangular islands (J. Barát, P. Hajnal, E.K. Horváth):
If $n \geq 2$, then $h_2(m,n) = [\frac{(m+1)n}{2}] + [\frac{(m-1)}{2}]$.

Torus board, rectangular islands (J. Barát, P. Hajnal, E.K. Horváth):
If $m, n \geq 2$, then $t(m,n) = [\frac{mn}{2}]$.

Peninsulas (semi islands) (J. Barát, P. Hajnal, E.K. Horváth):
$p(m,n) = f(m,n) = [(mn + m + n - 1)/2]$.

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Islands in Boolean algebras, i.e. in hypercubes

The board consists of all vertices of a hypercube, i.e. the elements of a Boolean algebra $BA = \{0,1\}^n$.

We consider two cells neighbouring if their Hamming distance is 1.

We denote the maximum number of islands in $BA = \{0,1\}^n$ by $b(n)$.

Island formula for Boolean algebras (J. Barát, P. Hajnal, E.K. Horváth)
$b(n) = 1 + 2^{n-1}$.

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Rectangular height functions/1

Joint work with Branimir Šešelja and Andreja Tepavčević

A *height function* h is a mapping from $\{1, 2, ..., m\} \times \{1, 2, ..., n\}$ to $\mathbb{N}$, $h : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$.

The co-domain of the height function is the lattice $(\mathbb{N}, \leq)$, where $\mathbb{N}$ is the set of natural numbers under the usual ordering $\leq$ and suprema and infima are max and min, respectively.

For every $p \in \mathbb{N}$, the *cut of the height function*, i.e. the *p-cut* of h is an ordinary relation h_p on $\{1, 2, ..., m\} \times \{1, 2, ..., n\}$ defined by

$$(x, y) \in h_p \text{ if and only if } h(x, y) \geq p.$$

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Rectangular height functions/2

We say that two rectangles $\{\alpha, ..., \beta\} \times \{\gamma, ..., \delta\}$ and $\{\alpha_1, ..., \beta_1\} \times \{\gamma_1, ..., \delta_1\}$ are *distant* if they are disjoint and for every two cells, namely (a, b) from the first rectangle and (c, d) from the second, we have $(a - c)^2 + (b - d)^2 \geq 4$.

The height function h is called *rectangular* if for every $p \in \mathbb{N}$, every nonempty p-cut of h is a union of distant rectangles.

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Rectangular height functions/3

5	5	3	5	5
4	4	2	4	4
2	2	1	2	2

$\Gamma_1 = \{1, 2, 3, 4, 5\} \times \{1, 2, 3\}$,
$\Gamma_2 = \{1, 2, 3, 4, 5\} \times \{1, 2, 3\} \setminus \{(3, 1)\}$,
$\Gamma_3 = \{(1, 2), (1, 3), (2, 2), (2, 3), (3, 3), (4, 2), (4, 3), (5, 2), (5, 3)\}$,
$\Gamma_4 = \{(1, 2), (1, 3), (2, 2), (2, 3), (4, 2), (4, 3), (5, 2), (5, 3)\}$ and
$\Gamma_5 = \{(1, 3), (2, 3), (4, 3), (5, 3)\}$

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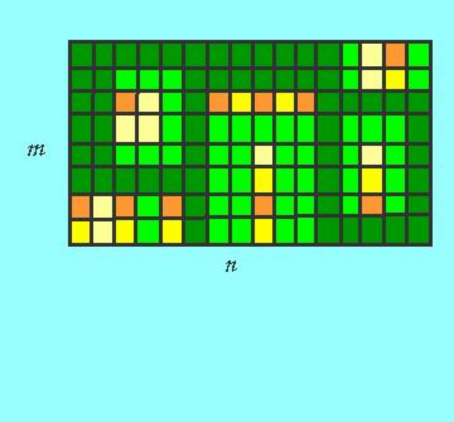
$\Gamma_4 = \{(1,2),(1,3),(2,2),(2,3),(4,2),(4,3),(5,2),(5,3)\}$ and
$\Gamma_5 = \{(1,3),(2,3),(4,3),(5,3)\}$

Theorem 2

For every height function $h : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \to \mathbb{N}$, there is a rectangular height function $h^* : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \to \mathbb{N}$, such that $\mathcal{I}_{rect}(h) = \mathcal{I}_{rect}(h^*)$.



Rectangular height functions/12

Theorem 4

For every rectangular height function

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$$h^{**} : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \to \mathbb{N},$$

such that $\mathcal{I}_{rect}(h^*) = \mathcal{I}_{rect}(h^{**})$ and in h^{**} every island appears exactly in one cut.

If a rectangular height function h^{**} has the property that each island appears exactly in one cut, then we call it *standard rectangular height function.*

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We denote by $\Lambda_{max}(m, n)$ the maximum number of different nonempty p-cuts of a standard rectangular height function on the rectangular table of size $m \times n$.

Theorem 5 $\Lambda_{max}(m, n) = m + n - 1.$

Rectangular height functions/13

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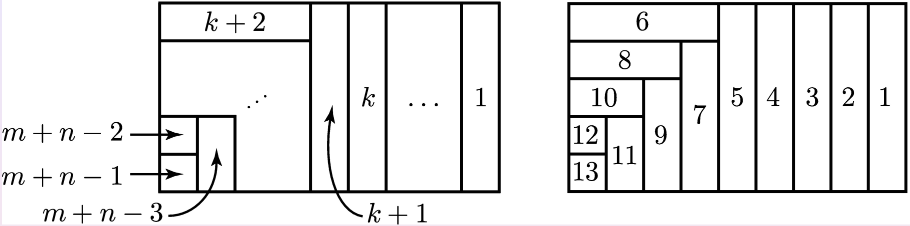
Rectangular height functions/14



The maximum number of different nonempty p-cuts of a standard rectangular height function is equal to the minimum cardinality of maximal systems of islands.

Rectangular height functions/15

Lemma 1
If $m \geq 3$ and $n \geq 3$ and a height function $h : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$ has maximally many islands, then it has exactly two maximal islands.

Lemma 2
If $m \geq 3$ or $n \geq 3$, then for any odd number $t = 2k + 1$ with $1 \leq t \leq max\{m - 2, n - 2\}$, there is a standard rectangular height function $h : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$ having the maximum number of islands f(m,n), such that one of the side-lengths of one of the maximal islands is equal to t.
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We denote by $\Lambda_h^{cz}(m,n)$ the number of different nonempty cuts of a standard rectangular height function h in the case h has maximally many islands, i.e., when the number of islands is

$$f(m,n) = \left\lfloor \frac{mn+m+n-1}{2} \right\rfloor .$$

Theorem 6

Let $h : \{1,2,...,m\} \times \{1,2,...,n\} \to \mathbb{N}$ be a standard rectangular height function having maximally many islands $f(m,n)$. Then, $\Lambda_h^{cz}(m,n) \geq \lceil log_2(m+1) \rceil + \lceil log_2(n+1) \rceil - 1$.

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CD-independent subsets in posets

Let $\mathbb{P} = (P, \leq)$ be a partially ordered set, and let $a, b \in P$. The elements a and b are called *disjoint* and we write $a \perp b$ if

either $\mathbb{P}$ has least element $0 \in P$ and $\inf\{a, b\} = 0$,
or $\mathbb{P}$ is without 0 and the elements a and b have no common lowerbound.

A nonempty set $X \subseteq P$ is called *CD-independent* if for any $x, y \in X$, $x \leq y$ or $y \leq x$, or $x \perp y$ holds. Maximal CD-independent sets (with respect to $\subseteq$) are called *CD-bases* in $\mathbb{P}$.

CD-independent subsets in distributive lattices

G. Czédli, M. Hartmann and E. T. Schmidt: CD-independent subsets in distributive lattices, Publicationes Mathematicae Debrecen, 74/1-2 (2009).

Any two CD-bases of a finite distributive lattice have the same number of elements.

If all finite lattices in a lattice variety have this property, then the variety must coincide with the variety of distributive lattices.

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Sets of pairwise disjoint elements

Definition

A nonempty set D of nonzero elements of P is called a *a set of pairwise disjoint elements* in $\mathbb{P}$ if $x \perp y$ holds for all $x, y \in D$, $x \neq y$; if $\mathbb{P}$ has 0-element, then $\{0\}$ is considered to be a set of pairwise disjont elements, too.

Remark

D is a set of pairwise disjoint elemets, if and only if it is a CD-independent antichain in $\mathbb{P}$.

Order ideals

For $X \subseteq P$, the order ideal $\{y \in P \mid y \leq x \text{ for some } x \in X\}$ is denoted by $\downarrow X$. The order-ideals of any poset form a (distributive) lattice with respect to $\subseteq$. So, the antichains of a poset can be ordered as follows:

Definition

If A_1, A_2 are antichains in $\mathbb{P}$, then we say that A_1 *is dominated by* A_2, and we denote it by $A_1 \leqslant A_2$ if $\downarrow A_1 \subseteq \downarrow A_2$.

Remarks

- $\leqslant$ is a partial order
- $A_1 \leqslant A_2$ is satisfied if and only if

$$\text{for each } x \in A_1 \text{ there exists an } y \in A_2, \text{ with } x \leq y. \qquad \text{(A)}$$

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The poset $\mathcal{D}(P)$

Let $\mathcal{D}(P)$ denote the set of all sets of pairwise disjont elements of P.

As sets of pairwise disjont elements of $\mathbb{P}$ are also antichains, restricting $\leqslant$ to $\mathcal{D}(P)$, we obtain a poset $(\mathcal{D}(P), \leqslant)$.

The connection between the poset $(\mathcal{D}(P), \leqslant)$ and the CD-bases of the poset P is shown by the next theorem:

Theorem

Let B *be a CD-base of a finite poset* $(P, \leq)$, *and* let $|B| = n$.

Then there exists a maximal chain $\{D_i\}_{1 \leq i \leq n}$ *in* $\mathcal{D}(P)$ *such that* $B = \bigcup_{i=1}^{n} D_i$.

Moreover, for any maximal chain $\{D_i\}_{1 \leq i \leq m}$ *in* $\mathcal{D}(P)$ *the set* $D = \bigcup_{i=1}^{m} D_i$ *is a CD-base in* $(P, \leq)$ *with* $|D| = m$.

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Illustration

A poset and a maximal chain of sets of disjoint elements.

Proof of the Theorem

Proposition

If B is a CD-base and $D \subseteq B$ is a set of pairwise disjont elements in the poset $(P, \leq)$, then $\downarrow D \cap B$ is also a CD-base in the subposet $(\downarrow D, \leq)$.

Lemma

If $D_1 \prec D_2$ in $\mathcal{D}(P)$, then $D_2 = \{a\} \cup \left\{y \in D_1 \setminus \{0\} \mid y \perp a\right\}$ for some minimal element a of the set
$S = \left\{s \in P \setminus (D_1 \cup \{0\}) \mid y \perp s \text{ or } y < s \text{ for all } y \in D_1\right\}$.
Moreover, $D_1 \prec \{a\} \cup \left\{y \in D_1 \setminus \{0\} \mid y \perp a\right\}$ holds for any minimal element a of the set S.

Lemma

Assume that B is a CD-base with at least two elements in a finite poset $\mathbb{P} = (P, \leq)$, $M = \max(B)$, *and* $m \in M$. Then M and $N := \max(B \setminus \{m\})$ are sets of pairwise disjoint elements. Moreover M is a maximal element in $\mathcal{D}(P)$, and $N \prec M$ holds in $\mathcal{D}(P)$.

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Corollary

Let $\mathbb{P} = (P, \leq)$ be a finite poset.

The CD-bases of $\mathbb{P}$ have the same number of elements if and only if the poset $\mathcal{D}(P)$ is graded.

Let $B \subseteq P$ be a CD-base of $\mathbb{P}$, and $(B, \leq)$ the poset under the restricted ordering. Then any maximal chain $\mathcal{C} = \{D_i\}_{1 \leq i \leq m}$ in $\mathcal{D}(B)$ is also a maximal chain in $\mathcal{D}(P)$.

If D is a disjoint set in $\mathbb{P}$ and the CD-bases of $\mathbb{P}$ have the same number of elements, then the CD-bases of the subposet $(I(D), \leq)$ also have the same number of elements.

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$\mathcal{D}(P)$ is graded

The poset $\mathbb{P}$ is called *graded*, if all its maximal chains have the same cardinality.

Let $\mathbb{P} = (P, \leq)$ be a finite poset with 0. Then the following conditions are equivalent:

(i) The CD-bases of $\mathbb{P}$ have the same number of elements,

(ii) $\mathcal{D}(P)$ is graded.

A set of pairwise disjoint elements D of a poset $(P, \leq)$ is called *complete*, if there is no $p \in P \setminus D$ such that $D \cup \{p\}$ is also a disjoint system.

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If $\mathbb{P}$ is a finite poset with 0

If all the principal ideals $(a]$ of $\mathbb{P}$ are weakly 0-modular, then $A(P) \cup C$ is a CD-base for every maximal chain C *in* $\mathbb{P}$.

If $\mathbb{P}$ has weakly 0-modular principal ideals and $\mathcal{D}(P)$ is graded, then $\mathbb{P}$ is also graded, and any CD-base of $\mathbb{P}$ contains $|A(P)| + l(P)$ elements.

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Lemma

Let $\mathbb{P}$ be a poset with 0 and D_k, $k \in K$ $(K \neq \emptyset)$ sets of pairwise disjoint elements in $\mathbb{P}$. If the meet $\bigwedge_{k\in K} a^{(k)}$ of any system of elements $a^{(k)} \in D_k$, $k \in K$ exist in $\mathbb{P}$, then $\bigwedge_{k\in K} D_k$ also exists in $\mathcal{D}(P)$.

A pair $a, b \in P$ with least upperbound $a \vee b$ in $\mathbb{P}$ is called a *distributive pair*, if $(c \wedge a) \vee (c \wedge b)$ exists in $\mathbb{P}$ for any $c \in P$, and $c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$.
We say that $(P, \wedge)$ is *dp-distributive*, if any $a, b \in P$ with $a \wedge b = 0$ is a distributive pair.

Theorem

(i) *If* $\mathbb{P} = (P, \wedge)$ is a semilattice with 0, then $\mathcal{D}(P)$ is a dp-distributive semilattice; if $D_1 \cup D_2$ is a CD-independent set for some $D_1, D_2 \in \mathcal{D}(P)$, then D_1, D_2 is a distributive pair in $\mathcal{D}(P)$.

(ii) If $\mathbb{P}$ is a complete lattice, then $\mathcal{D}(P)$ is a dp-distributive complete lattice.

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(ii) If $\mathbb{P}$ is a complete lattice, then $\mathcal{D}(P)$ is a dp-distributive complete lattice.

Let $(P, \leq)$ be a poset and $A \subseteq P$. $(A, \leq)$ is called a *sublattice* of $(P, \leq)$, if $(A, \leq)$ is a lattice such that for any $a, b \in A$ the infimum and the supremum of $\{a, b\}$ is the same in the subposet $(A, \leq)$ and in $(P, \leq)$. If the relation $x \prec y$ in $(A, \leq)$ for some $x, y \in A$ implies $x \prec y$ in the poset $(P, \leq)$, then we say that $(A, \leq)$ is a *cover-preserving subposet* of $(P, \leq)$.

Theorem

Let $\mathbb{P} = (P, \leq)$ be a poset with 0 and B a CD-base of it. Then $(\mathcal{D}(B), \leqslant)$ is a distributive cover-preserving sublattice of the poset $(\mathcal{D}(P), \leqslant)$. *If* $\mathbb{P}$ is a $\wedge$-semilattice, then for any $D \in \mathcal{D}(P)$ and $D_1, D_2 \in \mathcal{D}(B)$ we have $(D_1 \vee D_2) \wedge D = (D_1 \wedge D) \vee (D_2 \wedge D)$ in $(\mathcal{D}(P), \leqslant)$.

CD-bases in particular lattice classes

Lemma

Let L be a finite weakly 0-distributive lattice and D a dual atom in $\mathcal{D}(L)$. Then either $D = \{d\}$, for some $d \in L$ with $d \prec 1$, or D consist of two different elements $d_1, d_2 \in L$ and $d_1 \vee d_2 = 1$.

Theorem

Let L be a finite, weakly 0-distributive lattice. Then the following are equivalent:

- (i) L is graded, and $l(a) + l(b) = l(a \vee b)$ holds for all $a, b \in L$ with $a \wedge b = 0$.
- (ii) L is 0-modular and the CD-bases of L have the same number of elements.

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