

CD-independent subsets

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CD-independent subsets in distributive lattices

G. Czédli, M. Hartmann and E. T. Schmidt: CD-independent subsets in distributive lattices, Publicationes Mathematicae Debrecen, 74/1-2 (2009).

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If all finite lattices in a lattice variety have this property, then the variety must coincide with the variety of distributive lattices.

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Let $\mathbb{P} = (P, \leq)$ be a partially ordered set, and let $a, b \in P$. The elements a and b are called *disjoint* and we write $a \perp b$ if

either \mathbb{P} has least element $0 \in P$ and $\inf\{a, b\} = 0$,
or \mathbb{P} is without 0 and the elements a and b have no common lowerbound.

A nonempty set $X \subseteq P$ is called *CD-independent* if for any $x, y \in X$, $x \leq y$ or $y \leq x$, or $x \perp y$ holds. Maximal CD-independent sets (with respect to \subseteq) are called *CD-bases* in \mathbb{P} .

Tolerance relation

Now on the set P we define a relation $\rho \subseteq P \times P$ as follows: For any $x, y \in P$

$$(x, y) \in \rho \Leftrightarrow x \leq y \text{ or } y \leq x \text{ or } x \perp y.$$

Then ρ is reflexive and symmetric by its definition, i.e. it is a tolerance relation on P .

It is easy to see that the CD-bases of \mathbb{P} are exactly the tolerance classes of ρ .

As any tolerance relation has at least one tolerance class, and its classes form a covering of the underlying set, we obtain:

Proposition *Any poset $\mathbb{P} = (P, \leq)$ has at least one CD-base, and the set P is covered by the CD-bases of \mathbb{P} .*

Definition

A nonempty set D of nonzero elements of P is called a *set of pairwise disjoint element* in \mathbb{P} if $x \perp y$ holds for all $x, y \in D$, $x \neq y$; if \mathbb{P} has 0-element, then $\{0\}$ is considered to be a set of pairwise disjoint elements, too.

Remark

- D is a set of pairwise disjoint elements if and only if it is a CD-independent antichain in \mathbb{P} .

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Order ideals

Let $X \subseteq P$.

The order ideal $\{y \in P \mid y \leq x \text{ for some } x \in X\}$ is denoted by $\downarrow X$.

The order-ideals of any poset form a (distributive) lattice with respect to \subseteq .

So, the antichains of a poset can be ordered as follows:

Definition

If A_1, A_2 are antichains in \mathbb{P} , then we say that A_1 is *dominated by* A_2 , and we denote it by $A_1 \leq A_2$ if $\downarrow A_1 \subseteq \downarrow A_2$.

Remarks

- \leq is a partial order
- $A_1 \leq A_2$ is satisfied if and only if

for each $x \in A_1$ there exists an $y \in A_2$, with $x \leq y$. (A)

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Let $\mathcal{D}(\mathbb{P})$ denote the set of all sets of pairwise disjoint elements of \mathbb{P} .

As sets of pairwise disjoint elements of \mathbb{P} are also antichains, restricting \leq to $\mathcal{D}(\mathbb{P})$, we obtain a poset $(\mathcal{D}(\mathbb{P}), \leq)$.

The connection between CD-bases of a poset \mathbb{P} and the poset $(\mathcal{D}(\mathbb{P}), \leq)$ is shown by the next theorem:

Theorem

Let B be a CD-base of a finite poset (P, \leq) , and let $|B| = n$.

Then there exists a maximal chain $\{D_i\}_{1 \leq i \leq n}$ in $\mathcal{D}(P)$ such that

$$B = \bigcup_{i=1}^n D_i.$$

Moreover, for any maximal chain $\{D_i\}_{1 \leq i \leq m}$ in $\mathcal{D}(P)$ the set $D = \bigcup_{i=1}^m D_i$ is a CD-base in (P, \leq) with $|D| = m$.

Theorem

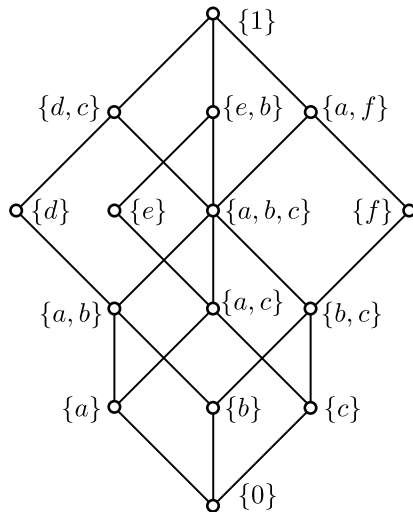
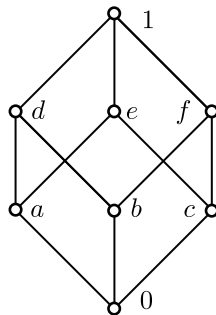
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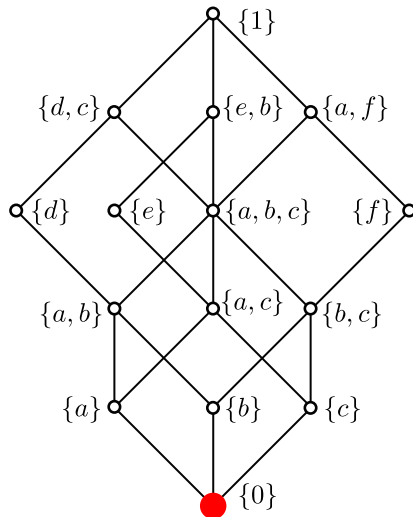
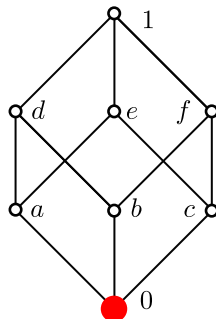
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Moreover, for any maximal chain $\{D_i\}_{1 \leq i \leq m}$ in $\mathcal{D}(P)$ the set $D = \bigcup_{i=1}^m D_i$ is a CD-base in (P, \leq) with $|D| = m$.

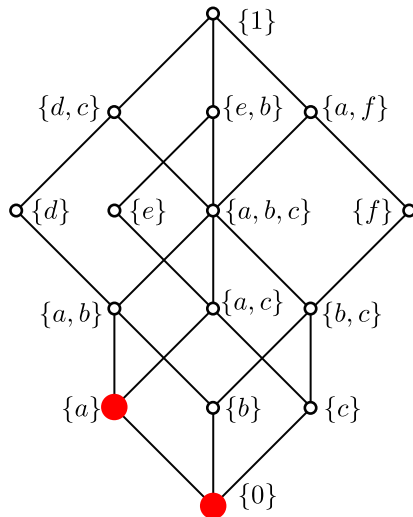
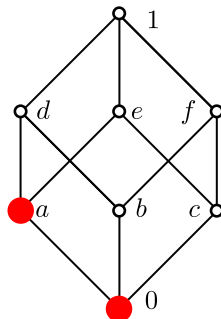
P and $\mathcal{D}(P)$



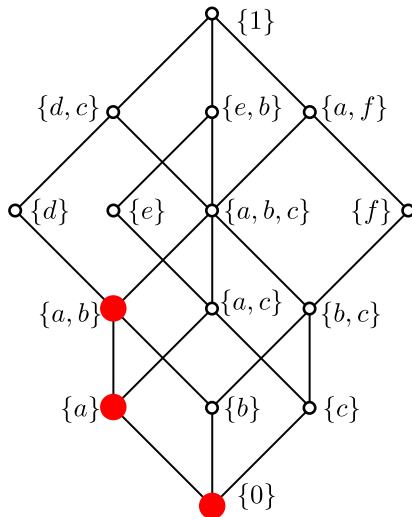
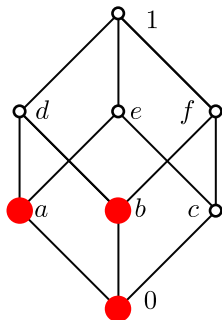
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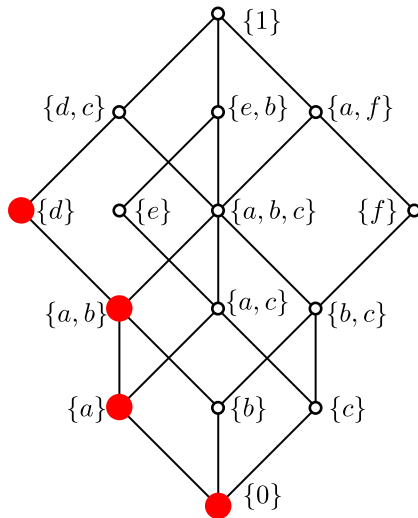
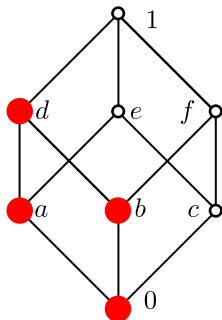
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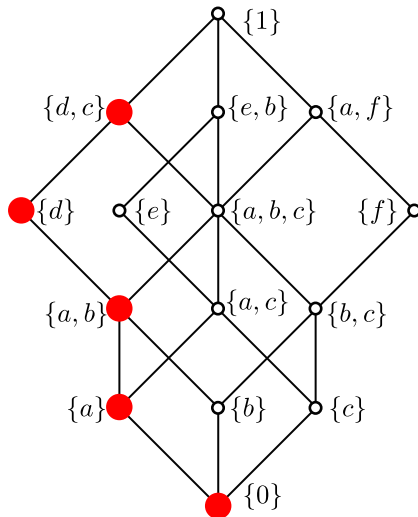
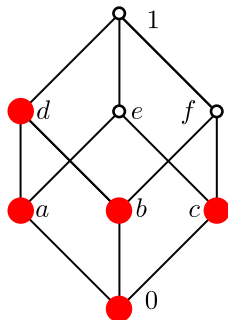
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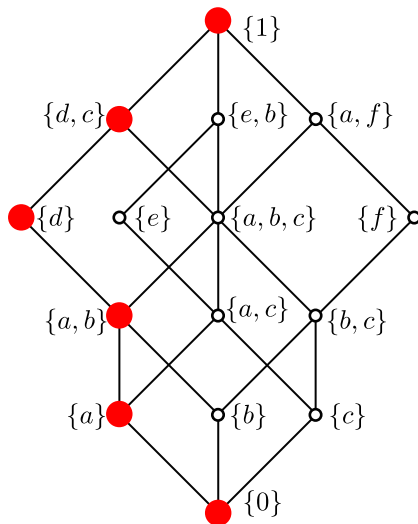
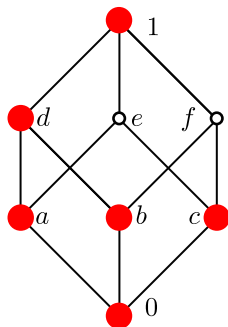
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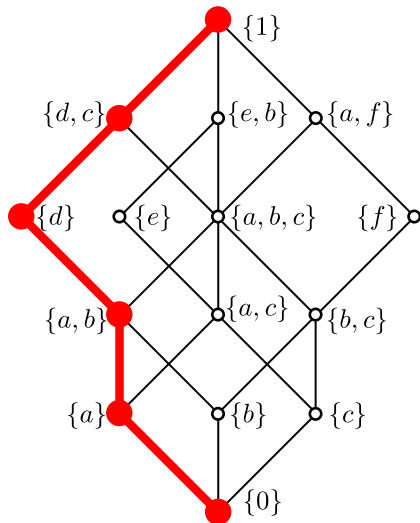
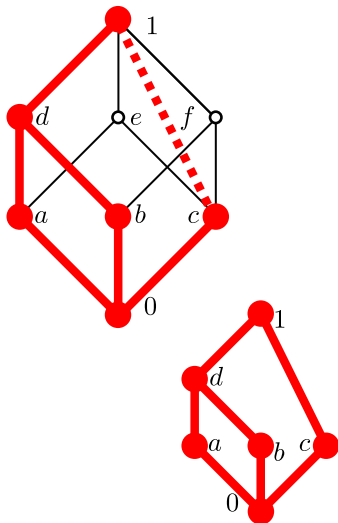
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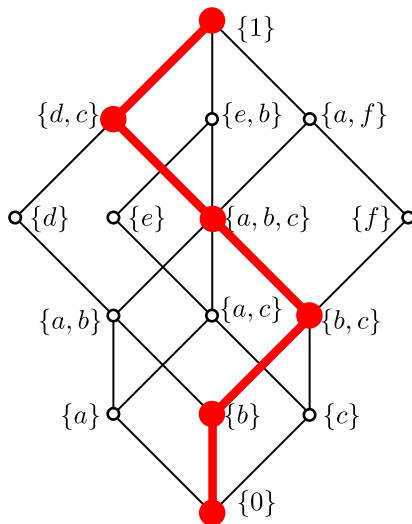
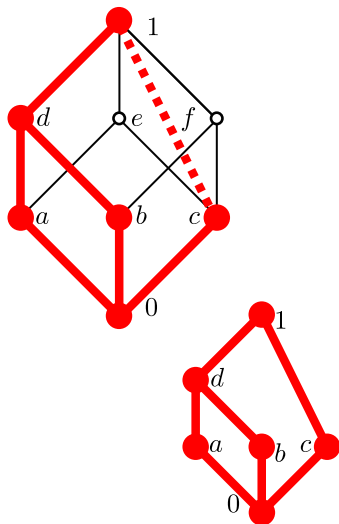
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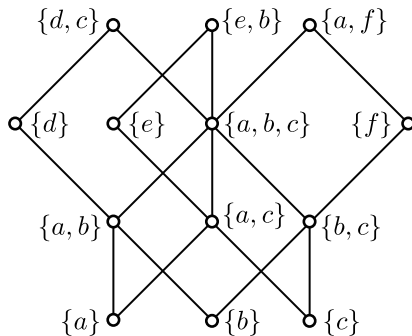
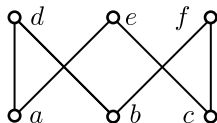
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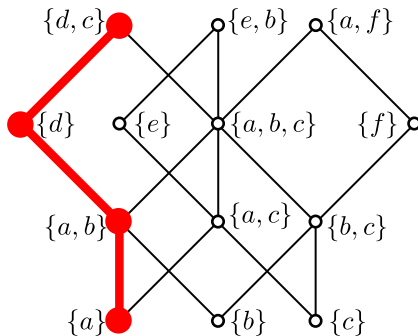
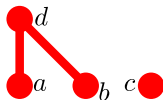
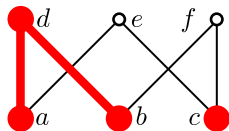
P and $\mathcal{D}(P)$



P and $\mathcal{D}(P)$, another example



P and $\mathcal{D}(P)$, another example



Lemma 1

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Lemma 2

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Lemma 1

If $D_1 \prec D_2$ in $\mathcal{D}(P)$, then $D_2 = \{a\} \cup \{y \in D_1 \setminus \{0\} \mid y \perp a\}$ for some minimal element a of the set

$$S = \{s \in P \setminus (D_1 \cup \{0\}) \mid y \perp s \text{ or } y < s \text{ for all } y \in D_1\}.$$

Moreover, $D_1 \prec \{a\} \cup \{y \in D_1 \setminus \{0\} \mid y \perp a\}$ holds for any minimal element a of the set S .

Illustration for Lemma 1

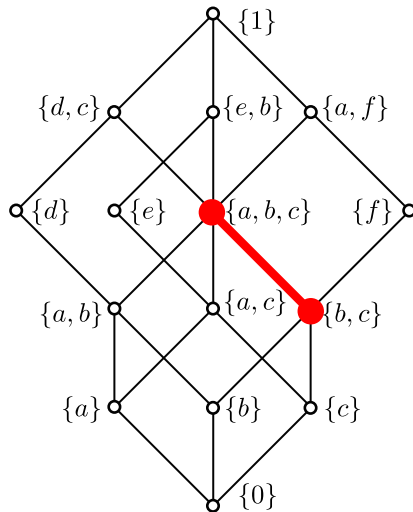
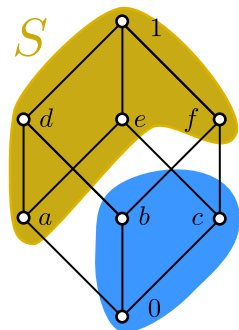


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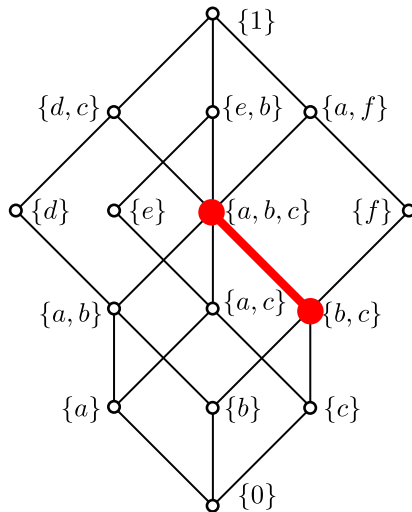
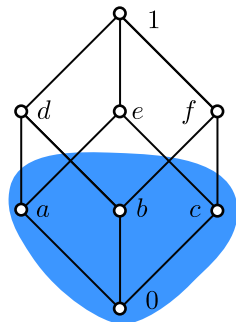


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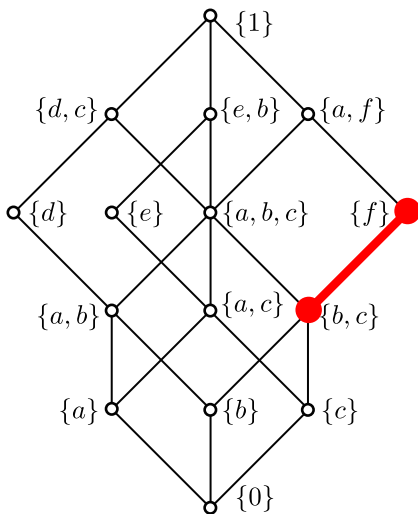
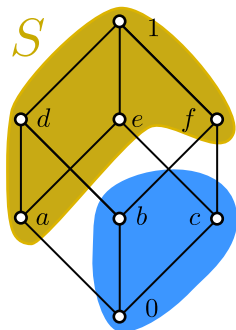
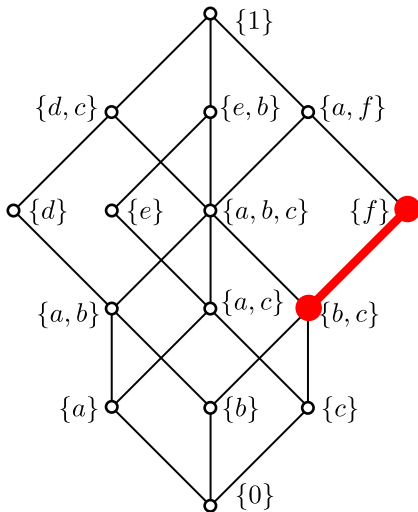
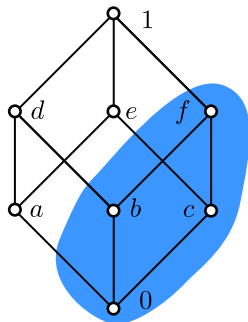


Illustration for Lemma 1



Lemma 2

Assume that B is a CD-base with at least two elements in a finite poset $\mathbb{P} = (P, \leq)$, $M = \max(B)$, and $m \in M$. Then M and $N := \max(B \setminus \{m\})$ are disjoint sets.

Moreover M is a maximal element in $\mathcal{D}(P)$, and $N \prec M$ holds in $\mathcal{D}(P)$.

Illustration for Lemma 2

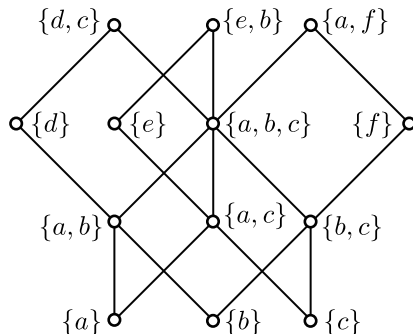
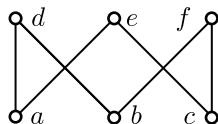


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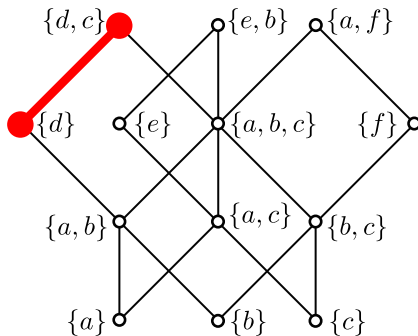
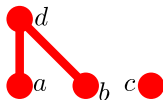
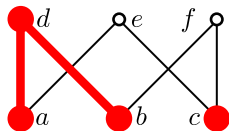


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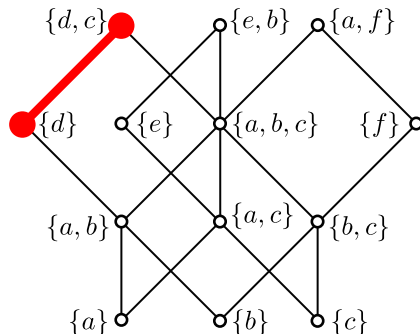
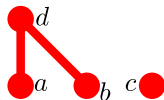
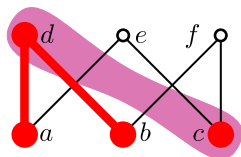
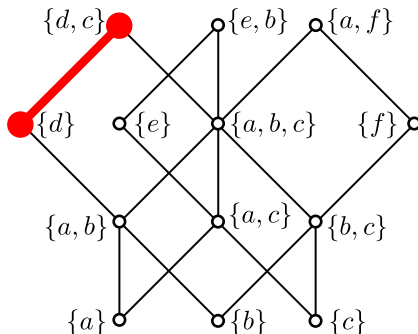
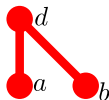
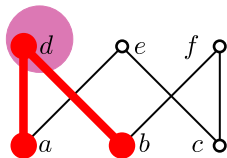


Illustration for Lemma 2



Let $\mathbb{P} = (P, \leq)$ be a finite poset.

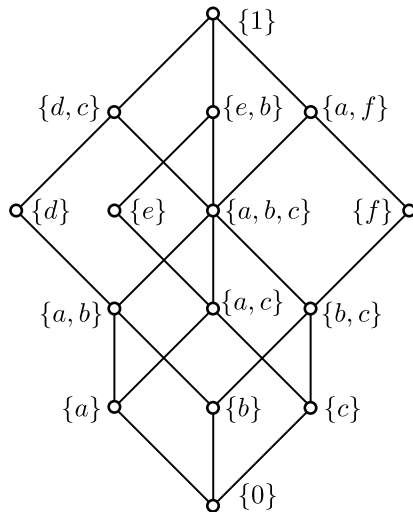
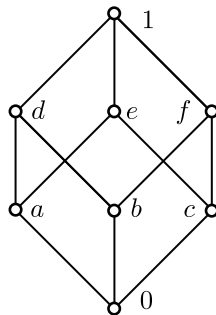
The poset \mathbb{P} is called *graded*, if all its maximal chains have the same cardinality.

The CD-bases of \mathbb{P} have the same number of elements if and only if the poset $\mathcal{D}(P)$ is graded.

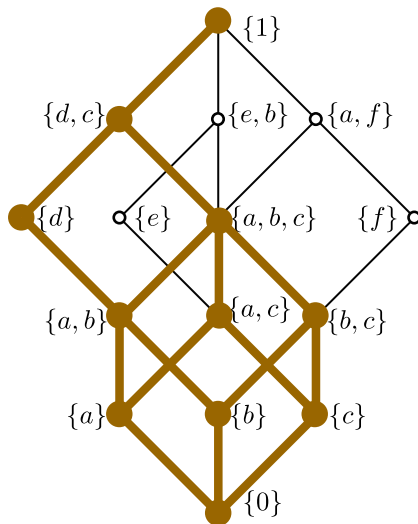
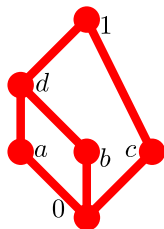
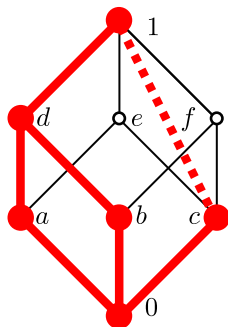
Corollary

Let $B \subseteq P$ be a CD-base of \mathbb{P} , and (B, \leq) the poset under the restricted ordering. Then any maximal chain $\mathcal{C} = \{D_i\}_{1 \leq i \leq m}$ in $\mathcal{D}(B)$ is also a maximal chain in $\mathcal{D}(P)$.

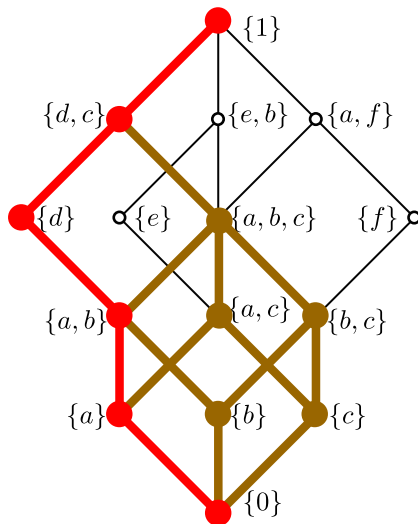
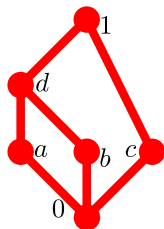
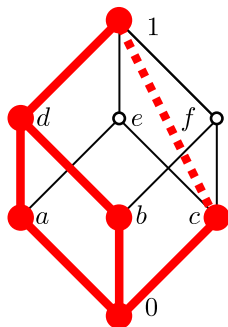
P , B $\mathcal{D}(P)$ and $\mathcal{D}(B)$



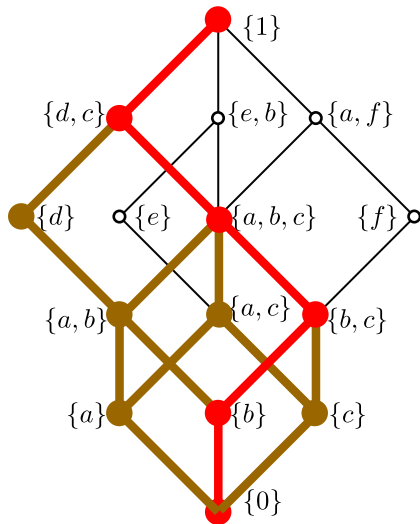
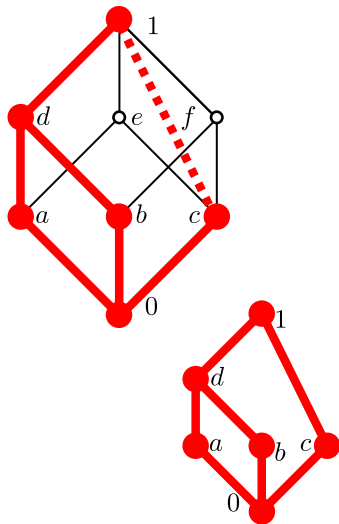
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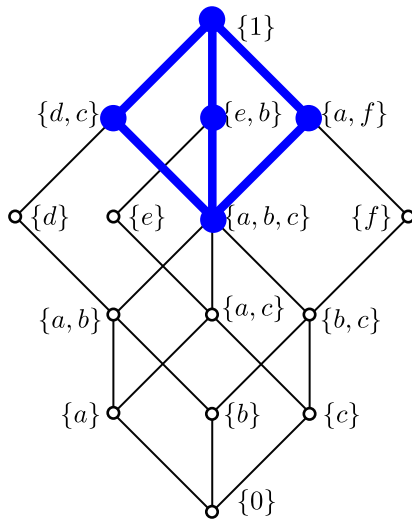
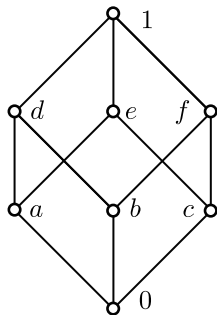
P , B $\mathcal{D}(P)$ and $\mathcal{D}(B)$



If D is a disjoint set in \mathbb{P} and the CD-bases of \mathbb{P} have the same number of elements, then the CD-bases of the subposet $(\downarrow D, \leq)$ also have the same number of elements.

A set of pairwise disjoint elements D of a poset (P, \leq) is called *complete*, if there is no $p \in P \setminus D$ such that $D \cup \{p\}$ is also a set of pairwise disjoint elements.

P , $\mathcal{D}(P)$ and $\mathcal{DC}(P)$



Let $\mathbb{P} = (P, \leq)$ be a finite poset with 0. Then the following conditions are equivalent:

(i) The CD-bases of \mathbb{P} have the same number of elements,

(ii) $\mathcal{D}(P)$ is graded.

(iii) $\mathcal{DC}(P)$ is graded.

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Weakly 0-modular lattices

Now, we will see that under some weak conditions, the condition that $\mathcal{D}(\mathbb{P})$ graded implies that \mathbb{P} itself is graded.

A poset with least element 0 and greatest element 1 is called *bounded*. A lattice $\mathbb{L} = (L, \leq)$ with 0 is called *0-modular* if for all $a, b, c \in L$

$$a \leq b \text{ and } b \wedge c = 0 \text{ imply } b \wedge (a \vee c) = a \quad (M_0)$$

where (and everywhere later) we use the usual notation $x \vee y = \sup\{a, b\}$ and $x \wedge y = \inf\{a, b\}$. Equivalently, L has no pentagon sublattice N_5 that contains $0 = 0_L$.

We know that that the tolerance lattices of algebras belonging to congruence distributive varieties are *0-modular* (but not necessarily modular).

If (M_0) is satisfied under the assumptions that a is an atom and $c \prec b \vee c$, then L is called *weakly 0-modular*.

L is *lower-semimodular* if for all $a, b, c \in L$, $b \prec c$ implies $a \wedge b \preceq a \wedge c$.

It belongs to the folklore that join-semidistributivity and lower semimodularity characterize the closure lattices of finite convex geometries.

It is easy to see that any lower-semimodular lattice and any 0-modular lattice is weakly 0-modular.

We say that a poset \mathbb{P} with 0 is *weakly 0-modular* if the above weak form of (M_0) holds whenever $\sup\{a, c\}$ and $\sup\{b, c\}$ exist in \mathbb{P} .

If \mathbb{P} is a finite bounded poset

Let \mathbb{P} be a finite bounded poset.

If all the principal ideals $\downarrow a$ of \mathbb{P} are weakly 0-modular, then $A(\mathbb{P}) \cup C$ is a CD-base for every maximal chain C in \mathbb{P} .

If each principal ideal of \mathbb{P} is weakly 0-modular and $\mathcal{D}(\mathbb{P})$ is graded, then \mathbb{P} is also graded, and any CD-base of \mathbb{P} contains $|A(\mathbb{P})| + l(\mathbb{P})$ elements.

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Lemma

We will see that if \mathbb{P} is a \wedge -semilattice, B is a CD-base of it, and $\mathbb{B} = (B, \leq)$ is the poset under the restricted ordering, then for appropriately chosen elements of $\mathcal{D}(\mathbb{B})$ and $\mathcal{D}(\mathbb{P})$ a special kind of distributivity holds. The next Lemma shows a property of $\mathcal{D}(\mathbb{P})$ for any poset \mathbb{P} with 0.

Let \mathbb{P} be a poset with 0. Assume that $K \neq \emptyset$ is an index set and, for each $k \in K$, D_k is a set of pairwise disjoint elements in \mathbb{P} . If for every choice function $f \in \prod_{k \in K} D_k$ the meet $\bigwedge_{k \in K} f(k)$ exists in \mathbb{P} , then $\bigwedge_{k \in K} D_k$ exists in $\mathcal{D}(\mathbb{P})$. In particular, for $K = \{1, 2\}$ and $D_1 = \{a_i \mid i \in I\}$, $D_2 = \{b_j \mid j \in J\} \in \mathcal{D}(\mathbb{P})$ such that all the $a_i \wedge b_j$ exists, we have

$$D_1 \wedge D_2 = \begin{cases} M := \{a_i \wedge b_j \mid i \in I, j \in J, a_i \wedge b_j \neq 0\} & \text{if } M \neq \emptyset; \\ \{0\} & \text{otherwise.} \end{cases}$$

CD-bases in semilattices and lattices / 2

Let $\mathbb{P} = (P, \wedge)$ be a semilattice with 0. Now, for all $a, b \in P$, the relation $a \perp b$ means that $a \wedge b = 0$. Hence, a set $\{a_i \mid i \in I\}$ of nonzero elements is a set of pairwise disjoint elements if and only if $a_i \wedge a_j = 0$ for all $i, j \in I, i \neq j$. A pair $a, b \in P$ with least upperbound $a \vee b$ in \mathbb{P} is called a *distributive pair* if $(c \wedge a) \vee (c \wedge b)$ exists in \mathbb{P} for all $c \in P$, and $c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$. We say that (P, \wedge) is *dp-distributive* (*distributive with respect to disjoint pairs*) if any pair $a, b \in P$ with $a \wedge b = 0$ is a distributive pair.

Theorem

(i) If $\mathbb{P} = (P, \wedge)$ is a semilattice with 0, then $\mathcal{D}(\mathbb{P})$ is a dp-distributive semilattice. Furthermore, for all $D_1, D_2 \in \mathcal{D}(\mathbb{P})$, if $D_1 \cup D_2$ is a CD-independent set, then D_1, D_2 is a distributive pair in $\mathcal{D}(\mathbb{P})$.

(ii) If \mathbb{P} is a complete lattice, then $\mathcal{D}(\mathbb{P})$ is a dp-distributive complete lattice.

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(ii) If \mathbb{P} is a complete lattice, then $\mathcal{D}(\mathbb{P})$ is a dp-distributive complete lattice.

Let (P, \leq) be a poset and $A \subseteq P$. (A, \leq) is called a *sublattice* of (P, \leq) , if (A, \leq) is a lattice such that for any $a, b \in A$ the infimum and the supremum of $\{a, b\}$ is the same in the subposet (A, \leq) and in (P, \leq) . If the relation $x \prec y$ in (A, \leq) for some $x, y \in A$ implies $x \prec y$ in the poset (P, \leq) , then we say that (A, \leq) is a *cover-preserving subposet* of (P, \leq) .

Theorem

Let $\mathbb{P} = (P, \leq)$ be a poset with 0 and B a CD-base of it. Then $(\mathcal{D}(B), \leq)$ is a distributive cover-preserving sublattice of the poset $(\mathcal{D}(P), \leq)$. If \mathbb{P} is a \wedge -semilattice, then for any $D \in \mathcal{D}(P)$ and $D_1, D_2 \in \mathcal{D}(B)$ we have $(D_1 \vee D_2) \wedge D = (D_1 \wedge D) \vee (D_2 \wedge D)$ in $(\mathcal{D}(P), \leq)$.

CD-bases in particular lattice classes

We investigate CD-bases in two particular classes of lattices. The properties of the first class generalize the properties of tolerance lattices of majority algebras. It was proved that the tolerance lattice of arbitrary majority algebra is a pseudocomplemented, 0-modular and dp-distributive lattice. These properties are not independent, since, for instance, it can be shown that dp-distributivity implies 0-modularity.

Let $\mathbb{L} = (L, \leq)$ a lattice. A lattice \mathbb{L} with 0 is called *pseudocomplemented* if for each $x \in L$ there exists an element $x^* \in L$ such that, for all $y \in L$, $y \wedge x = 0 \Leftrightarrow y \leq x^*$. It is known that an algebraic lattice L is pseudocomplemented if and only if it is *0-distributive*, that is, for all $a, b, x \in L$, $x \wedge a = 0$ and $x \wedge b = 0$ imply $x \wedge (a \vee b) = 0$. We say that \mathbb{L} is *weakly 0-distributive* if this implication holds under the condition $a \wedge b = 0$. Clearly, any 0-distributive lattice is weakly 0-distributive. If D is a set of pairwise disjoint elements in a weakly 0-distributive lattice and $|D| \geq 2$, then it is easy to see that replacing two different elements $d_1, d_2 \in D$ by their join $d_1 \vee d_2$, we obtain again a disjoint set.

Lemma

Let \mathbb{L} be a finite weakly 0-distributive lattice and D a dual atom of the poset $\mathcal{D}(\mathbb{L})$. Then either $D = \{d\}$ for some $d \in L$ with $d \prec 1$, or D consist of two different elements $d_1, d_2 \in L$ with $d_1 \vee d_2 = 1$.

Let \mathbb{L} be a graded lattice, and $a \in L$. Then the *height* of a is the length of the interval $[0, a]$, denoted by $l(a)$. (In the literature, it is also denoted by $h(a)$.) A graded lattice always has 0 and 1.

A graded lattice \mathbb{L} is 0-modular, whenever $l(a) + l(b) = l(a \vee b)$ holds for all $a, b \in L$ with $a \wedge b = 0$.

Theorem

Let L be a finite, weakly 0-distributive lattice. Then the following are equivalent:

- (i) L is graded, and $l(a) + l(b) = l(a \vee b)$ holds for all $a, b \in L$ with $a \wedge b = 0$.*
- (ii) L is 0-modular, and the CD-bases of L have the same number of elements.*

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- (ii) *L is 0-modular, and the CD-bases of L have the same number of elements.*

We say that two elements $a, b \in L$ form a *modular pair* in the lattice L and we write $(a, b)M$ if for all $x \in L$, $x \leq b$ implies $x \vee (a \wedge b) = (x \vee a) \wedge b$.

Also, a and b form a *dual-modular pair* if for all $x \in L$, $x \geq b$ implies $x \wedge (a \vee b) = (x \wedge a) \vee b$. This is denoted by $(a, b)M^*$.

Clearly, if a and b form a distributive pair, then $(a, b)M^*$ is satisfied.

By means of modular pairs, the 0-modularity condition can be reformulated as follows): For all $a, b \in L$,

Lemma (M. Stern) In a graded lattice of finite length, $(a, b)M$ implies $l(a) + l(b) \leq l(a \wedge b) + l(a \vee b)$.

With the help of Lemma of M. Stern above, using an N_5 sublattice containing 0 as well as the dual lattice, we obtain

Proposition If \mathbb{L} is a lattice with 0 such that $(a, b)M^*$ holds for all $a, b \in L$ with $a \wedge b = 0$, then L is 0-modular. If in addition \mathbb{L} is a graded lattice of finite length, then $l(a \vee b) = l(a) + l(b)$ holds for all $a, b \in L$ with $a \wedge b = 0$.

Corollary (i) Let \mathbb{L} be a finite, weakly 0-distributive lattice such that for each $a, b \in L$ with $a \wedge b = 0$, condition $(a, b)M^*$ holds. Then the CD-bases of \mathbb{L} have the same number of elements if and only if \mathbb{L} is graded.

(ii) If \mathbb{L} is a finite pseudocomplemented modular lattice, then the CD-bases of \mathbb{L} have the same number of elements.

As any dp-distributive lattice \mathbb{L} is weakly 0-distributive, and $(a, b)M^*$ holds for all $a, b \in L$ with $a \wedge b = 0$ since (a, b) is a distributive pair, we obtain

Corollary

- (i) Any dp-distributive lattice is 0-modular. If \mathbb{L} is a dp-distributive graded lattice with finite length, then $l(a \vee b) = l(a) + l(b)$ holds for all $a, b \in L$ with $a \wedge b = 0$.
- (ii) The CD-bases in a finite dp-distributive lattice \mathbb{L} have the same number of elements if and only if \mathbb{L} is graded.

An *interval system* (V, \mathcal{I}) is an algebraic closure system satisfying the axioms:

(I₀) $\{x\} \in \mathcal{I}$ for all $x \in V$, and $\emptyset \in \mathcal{I}$;

(I₁) $A, B \in \mathcal{I}$ and $A \cap B \neq \emptyset$ imply $A \cup B \in \mathcal{I}$;

(I₂) For any $A, B \in \mathcal{I}$ the relations $A \cap B \neq \emptyset$, $A \not\subseteq B$ and $B \not\subseteq A$ imply $A \setminus B \in \mathcal{I}$ (and $B \setminus A \in \mathcal{I}$).

The *modules* (*X-sets*, or *autonomous sets*) of an undirected graph $G = (V, E)$, the *intervals* of an n -ary relation $R \subseteq V^n$ on the set V for $n \geq 2$ – in particular, the usual intervals of a linearly ordered set (V, \leq) – form interval systems.

Generalizing interval systems

Let us consider now the condition:

(II) If $a \wedge b \neq 0$, then $(x \leq a \vee b \text{ and } x \wedge a = 0) \Rightarrow x \leq b$ for all $a, b, x \in L$.

Lattices with 0 satisfying condition (II) and with the property that $\uparrow a$ is a modular lattice for all $a \in L$, $a \neq 0$, can be considered as a generalization of the lattice (\mathcal{I}, \subseteq) of an interval system (V, \mathcal{I}) . To study their CD-bases, first we proved:

Lemma Let \mathbb{L} be an atomic lattice satisfying condition (II). Assume $D \in \mathcal{D}(\mathbb{L})$ and define $S_D = \{s \in L \setminus (D \cup \{0\}) \mid d \wedge s = 0 \text{ or } d < s, \text{ for all } d \in D\}$. Then for all $b, c \in S_D$ with $b \wedge c \neq 0$ and all $d \in D$, $d \wedge (b \vee c) \neq 0$ if and only if $0 < d < b$ or $0 < d < c$ holds.

Remark Let \mathbb{L} be a finite lattice and $D = \{d_j \mid j \in J\} \in \mathcal{DC}(\mathbb{L})$. If $D \prec D'$ for some $D' \in \mathcal{D}(\mathbb{L})$; then, in view of Lemma ??, there is a minimal element $a \in S_D$ such that

$D' = \{a\} \cup \{d_j \in D \setminus \{0\} \mid d_j \wedge a = 0\}$. In this case there exists a set $K \subseteq J$ such that

It is well-known that a finite lattice \mathbb{L} is semimodular if and only if it satisfies *Birkhoff's condition*, namely, for all $a, b \in L$

$$(Bi) \quad a \wedge b \prec a, b \text{ implies } a, b \prec a \vee b.$$

We also say that a pair $a, b \in L$ satisfies Birkhoff's condition if the above implication (Bi) is valid for a, b . It is known that any distributive pair $a, b \in L$ satisfies Birkhoff's condition.

Theorem Let \mathbb{L} be a finite lattice satisfying condition (II) such that any principal filter $\uparrow a$ with $a \in L \setminus \{0\}$ is a modular lattice. Then $\mathcal{DC}(\mathbb{L})$ is a semimodular lattice.

- Corollary** (i) If \mathbb{L} is a finite distributive lattice, then $\mathcal{DC}(\mathbb{L})$ is a semimodular lattice.
- (ii) If \mathbb{L} is a finite lattice that satisfies the conditions in Theorem ??, then its CD-bases have the same number of elements.

By applying this to interval systems we obtain:

Corollary

If (V, \mathcal{I}) is a finite interval system, then the CD-bases of the lattice (\mathcal{I}, \subseteq) contain the same number of elements.

$$U \in \mathcal{C} \subseteq \mathcal{K} \subseteq \mathcal{P}(U)$$

Let $h: U \rightarrow \mathbb{R}$ be a height function and let $S \in \mathcal{C}$ be a nonempty set.

We say that S is an *pre-island* with respect to the triple $(\mathcal{C}, \mathcal{K}, h)$, if every $K \in \mathcal{K}$ with $S \prec K$ satisfies

$$\min h(K) < \min h(S).$$

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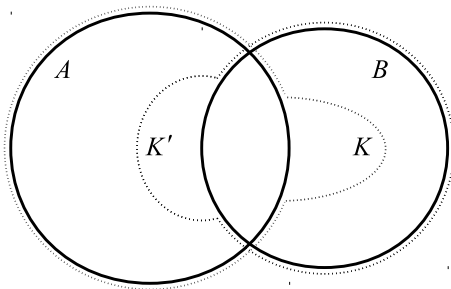
$$h(u) < \min h(S) \text{ for all } u \in K \setminus S.$$

Connective island domains

Definition

A pair $(\mathcal{C}, \mathcal{K})$ is an *connective island domain* if

$$\forall A, B \in \mathcal{C} : (A \cap B \neq \emptyset \text{ and } B \not\subseteq A) \implies \exists K \in \mathcal{K} : A \subset K \subseteq A \cup B.$$



Definition A family $\mathcal{H} \subseteq \mathcal{P}(U)$ is *weakly independent* if

$$H \subseteq \bigcup_{i \in I} H_i \implies \exists i \in I : H \subseteq H_i \quad (1)$$

holds for all $H \in \mathcal{H}, H_i \in \mathcal{H} (i \in I)$. If \mathcal{H} is both CD-independent and weakly independent, then we say that \mathcal{H} is *CDW-independent*.

Lemma

If $(\mathcal{C}, \mathcal{K})$ is a connective island domain, then every admissible subfamily of \mathcal{C} is CDW-independent. [[But not conversely.]]

Theorem

The following three conditions are equivalent for any pair $(\mathcal{C}, \mathcal{K})$:

(i) $(\mathcal{C}, \mathcal{K})$ is a connective island domain.

(ii) Every system of pre-islands corresponding to $(\mathcal{C}, \mathcal{K})$ is CD-independent.

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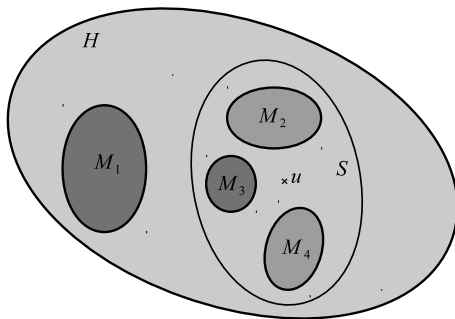
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(iii) Every system of pre-islands corresponding to $(\mathcal{C}, \mathcal{K})$ is CDW-independent.

Theorem

If $(\mathcal{C}, \mathcal{K})$ is a connective island domain, then a subfamily of \mathcal{C} is a system of pre-islands if and only if it is admissible.



Theorem

If $(\mathcal{C}, \mathcal{K})$ is a connective island domain and \mathcal{S} is a system of pre-islands corresponding to $(\mathcal{C}, \mathcal{K})$, then $|\mathcal{S}| \leq |U|$.

Proof.

Let $(\mathcal{C}, \mathcal{K})$ be a connective island domain and let $\mathcal{S} \subseteq \mathcal{C} \setminus \{\emptyset\}$ be a system of pre-islands corresponding to $(\mathcal{C}, \mathcal{K})$. \mathcal{S} is CDW-independent, and hence $\mathcal{S} \cup \{\emptyset\}$ is also CDW-independent. From the results of G. Czédli and E. T. Schmidt it follows that every maximal CDW-independent subset of $\mathcal{P}(U)$ has $|U| + 1$ elements. Thus we have $|\mathcal{S}| + 1 \leq |U| + 1$.

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