

Some enumerative and lattice theoretic aspects of islands (and lakes) and related investigations

Eszter K. Horváth, Szeged

Co-authors: János Barát, Péter Hajnal, Branimir Šešelja, Andreja Tepavčević, Sándor Radeleczki

Dresden, 2015, Jan 16 .

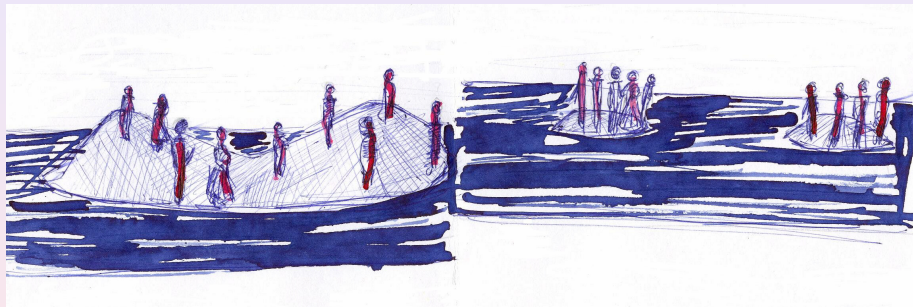
Islands? Alcatraz, berüchtigt



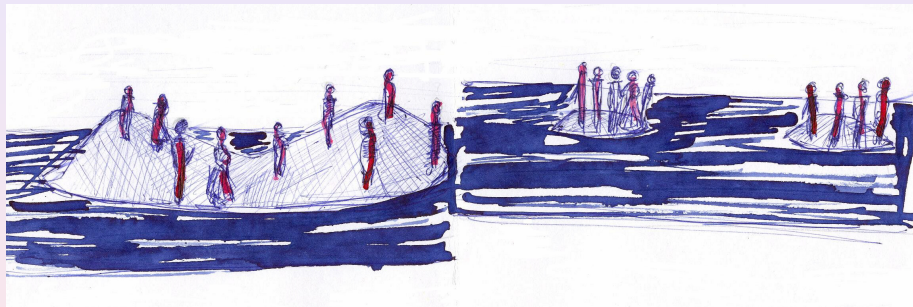
Islands?



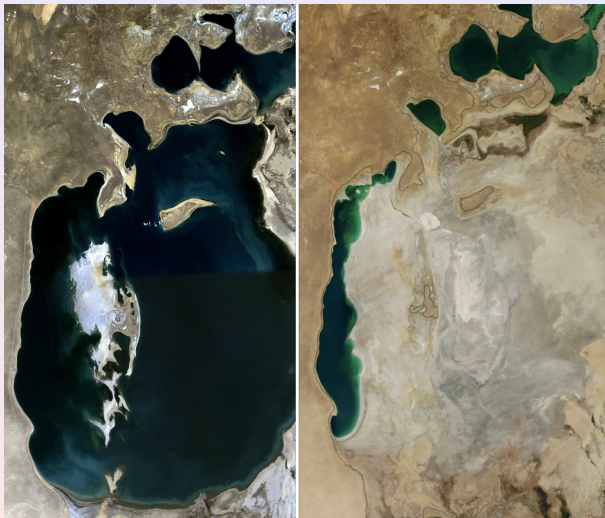
Islands?



Islands?



Lakes? (Aral sea, satellit photo)

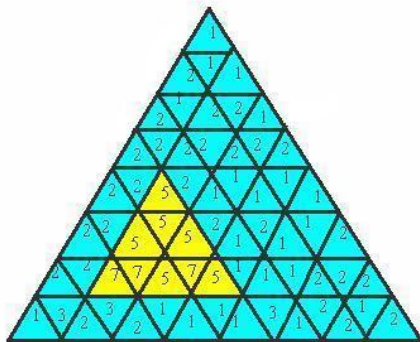
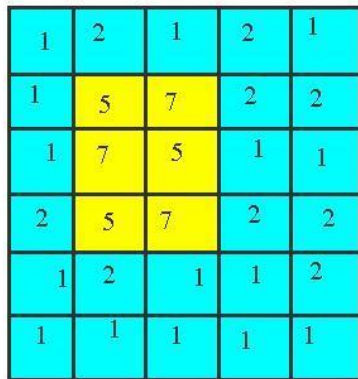


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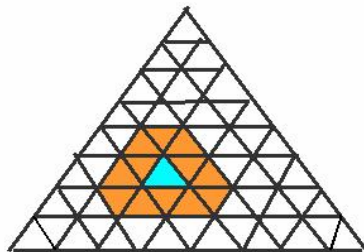
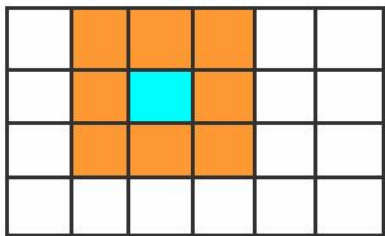
Definition

We call a rectangle/triangle a *rectangular/triangular island*, if for the cell t , if we denote its height by a_t , then for each cell \hat{t} neighbouring with a cell of the rectangle/triangle T , the inequality $a_{\hat{t}} < \min\{a_t : t \in T\}$ holds.



Definition

Grid, neighbourhood



The number of rectangular islands

We put heights into the cells.

How many rectangular islands do we have?

2	1	3	2
2	1	3	2
3	1	1	1

The number of rectangular islands

Water level: 0,5

Number of rectangular islands: 1

2	1	3	2
2	1	3	2
3	1	1	1

2	1	3	2
2	1	3	2
3	1	1	1

The number of rectangular islands

Water level: 1,5

Number of rectangular islands: 2

2	1	3	2
2	1	3	2
3	1	1	1

2	1	3	2
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The number of rectangular islands

Water level: 2,5

Number of rectangular islands: 2

2	1	3	2
2	1	3	2
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2	1	3	2
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The number of rectangular islands

Altogether: $1 + 2 + 2 = 5$ rectangular islands.

2	1	3	2
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2	1	3	2
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2	1	3	2
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Could we put more rectangular islands onto this grid? (With other heights?)

The number of rectangular islands

Yes, we could put more rectangular islands, here we have $1 + 2 + 4 + 2 = 9$ rectangular islands.

3	1	4	3
2	1	2	2
3	1	3	4

3	1	4	3
2	1	2	2
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3	1	4	3
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HOWEVER, WE CANNOT PUT MORE RECTANGULAR ISLANDS!!!

The maximum number of rectangular islands on the $m \times n$ size grid (Gábor Czédli , Szeged, 2007. june 17.)

$$f^{\text{cz}}(m, n) = \left\lceil \frac{mn + m + n - 1}{2} \right\rceil$$

Coding theory

S. Földes and N. M. Singhi: On instantaneous codes, J. of Combinatorics, Information and System Sci., 31 (2006), 317-326.

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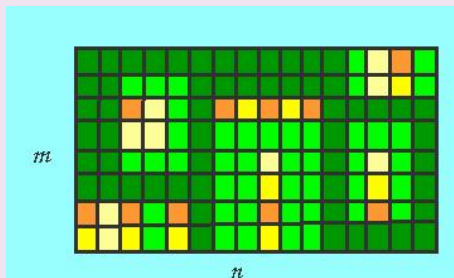
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Rectangular islands

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The maximum number of rectangular islands in a $m \times n$ rectangular board on square grid:

$$f^{cz}(m, n) = \left\lceil \frac{mn + m + n - 1}{2} \right\rceil.$$



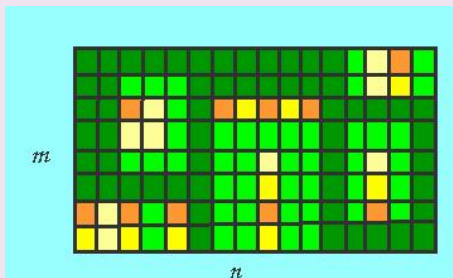
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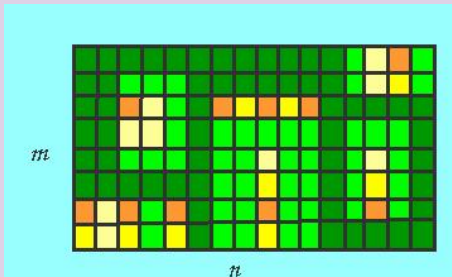
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Proving $f(m, n) = \left\lceil \frac{mn+m+n-1}{2} \right\rceil$

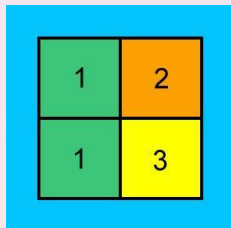
THERE EXISTS:

By induction on the number of the cells: $f(m, n) \geq \left\lceil \frac{mn+m+n-1}{2} \right\rceil$.

If $m = 1$, then $\left\lceil \frac{n+1+n-1}{2} \right\rceil = n$, we put the numbers $1, 2, 3, \dots, n$ in the cells and we will have exactly n islands.

If $n = 1$, then $\left\lceil \frac{m+m+1-1}{2} \right\rceil = m$.

If $m = n = 2$:



Proving $f(m, n) = \left\lceil \frac{mn+m+n-1}{2} \right\rceil$
THERE EXISTS:

Let $m, n > 2$.

$$\begin{aligned} f(m, n) &\geq f(m-2, n) + f(1, n) + 1 \geq \left\lceil \frac{(m-2)n + (m-2) + n - 1}{2} \right\rceil + \left\lceil \frac{n+1+n-1}{2} \right\rceil + 1 = \\ &= \left\lceil \frac{(m-2)n + (m-2) + n - 1 + 2n}{2} \right\rceil + 1 = \left\lceil \frac{mn+m+n-1}{2} \right\rceil. \end{aligned}$$

LATTICE METHOD

G. Czédli, A. P. Huhn and E. T. Schmidt: Weakly independent subsets in lattices, *Algebra Universalis* 20 (1985), 194-196.

Any two weak bases of a finite distributive lattice have the same number of elements.

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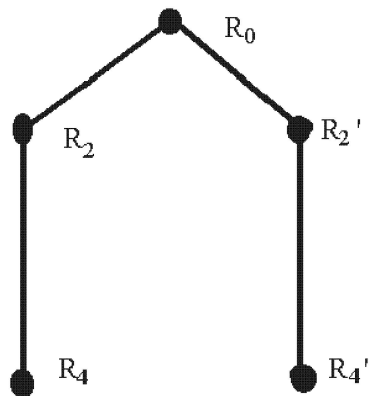
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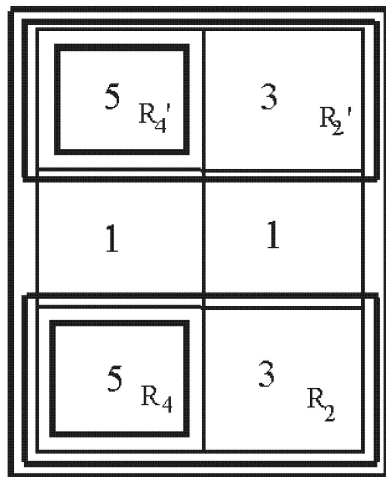
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TREE-GRAPH METHOD



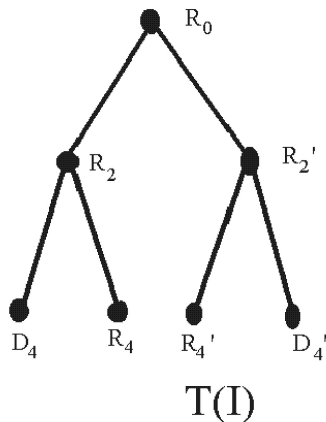
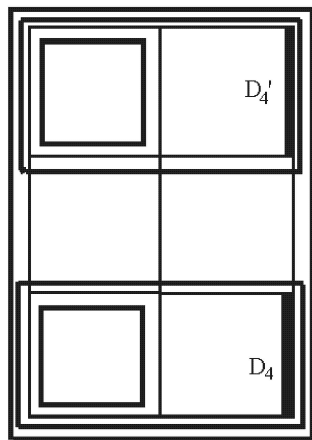
$T_0(I)$



R_0

T

TREE-GRAPH METHOD



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Lemma 2 (folklore)

Let T be a rooted tree such that any non-leaf node has at least 2 sons. Let ℓ be the number of leaves in T . Then $|V| \leq 2\ell - 1$.

We have $4s + 2d \leq (n + 1)(m + 1)$.

The number of leaves of $T(\mathcal{I})$ is $\ell = s + d$. Hence by Lemma 2 the number of islands is

$$|V| - d \leq (2\ell - 1) - d = 2s + d - 1 \leq \frac{1}{2}(n + 1)(m + 1) - 1.$$

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ELEMENTARY METHOD

We define

$$\mu(R) = \mu(u, v) := (u + 1)(v + 1).$$

Now

$$\begin{aligned} f(m, n) &= 1 + \sum_{R \in \max \mathcal{I}} f(R) = 1 + \sum_{R \in \max \mathcal{I}} \left(\left\lceil \frac{(u+1)(v+1)}{2} \right\rceil - 1 \right) \\ &= 1 + \sum_{R \in \max \mathcal{I}} \left(\left\lceil \frac{\mu(u, v)}{2} \right\rceil - 1 \right) \leq 1 - |\max \mathcal{I}| + \left\lceil \frac{\mu(C)}{2} \right\rceil. \end{aligned}$$

If $|\max \mathcal{I}| \geq 2$, then the proof is ready. Case $|\max \mathcal{I}| = 1$ is an easy exercise.

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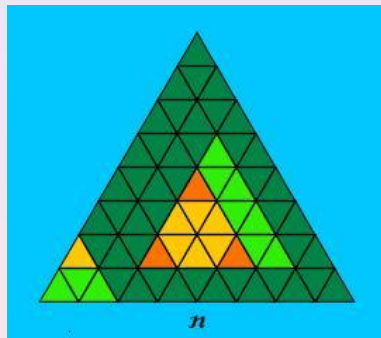
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Triangular islands

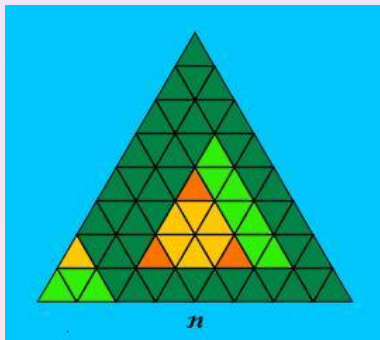
For the maximum number of triangular islands in an equilateral triangle of side length n , $\frac{n^2+3n}{5} \leq f(n) \leq \frac{3n^2+9n+2}{14}$ holds.



E. K. Horváth, Z. Németh and G. Pluhár: The number of triangular islands on a triangular grid, *Periodica Mathematica Hungarica*, 58 (2009), 25–34.

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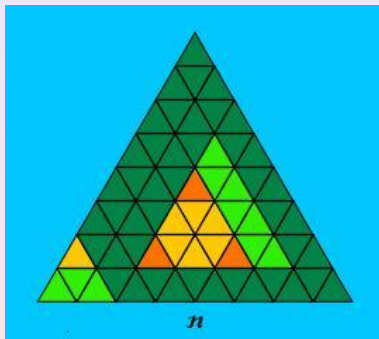
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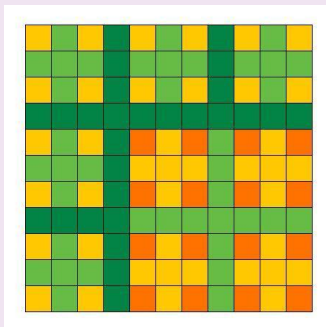
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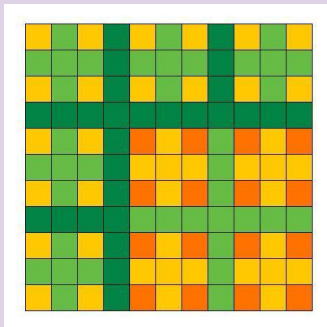
Square islands (also in higher dimensions)

$$\frac{1}{3}(rs - 2r - 2s) \leq f(r, s) \leq \frac{1}{3}(rs - 1)$$



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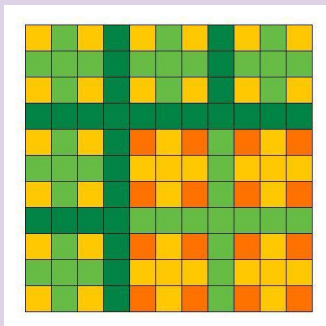
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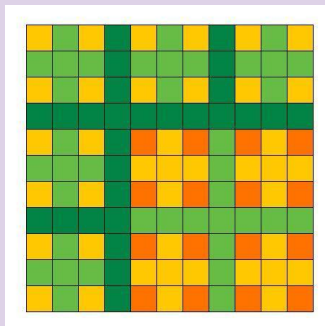
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Exact results

Cylindric board, rectangular islands (J. Barát, P. Hajnal, E.K. Horváth):
If $n \geq 2$, then $h_1(m, n) = \lfloor \frac{(m+1)n}{2} \rfloor$.

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Peninsulas (semi islands) (J. Barát, P. Hajnal, E.K. Horváth):
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Maximal systems of rectangular islands

Further results on rectangular islands

Zs. Lengvárszky: The minimum cardinality of maximal systems of rectangular islands, *European Journal of Combinatorics*, **30** (2009), 216-219.

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Islands in Boolean algebras, i.e. in hypercubes

The board consists of all vertices of a hypercube, i.e. the elements of a Boolean algebra $BA = \{0, 1\}^n$.

We consider two cells neighbouring if their Hamming distance is 1.

We denote the maximum number of islands in $BA = \{0, 1\}^n$ by $b(n)$.

Island formula for Boolean algebras (J. Barát, P. Hajnal, E.K. Horváth)
 $b(n) = 1 + 2^{n-1}$.

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High school competition exercise

Determine the maximum number of islands on n consecutive cells, if the possible heights on the grid are the following: $0, 1, 2, \dots, h$; where $h \geq 1$.

The solution:

$$I(n, h) = n - \left\lceil \frac{n}{2^h} \right\rceil.$$

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Rectangular height functions/1

Joint work with Branimir Šešelja and Andreja Tepavčević

A *height function* h is a mapping from $\{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ to \mathbb{N} , $h : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow \mathbb{N}$.

The co-domain of the height function is the lattice (\mathbb{N}, \leq) , where \mathbb{N} is the set of natural numbers under the usual ordering \leq and suprema and infima are max and min, respectively.

For every $p \in \mathbb{N}$, the *cut of the height function*, i.e. the p -cut of h is an ordinary relation h_p on $\{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ defined by

$$(x, y) \in h_p \text{ if and only if } h(x, y) \geq p.$$

Rectangular height functions/1

Joint work with Branimir Šešelja and Andreja Tepavčević

A *height function* h is a mapping from $\{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ to \mathbb{N} , $h : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow \mathbb{N}$.

The co-domain of the height function is the lattice (\mathbb{N}, \leq) , where \mathbb{N} is the set of natural numbers under the usual ordering \leq and suprema and infima are max and min, respectively.

For every $p \in \mathbb{N}$, the *cut of the height function*, i.e. the p -cut of h is an ordinary relation h_p on $\{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ defined by

$$(x, y) \in h_p \text{ if and only if } h(x, y) \geq p.$$

Rectangular height functions/2

We say that two rectangles $\{\alpha, \dots, \beta\} \times \{\gamma, \dots, \delta\}$ and $\{\alpha_1, \dots, \beta_1\} \times \{\gamma_1, \dots, \delta_1\}$ are *distant* if they are disjoint and for every two cells, namely (a, b) from the first rectangle and (c, d) from the second, we have $(a - c)^2 + (b - d)^2 \geq 4$.

The height function h is called *rectangular* if for every $p \in \mathbb{N}$, every nonempty p -cut of h is a union of distant rectangles.

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Rectangular height functions/3

5	5	3	5	5
4	4	2	4	4
2	2	1	2	2

$$\Gamma_1 = \{1, 2, 3, 4, 5\} \times \{1, 2, 3\},$$

$$\Gamma_2 = \{1, 2, 3, 4, 5\} \times \{1, 2, 3\} \setminus \{(3, 1)\},$$

$$\Gamma_3 = \{(1, 2), (1, 3), (2, 2), (2, 3), (3, 3), (4, 2), (4, 3), (5, 2), (5, 3)\},$$

$$\Gamma_4 = \{(1, 2), (1, 3), (2, 2), (2, 3), (4, 2), (4, 3), (5, 2), (5, 3)\} \text{ and}$$

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Rectangular height functions/4

CHARACTERIZATION THEOREM

Theorem 1

A height function $h_{\mathbb{N}} : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow \mathbb{N}$ is rectangular if and only if for all $(\alpha, \gamma), (\beta, \delta) \in \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ either

- these are not neighboring cells and there is a cell (μ, ν) between (α, γ) and (β, δ) such that $h_{\mathbb{N}}(\mu, \nu) < \min\{h_{\mathbb{N}}(\alpha, \gamma), h_{\mathbb{N}}(\beta, \delta)\}$, or
- for all $(\mu, \nu) \in [\min\{\alpha, \beta\}, \max\{\alpha, \beta\}] \times [\min\{\gamma, \delta\}, \max\{\gamma, \delta\}]$,

$$h_{\mathbb{N}}(\mu, \nu) \geq \min\{h_{\mathbb{N}}(\alpha, \gamma), h_{\mathbb{N}}(\beta, \delta)\}.$$

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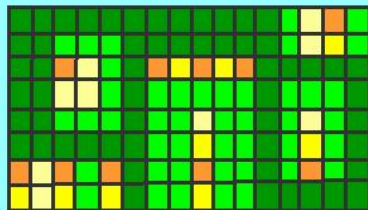
$$h_{\mathbb{N}}(\mu, \nu) \geq \min\{h_{\mathbb{N}}(\alpha, \gamma), h_{\mathbb{N}}(\beta, \delta)\}.$$

Rectangular height functions/5

Theorem 2

For every height function $h : \{1, 2, \dots, n\} \times \{1, 2, \dots, m\} \rightarrow \mathbb{N}$, there is a rectangular height function $h^* : \{1, 2, \dots, n\} \times \{1, 2, \dots, m\} \rightarrow \mathbb{N}$, such that $\mathcal{I}_{rect}(h) = \mathcal{I}_{rect}(h^*)$.

m



n

Rectangular height functions/6

CONSTRUCTING ALGORITHM

1. FOR $i = t$ TO 0
2. FOR $y = 1$ TO n
3. FOR $x = 1$ TO m
4. IF $h(x, y) = a_i$ THEN
5. $j := i$
6. WHILE there is no island of h which is a subset of h_{a_j} that contains (x, y) DO $j := j - 1$
7. ENDWHILE
8. Let $h^*(x, y) := a_j$.
9. ENDIF
10. NEXT x
11. NEXT y
12. NEXT i
13. END.

Rectangular height functions/7

LATTICE-VALUED REPRESENTATION

Theorem 3

Let $h : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow \mathbb{N}$ be a rectangular height function. Then there is a lattice L and an L -valued mapping Φ , such that the cuts of Φ are precisely all islands of h .

Rectangular height functions/8

Let $h : \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4\} \rightarrow \mathbb{N}$ be a height function.

4	9	8	7	1	5
3	8	8	7	1	4
2	7	7	7	1	5
1	2	2	2	1	6
	1	2	3	4	5

Rectangular height functions/9

h is a rectangular height function. Its islands are:

$$I_1 = \{(1, 4)\},$$

$$I_2 = \{(1, 3), (1, 4), (2, 3), (2, 4)\},$$

$$I_3 = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\},$$

$$I_4 = \{(5, 1)\},$$

$$I_5 = \{(5, 1), (5, 2)\},$$

$$I_6 = \{(5, 4)\},$$

$$I_7 = \{(5, 1), (5, 2), (5, 3), (5, 4)\},$$

$$I_8 = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4), (1, 1), (2, 1), (3, 1)\},$$

$$I_9 = \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4\}.$$

Rectangular height functions/10

Its cut relations are:

$$h_{10} = \emptyset$$

$$h_9 = I_1 \text{ (one-element island)}$$

$$h_8 = I_2 \text{ (four-element square island)}$$

$$h_7 = I_3 \text{ (nine-element square island)}$$

$$h_6 = I_3 \cup I_4 \text{ (this cut is a disjoint union of two islands)}$$

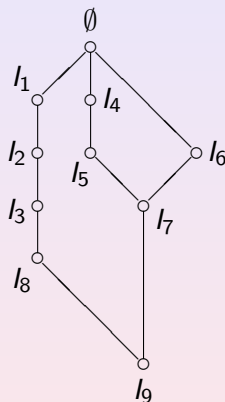
$$h_5 = I_3 \cup I_5 \cup I_6 \text{ (union of three islands)}$$

$$h_4 = I_3 \cup I_7 \text{ (union of two islands)}$$

$$h_2 = I_7 \cup I_8 \text{ (union of two islands)}$$

$$h_1 = \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4\} = I_9 \text{ (the whole domain)}$$

Rectangular height functions/11



$$L = (\mathcal{I}_0(\Gamma), \supseteq)$$

Theorem 4

For every rectangular height function

$$h^* : \{1, 2, \dots, n\} \times \{1, 2, \dots, m\} \rightarrow \mathbb{N},$$

there is a rectangular height function

$$h^{**} : \{1, 2, \dots, n\} \times \{1, 2, \dots, m\} \rightarrow \mathbb{N},$$

such that $\mathcal{I}_{rect}(h^*) = \mathcal{I}_{rect}(h^{**})$ and in h^{**} every island appears exactly in one cut.

If a rectangular height function h^{**} has the property that each island appears exactly in one cut, then we call it *standard rectangular height function*.

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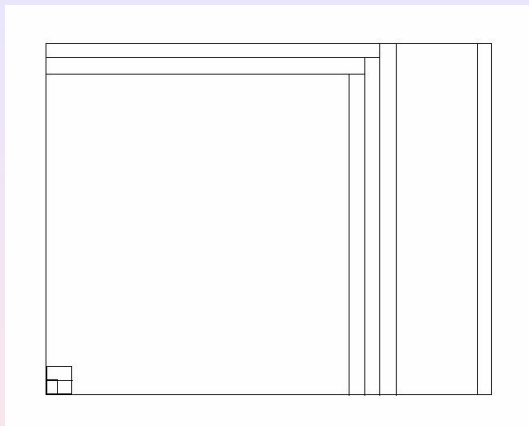
We denote by $\Lambda_{\max}(m, n)$ the maximum number of different nonempty p -cuts of a standard rectangular height function on the rectangular table of size $m \times n$.

Theorem 5 $\Lambda_{\max}(m, n) = m + n - 1$.

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Rectangular height functions/14



The maximum number of different nonempty p -cuts of a standard rectangular height function is equal to the minimum cardinality of maximal systems of islands.

Lemma 1

If $m \geq 3$ and $n \geq 3$ and a height function $h : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow \mathbb{N}$ has $f^{\text{cz}}(m, n)$ many rectangular islands (that is, maximally many), then it has exactly two maximal rectangular islands.

Lemma 2

If $m \geq 3$ or $n \geq 3$, then for any odd number $t = 2k + 1$ with $1 \leq t \leq \max\{m - 2, n - 2\}$, there is a standard rectangular height function $h : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow \mathbb{N}$ having the maximum number of islands $f(m, n)$, such that one of the side-lengths of one of the maximal islands is equal to t .

(Remark: The statement is not true for even side-lengths, one can construct counterexample easily!)

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Rectangular height functions/16

We denote by $\Lambda_h^{\text{cz}}(m, n)$ the number of different nonempty cuts of a standard rectangular height function h in the case h has maximally many islands, i.e., when the number of islands is

$$f^{\text{cz}}(m, n) = \left\lfloor \frac{mn + m + n - 1}{2} \right\rfloor.$$

Theorem 6

Let $h : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow \mathbb{N}$ be a standard rectangular height function having maximally many islands $f^{\text{cz}}(m, n)$. Then,

$$\Lambda_h^{\text{cz}}(m, n) \geq \lceil \log_2(m+1) \rceil + \lceil \log_2(n+1) \rceil - 1.$$

Furthermore, there exists a standard rectangular height function with maximally many rectangular islands such that number of different nonempty cuts is exactly $\lceil \log_2(m+1) \rceil + \lceil \log_2(n+1) \rceil - 1$.

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Proposition

If $m, n \geq 2$, then $\Lambda_h^{cz}(m, n) \leq \left\lfloor \frac{(m+n+3)}{2} \right\rfloor$. Furthermore, for $m, n \geq 3$ this bound is sharp.

Remark

If $n = 1$, then $\Lambda_h^{cz}(m, n) \leq m$ (similarly for $m = 1$).

CD-independent subsets in distributive lattices

G. Czédli, M. Hartmann and E. T. Schmidt: CD-independent subsets in distributive lattices, Publicationes Mathematicae Debrecen, 74/1-2 (2009).

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If all finite lattices in a lattice variety have this property, then the variety must coincide with the variety of distributive lattices.

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CD-independent subsets in posets

Let $\mathbb{P} = (P, \leq)$ be a partially ordered set, and let $a, b \in P$. The elements a and b are called *disjoint* and we write $a \perp b$ if

either \mathbb{P} has least element $0 \in P$ and $\inf\{a, b\} = 0$,
or \mathbb{P} is without 0 and the elements a and b have no common lowerbound.

A nonempty set $X \subseteq P$ is called *CD-independent* if for any $x, y \in X$, $x \leq y$ or $y \leq x$, or $x \perp y$ holds. Maximal CD-independent sets (with respect to \subseteq) are called *CD-bases* in \mathbb{P} .

Definition

A nonempty set D of nonzero elements of P is called a *set of pairwise disjoint element* in \mathbb{P} if $x \perp y$ holds for all $x, y \in D$, $x \neq y$; if \mathbb{P} has 0-element, then $\{0\}$ is considered to be a set of pairwise disjoint elements, too.

Remark

- D is a set of pairwise disjoint elements if and only if it is a CD-independent antichain in \mathbb{P} .

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Order ideals

Any antichain $A = \{a_i \mid i \in I\}$ of a poset \mathbb{P} determines a unique order-ideal $I(A)$ of \mathbb{P} :

$$I(A) = \bigcup_{i \in I} (a_i] = \{x \in P \mid x \leq a_i, \text{ for some } i \in I\},$$

where $(a]$ stands for the principal ideal of an element $a \in P$.

Definition

If A_1, A_2 are antichains in \mathbb{P} , then we say that A_1 is dominated by A_2 , and we denote it by $A_1 \leq A_2$ if

$$I(A_1) \subseteq I(A_2).$$

Remarks

- \leq is a partial order
- $A_1 \leq A_2$ is satisfied if and only if

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Let $\mathcal{D}(\mathbb{P})$ denote the set of all sets of pairwise disjoint elements of \mathbb{P} .
As sets of pairwise disjoint elements of \mathbb{P} are also antichains, restricting \leq to $\mathcal{D}(\mathbb{P})$, we obtain a poset $(\mathcal{D}(\mathbb{P}), \leq)$.
The connection between CD-bases of a poset \mathbb{P} and the poset $(\mathcal{D}(\mathbb{P}), \leq)$ is shown by the next theorem:

Theorem

Let B be a CD-base of a finite poset (P, \leq) , and let $|B| = n$.

Then there exists a maximal chain $\{D_i\}_{1 \leq i \leq n}$ in $\mathcal{D}(P)$ such that

$$B = \bigcup_{i=1}^n D_i.$$

Moreover, for any maximal chain $\{D_i\}_{1 \leq i \leq m}$ in $\mathcal{D}(P)$ the set $D = \bigcup_{i=1}^m D_i$ is a CD-base in (P, \leq) with $|D| = m$.

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Graded posets

The poset \mathbb{P} is called *graded*, if all its maximal chains have the same cardinality.

Let $\mathbb{P} = (P, \leq)$ be a finite poset with 0. Then the following conditions are equivalent:

(i) *The CD-bases of \mathbb{P} have the same number of elements,*

(ii) *$\mathcal{D}(P)$ is graded.*

A set of pairwise disjoint elements D of a poset (P, \leq) is called *complete*, if there is no $p \in P \setminus D$ such that $D \cup \{p\}$ is also a set of pairwise disjoint elements.

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If \mathbb{P} is a finite bounded poset

If all the principal ideals $(a]$ of \mathbb{P} are weakly 0-modular, then $A(P) \cup C$ is a CD-base for every maximal chain C in \mathbb{P} .

If \mathbb{P} has weakly 0-modular principal ideals and $\mathcal{D}(P)$ is graded, then \mathbb{P} is also graded, and any CD-base of \mathbb{P} contains $|A(P)| + l(P)$ elements.

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Weakly 0-modular lattices

A poset with least element 0 and greatest element 1 is called *bounded*.

A lattice L with 0 is called *0-modular* if for any $a, b, c \in L$

$$a \leq b \text{ and } b \wedge c = 0 \text{ imply } b \wedge (a \vee c) = a \quad (M_0)$$

Equivalently, L has no pentagon sublattice N_5 that contains $0 = 0_L$.

If (M_0) is satisfied under the assumptions that a is an atom and $c \prec b \vee c$, then L is called *weakly 0-modular*.

The lattice L is *lower-semimodular* if for any $a, b, c \in L$, $b \prec c$ implies $a \wedge b \preceq a \wedge c$.

It is easy to see that any lower-semimodular lattice and any 0-modular lattice is weakly 0-modular.

We say that a poset \mathbb{P} with 0 is *weakly 0-modular* if the above weak form of (M_0) holds whenever $\sup\{a, c\}$ and $\sup\{b, c\}$ exist in \mathbb{P} .

Lemma

Let \mathbb{P} be a poset with 0 and D_k , $k \in K$ ($K \neq \emptyset$) disjoint sets in \mathbb{P} . If the meet $\bigwedge_{k \in K} a^{(k)}$ of any system of elements $a^{(k)} \in D_k$, $k \in K$ exist in \mathbb{P} , then $\bigwedge_{k \in K} D_k$ also exists in $\mathcal{D}(P)$.

A pair $a, b \in P$ with least upperbound $a \vee b$ in \mathbb{P} is called a *distributive pair*, if $(c \wedge a) \vee (c \wedge b)$ exists in \mathbb{P} for any $c \in P$, and $c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$.

We say that (P, \wedge) is *dp-distributive*, if any $a, b \in P$ with $a \wedge b = 0$ is a distributive pair.

Theorem

(i) If $\mathbb{P} = (P, \wedge)$ is a semilattice with 0, then $\mathcal{D}(P)$ is a dp-distributive semilattice; if $D_1 \cup D_2$ is a CD-independent set for some $D_1, D_2 \in \mathcal{D}(P)$, then D_1, D_2 is a distributive pair in $\mathcal{D}(P)$.

(ii) If \mathbb{P} is a complete lattice, then $\mathcal{D}(P)$ is a dp-distributive complete lattice.

A pair $a, b \in P$ with least upperbound $a \vee b$ in \mathbb{P} is called a *distributive pair*, if $(c \wedge a) \vee (c \wedge b)$ exists in \mathbb{P} for any $c \in P$, and $c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$.

We say that (P, \wedge) is *dp-distributive*, if any $a, b \in P$ with $a \wedge b = 0$ is a distributive pair.

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(ii) If \mathbb{P} is a complete lattice, then $\mathcal{D}(P)$ is a dp-distributive complete lattice.

Let (P, \leq) be a poset and $A \subseteq P$. (A, \leq) is called a *sublattice* of (P, \leq) , if (A, \leq) is a lattice such that for any $a, b \in A$ the infimum and the supremum of $\{a, b\}$ is the same in the subposet (A, \leq) and in (P, \leq) . If the relation $x \prec y$ in (A, \leq) for some $x, y \in A$ implies $x \prec y$ in the poset (P, \leq) , then we say that (A, \leq) is a *cover-preserving subposet* of (P, \leq) .

Theorem

Let $\mathbb{P} = (P, \leq)$ be a poset with 0 and B a CD-base of it. Then $(\mathcal{D}(B), \leq)$ is a distributive cover-preserving sublattice of the poset $(\mathcal{D}(P), \leq)$. If \mathbb{P} is a \wedge -semilattice, then for any $D \in \mathcal{D}(P)$ and $D_1, D_2 \in \mathcal{D}(B)$ we have $(D_1 \vee D_2) \wedge D = (D_1 \wedge D) \vee (D_2 \wedge D)$ in $(\mathcal{D}(P), \leq)$.

Lemma

Let L be a finite weakly 0-distributive lattice and D a dual atom in $\mathcal{D}(L)$. Then either $D = \{d\}$, for some $d \in L$ with $d \prec 1$, or D consist of two different elements $d_1, d_2 \in L$ and $d_1 \vee d_2 = 1$.

Theorem

Let L be a finite, weakly 0-distributive lattice. Then the following are equivalent:

- (i) L is graded, and $l(a) + l(b) = l(a \vee b)$ holds for all $a, b \in L$ with $a \wedge b = 0$.*

CD-bases in particular lattice classes

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- (i) L is graded, and $l(a) + l(b) = l(a \vee b)$ holds for all $a, b \in L$ with $a \wedge b = 0$.*
- (ii) L is 0-modular, and the CD-bases of L have the same number of elements.*

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An *interval system* (V, \mathcal{I}) is an algebraic closure system satisfying the axioms:

(I₀) $\{x\} \in \mathcal{I}$ for all $x \in V$, and $\emptyset \in \mathcal{I}$;

(I₁) $A, B \in \mathcal{I}$ and $A \cap B \neq \emptyset$ imply $A \cup B \in \mathcal{I}$;

(I₂) For any $A, B \in \mathcal{I}$ the relations $A \cap B \neq \emptyset$, $A \not\subseteq B$ and $B \not\subseteq A$ imply $A \setminus B \in \mathcal{I}$ (and $B \setminus A \in \mathcal{I}$).

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