Lattice-valued functions

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Lattice-valued functions

Let S be a nonempty set and L a complete lattice. Every mapping $\mu : S \to L$ is called a **lattice-valued** (L-**valued**) **function** on S.

Cut set (p-cut)

Let $p \in L$. A **cut set** of an L-valued function $\mu : S \to L$ (a p-cut) is a subset $\mu_p \subseteq S$ defined by:

$$x \in \mu_p \text{ if and only if } \mu(x) \geq p. \tag{1}$$

In other words, a p-cut of $\mu : S \to L$ is the inverse image of the principal filter $\uparrow p$, generated by $p \in L$:

$$\mu_p = \mu^{-1}(\uparrow p). \tag{2}$$

It is obvious that for every $p, q \in L$, $p \leq q$ implies $\mu_q \subseteq \mu_p$.

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Cuts and closure systems

If $\mu : S \to L$ is an L-valued function on S, then the collection μ_L of all cuts of μ is a closure system on S under the set-inclusion.

Let $\mathcal{F}$ be a closure system on a set S. Then there is a lattice L and an L-valued function $\mu : S \to L$, such that the collection μ_L of cuts of μ is $\mathcal{F}$.

A required lattice L is the collection $\mathcal{F}$ ordered by the *reversed-inclusion*, and that $\mu : S \to L$ can be defined as follows:

$$\mu(x) = \bigcap\{f \in \mathcal{F} \mid x \in f\}. \tag{3}$$

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The relation $\approx$ on L

Given an L-valued function $\mu : S \to L$, we define the relation $\approx$ on L: for $p, q \in L$

$$p \approx q \ \text{ if and only if } \ \mu_p = \mu_q. \tag{4}$$

The relation $\approx$ is an equivalence on L, and

$$p \approx q \ \text{ if and only if } \ \uparrow p \cap \mu(S) = \uparrow q \cap \mu(S), \tag{5}$$

where $\mu(S) = \{r \in L \mid r = \mu(x) \ \text{for some} \ x \in S\}$.

We denote by $L/\approx$ the collection of equivalence classes under $\approx$.

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The collection of cuts

Let $(\mu_L, \leq)$ be the poset with $\mu_L = \{\mu_p \mid p \in L\}$ (the collection of cuts of μ) and the order $\leq$ being the inverse of the set-inclusion: for $\mu_p, \mu_q \in \mu_L$,

$$\mu_p \leq \mu_q \text{ if and only if } \mu_q \subseteq \mu_p.$$

$(\mu_L, \leq)$ is a complete lattice and for every collection $\{\mu_p \mid p \in L_1\}$, $L_1 \subseteq L$ of cuts of μ, we have

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The quotient $L/\approx$

Each $\approx$-class contains its supremum:

$$\bigvee [p]_\approx \in [p]_\approx . \tag{7}$$

The mapping $p \mapsto \bigvee [p]_\approx$ is a closure operator on L.

The quotient $L/\approx$ can be ordered by the relation $\leq_{L/\approx}$ defined as follows:

$$[p]_\approx \leq_{L/\approx} [q]_\approx \text{ if and only if } \uparrow q \cap \mu(S) \subseteq \uparrow p \cap \mu(S).$$

The order $\leq_{L/\approx}$ of classes in $L/\approx$ corresponds to the order of suprema of classes in L (we denote the order in L by $\leq_L$):

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(i) $[p]_\approx \leq_{L/\approx} [q]_\approx$ *if and only if* $\bigvee [p]_\approx \leq_L \bigvee [q]_\approx$.

(ii) *The mapping* $[p]_\approx \mapsto \bigvee [p]_\approx$ *is an injection of* $L/\approx$ *into* L.

The sub-poset $(\bigvee [p]_\approx, \leq_L)$ *of* L *is isomorphic to the lattice* $(L/\approx, \leq_{L/\approx})$ *under* $\bigvee [p]_\approx \mapsto [p]_\approx$.

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Canonical representation of lattice-valued functions

We take the lattice $(\mathcal{F}, \leq)$, where $\mathcal{F} = \mu_L \subseteq \mathcal{P}(S)$ is the collection of cuts of μ, and the order $\leq$ is the dual of the set inclusion.

Let $\widehat{\mu} : S \to \mathcal{F}$, where

$$\widehat{\mu}(x) := \bigcap \{\mu_p \in \mu_L \mid x \in \mu_p\}. \qquad (8)$$

Properties:

$\widehat{\mu}$ has the same cuts as μ.

$\widehat{\mu}$ has one-element classes of the equivalence relation $\approx$.

Every $f \in \mathcal{F}$ is equal to the corresponding cut of $\widehat{\mu}$.

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Canonical representation of $\mu : S \to L$

By the definition, every element of the codomain lattice of $\widehat{\mu}$ is a cut of μ. Therefore, if $f \in \mathcal{F}$, then $f = \mu_p$ for some $p \in L$, and for the cut $\widehat{\mu}_f$ of $\widehat{\mu}$, by the definition of a cut and by (8), we have

$$\begin{aligned} \widehat{\mu}_f &= \{x \in S \mid \widehat{\mu}(x) \geq f\} = \{x \in S \mid \widehat{\mu}(x) \subseteq \mu_p\} \\ &= \{x \in S \mid \bigcap\{\mu_q \mid x \in \mu_q\} \subseteq \mu_p\} = \mu_p = f. \end{aligned}$$

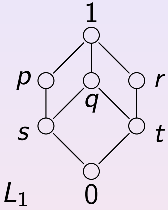
Therefore, the collection of cuts of $\widehat{\mu}$ is

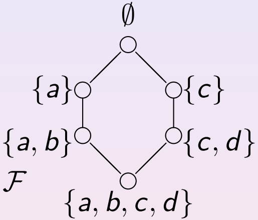
$$\widehat{\mu}_{\mathcal{F}} = \{Y \subseteq S \mid Y = \widehat{\mu}_{\mu_p}, \text{ for some } \mu_p \in \mu_L\}.$$

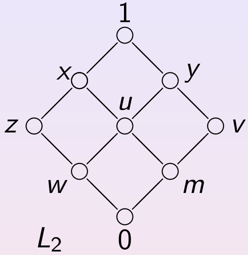
The lattices of cuts of a lattice-valued function μ and of its canonical representation $\widehat{\mu}$ coincide.

Example

$S = \{a, b, c, d\}$







$$\mu = \begin{pmatrix} a & b & c & d \\ p & s & r & t \end{pmatrix} \qquad \nu = \begin{pmatrix} a & b & c & d \\ z & w & m & v \end{pmatrix}$$

$$\widehat{\mu} = \widehat{\nu} = \begin{pmatrix} a & b & c & d \\ \{a\} & \{a, b\} & \{c\} & \{c, d\} \end{pmatrix}$$

Lattice-valued Boolean functions

A **Boolean function** is a mapping $f : \{0,1\}^n \to \{0,1\}$, $n \in \mathbb{N}$.

A **lattice-valued Boolean function** is a mapping

$$f : \{0,1\}^n \to L,$$

where L is a complete lattice.

We also deal with **lattice-valued n-variable functions** on a finite domain $\{0,1,\dots,k-1\}$:

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We use also p-**cuts** of lattice-valued functions as characteristic functions: for $f : \{0,1,\dots,k-1\}^n \to L$ and $p \in L$, we have

$$f_p : \{0,1,\dots,k-1\}^n \to \{0,1\},$$

such that $f_p(x_1,\dots,x_n) = 1$ if and only if $f(x_1,\dots,x_n) \geq p$.

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Invariance group

As usual, by S_n we denote the symmetric group of all permutations over an n-element set. If f is an n-variable function on a finite domain X and $\sigma \in S_n$, then f is **invariant** under σ, symbolically $\sigma \vdash f$, if for all $(x_1, \ldots, x_n) \in X^n$

$$f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

If f is invariant under all permutations in $G \leq S_n$ and not invariant under any permutation from $S_n \setminus G$, then G is called **the invariance group** of f, and it is denoted by $G(f)$.

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Representability

A group $G \leq S_n$ is said to be (k,m)-*representable* if there is a function $f : \{0,1,\dots,k-1\}^n \to \{1,\dots,m\}$ whose invariance group is G.

If G is the invariance group of a function $f : \{0,1,\dots,k-1\}^n \to \mathbb{N}$, then it is called (k,∞)-*representable*.

$G \leq S_n$ is called *m-representable* if it is the invariance group of a function $f : \{0,1\}^n \to \{1,\dots,m\}$;

it is called *representable* if it is m-representable for some $m \in \mathbb{N}$.

By the above, representability is equivalent with $(2,\infty)$-representability.

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A Galois connection for invariance groups

Let $O_k^{(n)} = \{f \mid f\colon \mathbf{k}^n \to \mathbf{k}\}$ denote the set of all n-ary operations on $\mathbf{k}$, and for $F \subseteq O_k^{(n)}$ and $G \subseteq S_n$ let

$$F^{\vdash} := \{\sigma \in S_n \mid \forall f \in F : \sigma \vdash f\}, \qquad \overline{F}^{(k)} := (F^{\vdash})^{\vdash},$$
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The assignment $G \mapsto \overline{G}^{(k)}$ is a closure operator on S_n, and it is easy to see that $\overline{G}^{(k)}$ is a subgroup of S_n for every subset $G \subseteq S_n$ (even if G is not a group). For $G \leq S_n$, we call $\overline{G}^{(k)}$ the *Galois closure of G over* $\mathbf{k}$, and we say that G is *Galois closed over* $\mathbf{k}$ if $\overline{G}^{(k)} = G$.

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$$\overline{G}^{(k)} = \bigcap_{a \in \mathbf{k}^n} (S_n)_a \cdot G.$$

For arbitrary $k, n \geq 2$, characterize those subgroups of S_n that are Galois closed over $\mathbf{k}$.

Theorem (H., Makay, Pöschel, Waldhauser) Let $n > \max\left(2^d, d^2 + d\right)$ and $G \leq S_n$. Then G is not Galois closed over $\mathbf{k}$ if and only if $G = A_B \times L$ or $G <_{\mathrm{sd}} S_B \times L$, where $B \subseteq \mathbf{n}$ is such that $D := \mathbf{n} \setminus B$ has less than d elements, and L is an arbitrary permutation group on D.

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Cuts of composition of functions

Theorem Let L be a complete lattice, let $A \neq \emptyset$ be a set and let $\sigma : A \to A$, $\mu : A \to L$, $\psi : L \to L$. Then, for every $p \in L$,

$$(\sigma \circ \mu \circ \psi)_p = \sigma \circ \mu \circ \psi_p.$$

Corollary Let L be a complete lattice, let $A \neq \emptyset$ and let $\mu : A \to L$. Then the following holds.

(i) $\mu_p = \mu \circ (\mathcal{I}_L)_p$, where $\mathcal{I}_L$ is the identity mapping $\mathcal{I}_L : L \to L$.

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Proposition Let $f : \{0, \dots, k-1\}^n \to L$ and $\sigma \in S_n$. Then

$$\sigma \vdash f \text{ if and only if for every } p \in L,\ \sigma \vdash f_p.$$

The invariance group of a lattice-valued function f depends only on the canonical representation of f.

If $f_1 : \{0, \dots, k-1\}^n \to L_1$ and $f_2 : \{0, \dots, k-1\}^n \to L_2$ are two n-variable lattice-valued functions on the same domain, then $\widehat{f_1} = \widehat{f_2}$ implies $G(f_1) = G(f_2)$.

Invariance groups of lattice-valued functions via cuts

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Representation theorem

For every $n \in \mathbb{N}$, there is a lattice L and a lattice valued Boolean function $F : \{0,1\}^n \to L$ satisfying the following: If $G \leq S_n$ and $G = G(f)$ for a Boolean function f, then $G = G(F_p)$, for a cut F_p of F.

Representation theorem on the k-element set

Every subgroups of S_n is an invariance group of a function $\{0, \ldots, k-1\}^n \to \{0, \ldots, k-1\}$ if and only if $k \geq n$.

If $k \geq n$, then for every subgroup G of S_n there exists a function $f : \{0, \ldots, k-1\}^n \to \{0, 1\}$ such that the invariance group of f is exactly G.

For $k, n \in \mathbb{N}$ and $k \geq n$, there is a lattice L and a lattice valued function $F : \{0, \ldots, k-1\}^n \to L$ such that the following holds: If $G \leq S_n$, then $G = G(F_p)$ for a cut F_p of of F.

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