# Isotone lattice-valued Boolean functions and cuts 

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This paper is dedicated to Professor László Leindler on his 80th birthday

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#### Abstract

For an integer $n \geq 1$, an $n$-ary lattice-valued Boolean function is a map from the $n$-th direct power of the 2 -element Boolean lattice to a bounded lattice. In terms of closure systems and cuts, we characterize lattice-valued Boolean functions that can be given by linear combinations of elements of the co-domain lattice.


## 1. Introduction

### 1.1. Target

A lattice-valued function is a map $\mu:\{0,1\}^{n} \rightarrow L$ where $L$ is a bounded lattice and $n \in\{1,2,3, \ldots\}$. We always assume that 0 and 1 in the domain $\{0,1\}$ of $\mu$ belong to $L$ as its least and greatest elements, respectively. Throughout the paper, the Boolean lattice $\{0,1\}^{n}$ will be denoted by $B_{n}$. To emphasize that (1.1) below is analogous to a linear combination, we often write $u v$ or $u \cdot v$ instead of $u \wedge v$ for elements of $L$. We say that $\mu$ can be given by a linear combination (in $L$ ) if there are $w_{1}, \ldots, w_{n} \in L$ such that, for all $x=\left\langle x_{1}, \ldots, x_{n}\right\rangle \in B_{n}$,

$$
\begin{equation*}
\mu(x)=\bigvee_{i=1}^{n} w_{i} x_{i}, \quad \text { that is, } \quad \mu(x)=\bigvee_{i=1}^{n}\left(w_{i} \wedge x_{i}\right) \tag{1.1}
\end{equation*}
$$

Our goal is to characterize lattice-valued functions that can be given as in (1.1) by means of the following concepts. For $p \in L$, the set

$$
\begin{equation*}
\mu_{p}:=\left\{x \in B_{n}: \mu(x) \geq p\right\} \tag{1.2}
\end{equation*}
$$

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is called a cut of $\mu$. As usual, a closure system $\mathcal{F}$ over $B_{n}$ is a $\cap$-subsemilattice of the powerset lattice $P\left(B_{n}\right)=\left\langle P\left(B_{n}\right) ; \cup, \cap\right\rangle$ such that $B_{n} \in \mathcal{F}$. By finiteness, $\mathcal{F}$ is necessarily a complete $\cap$-semilattice. (The reader should not confuse $\mathcal{F}$ with the more abstract concept of closure systems as $\{\wedge, 1\}$-subsemilattices of $\left\langle B_{n} ; \vee, \wedge\right\rangle$, which do not occur in the paper.) A closure system $\mathcal{F}$ determines a closure operator in the standard way. In this paper, we only need the closures of singleton sets and we write $\bar{x}$ rather then $\overline{\{x\}}$. That is,

$$
\begin{equation*}
\text { for } x \in B_{n}, \text { we have } \bar{x}:=\bigcap\{f \in \mathcal{F}: x \in f\} \tag{1.3}
\end{equation*}
$$

If $\mu: B_{n} \rightarrow L$ is such that $\mu(0)=0$ and, for all $x, y \in B_{n}, \mu(x \vee y)=\mu(x) \vee \mu(y)$, then $\mu$ is a $\{\vee, 0\}$-homomorphism. As a first characterization of our functions, it will be quite easy to prove the following statement.

Proposition 1.1. A lattice-valued function $B_{n} \rightarrow L$ can be given by a linear combination in $L$ iff it is a $\{\vee, 0\}$-homomorphism.

Our main result below characterizes the same functions by cuts. If $\varnothing \neq X \subseteq B_{n}$ is such that $(\forall x \in X)\left(\forall y \in B_{n}\right)(x \leq y \Rightarrow y \in X)$, then $X$ is an up-set of $B_{n}$. For example, every cut $\mu_{p}$ in (1.2) of an isotone $(x \leq y \Rightarrow \mu(x) \leq \mu(y))$ lattice-valued function $\mu$ is an up-set.

Theorem 1.2. Let $\mathcal{F}$ be a closure system over $B_{n}$ that consists of some up-sets of $B_{n}$. Then, adopting (1.3), the following three conditions are equivalent.
(i) For all $x, y \in B_{n}, \bar{x} \subseteq \bar{y} \Rightarrow \overline{x \vee y}=\bar{x}$.
(ii) For all $x, y \in B_{n}, \overline{x \vee y}=\bar{x} \cap \bar{y}$.
(iii) There exist a bounded lattice $L$ and a lattice-valued function $\mu: B_{n} \rightarrow L$ given by a linear combination such that $\mathcal{F}$ is the family of cuts of $\mu$.

By Proposition 1.1, we can also say " $\{\vee, 0\}$-homomorphism" in (iii) instead of "given by a linear combination".

### 1.2. Outline

We prove Proposition 1.1 and Theorem 1.2 in Section 2. Prior to this, we give some motivation.

### 1.3. Motivation

In the particular case $|L|=2$, our function $\mu$ is simply a Boolean function. If $\mu$ belongs to the scope of Theorem 1.2 , then it is isotone since it is a $\{\vee, 0\}$ homomorphism. Isotone Boolean functions are often applied in electrical engineering,
computer science and related fields. There are papers dealing with the number of these functions (Dedekind's problem, see e.g., [15]), or with conjunctive and disjunctive normal forms (e.g., [9], and others). Many fundamental properties of isotone and other Boolean functions can be found in the classical books Crama and Hammer [4] and Rudeanu [18]. For some recent results see, e.g., [2, 8, 14]. A threshold function is a Boolean function given by a linear combination like (1.1) but using $\left\langle\sum, \mathbb{R}\right\rangle$ instead of $\langle\bigvee, L\rangle$ in the sense that the preimage $\mu^{-1}(1)$ of 1 is a cut of the linear combination. As commented in [4], threshold functions can deal with questions investigated in electrical engineering, artificial intelligence, game theory and other areas. In [17], modeling neurons and political decisions are also mentioned as applications of classical threshold functions. Further results concerning threshold functions and related algebraic structures (groups, rings and others) can also be found in $[1,10,11]$. The linear combination in the sense of (1.1) was introduced in [13], where we proved that a Boolean function $\mu: B_{n} \rightarrow B_{1}$ is isotone iff $B_{1}=\{0,1\}$ has a $\{0,1\}$-extension to a lattice $L$ (equivalently, to the free bounded distributive lattice on $n$ generators) such that $\mu^{-1}(1)$ is a cut of $\mu$. Note that in this case and, more generally, for a distributive $L$, we know from Goodstein [7], see also Couceiro and Waldhauser [3], that $\mu$ extends to a lattice polynomial $L^{n} \rightarrow L$.

## 2. Proving our results

Proof of Proposition 1.1. To prove the "only if" part, assume that $\mu: B_{n} \rightarrow L$ is given by (1.1). Evidently, $\mu(0)=0$. For $x, y \in B_{n}$ and $i \in\{1, \ldots, n\}$, observe that $\left\{w_{i}, x_{i}, y_{i}\right\}$ is a distributive sublattice of $L$ since it is a subset of the chain $\left\{0, w_{i}, 1\right\}$. Hence $\mu(x \vee y)=\bigvee_{i} w_{i}\left(x_{i} \vee y_{i}\right)=\bigvee_{i}\left(w_{i} x_{i} \vee w_{i} y_{i}\right)=\bigvee_{i} w_{i} x_{i} \vee \bigvee_{i} w_{i} y_{i}=$ $\mu(x) \vee \mu(y)$, as required. Conversely, to prove the "if" part, assume that $\mu$ is a $\{\vee, 0\}$ - homomorphism. Let $e^{(i)}=\langle 0, \ldots, 0,1,0, \ldots, 0\rangle \in B_{n}$ where 1 stands in the $i$-th place. Define $w_{i}:=\mu\left(e^{(i)}\right)$. Observe that $\mu\left(e^{(i)} \cdot 1\right)=w_{i}=w_{i} \cdot 1$ and $\mu\left(e^{(i)} \cdot 0\right)=0=w_{i} \cdot 0$, that is, $\mu\left(e^{(i)} \cdot x_{i}\right)=w_{i} \cdot x_{i}$. Therefore, for $x \in B_{n}$, we obtain $\mu(x)=\mu\left(\bigvee_{i} e^{(i)} x_{i}\right)=\bigvee_{i} \mu\left(e^{(i)} x_{i}\right)=\bigvee_{i} w_{i} \cdot x_{i}$; proving (1.1) and the "if" part.

Proof of Theorem 1.2. To prove (i) $\Rightarrow$ (ii), assume that (i) holds for $\mathcal{F}$. Let $x, y \in$ $B_{n}$. Since the closure induced by $\mathcal{F}$ is clearly

$$
\begin{equation*}
\text { order-reversing in the sense that } x \leq y \Rightarrow \bar{x} \supseteq \bar{y} \tag{2.1}
\end{equation*}
$$

$\overline{x \vee y} \subseteq \bar{x}$ and $\overline{x \vee y} \subseteq \bar{y}$. Hence, $\overline{x \vee y} \subseteq \bar{x} \cap \bar{y}$. To show the converse inclusion, let $z \in \bar{x} \cap \bar{y}$. By well-known properties of closure operators, $\bar{z} \subseteq \bar{x}$ and $\bar{z} \subseteq \bar{y}$. Since we have assumed (i), $\bar{z}=\overline{z \vee x}$ and $\bar{z}=\overline{z \vee y}$. Using (i) again for the inclusion
$\overline{z \vee x} \subseteq \overline{z \vee y}$, which is actually an equality, and applying (2.1) thereafter, we obtain $z \in \bar{z}=\overline{z \vee x}=\overline{z \vee x \vee z \vee y} \subseteq \overline{x \vee y}$. Hence, $\overline{x \vee y}=\bar{x} \cap \bar{y}$ and (ii) holds.

To prove (ii) $\Rightarrow$ (iii), assume that $\mathcal{F}$ satisfies (ii). Since $\mathcal{F}$ is a finite $\cap$-closed family of subsets of $B_{n}$ and $B_{n} \in \mathcal{F},\langle\mathcal{F} ; \subseteq\rangle$ is a lattice. Let $L$ be the dual $\langle\mathcal{F} ; \supseteq\rangle$ of this lattice and define $\mu: B_{n} \rightarrow L$ by $x \mapsto \bar{x}$, where $\bar{x}$ is given by (1.3). We claim that the cuts of $\mu$ are exactly the members of $\mathcal{F}$. First, let $f \in \mathcal{F}$. Then

$$
\begin{equation*}
f=\left\{x \in B_{n}: x \in f\right\}=\left\{x \in B_{n}: \bar{x} \subseteq f\right\}=\left\{x \in B_{n}: \mu(x) \geq f\right\}=\mu_{f} \tag{2.2}
\end{equation*}
$$

is a cut of $\mu$; see (1.2). Second, every cut of $\mu$ is of the form $\mu_{f}$ for some $f \in \mathcal{F}$, and (2.2) gives that $\mu_{f}=f$, which is in $\mathcal{F}$. This proves that $\mathcal{F}$ is the family of cuts of $\mu$. Since $\mathcal{F}$ consists of up-sets of $B_{n}$, the only member of $\mathcal{F}$ containing 0 is $B_{n}$. Hence $\mu(0)=\overline{0}=B_{n}=0_{L}$. Finally, since $\cap$ is the meet in $\langle\mathcal{F}, \subseteq\rangle$, it is the join in $L$. Thus, (ii) yields that $\mu$ is a $\{\vee, 0\}$-homomorphism. By Proposition 1.1, $\mu$ can be given by a linear combination. Therefore, (iii) holds for $\mathcal{F}$.

Before the last part of the proof, we show that whenever $\mathcal{F}$ is the collection of cuts of an isotone lattice-valued function $\mu: B_{n} \rightarrow L$ and $x \in B_{n}$, then

$$
\begin{equation*}
\mathcal{F} \text { is a closure system and } \bar{x}=\left\{z \in B_{n}: \mu(z) \geq \mu(x)\right\}=\mu_{\mu(x)} \tag{2.3}
\end{equation*}
$$

where $\bar{x}$ is understood as in (1.3). Note that $B_{n}=\left\{x \in B_{n}: \mu(x) \geq 0\right\}=\mu_{0} \in \mathcal{F}$. For any two members of $\mathcal{F}$, say, $\mu_{p}, \mu_{q} \in \mathcal{F}$, we have $\mu_{p} \cap \mu_{q}=\left\{x \in B_{n}: x \geq\right.$ $p$ and $x \geq q\}=\left\{x \in B_{n}: x \geq p \vee q\right\}=\mu_{p \vee q} \in \mathcal{F}$. Hence, $\mathcal{F}$ is a closure system over $B_{n}$. Next, let $x \in B_{n}$, and denote $\mu(x)$ by $q$; we have to show that $\bar{x}$ equals $\left\{z \in B_{n}: \mu(z) \geq q\right\}$, which is $\mu_{q}$. Since $x \in \mu_{q} \in \mathcal{F}$ is clear, we have to verify that for all $p \in L, x \in \mu_{p} \Rightarrow \mu_{q} \subseteq \mu_{p}$. So consider an element $p \in L$ such that $x \in \mu_{p}$, that is, $\mu(x) \geq p$. For any $z \in \mu_{q}$, we have $\mu(z) \geq q=\mu(x)$, and $\mu(z) \geq p$ follows by transitivity. That is, $z \in \mu_{p}$, implying the required inclusion $\mu_{q} \subseteq \mu_{p}$. Consequently, (2.3) holds.

Now, we are in the position to prove (iii) $\Rightarrow((\mathrm{i}))$. Assume (iii). Since $\mu$ is a $\{\vee, 0\}$-homomorphism by Proposition 1.1, the standard trick $x \leq y \Rightarrow \mu(y)=$ $\mu(x \vee y)=\mu(x) \vee \mu(y) \Rightarrow \mu(x) \leq \mu(y)$ shows that $\mu$ is isotone; this allows us to use (2.3). Let $x, y \in B_{n}$ be such that $\bar{x} \subseteq \bar{y}$. Since we have $\overline{x \vee y} \subseteq \bar{x}$ by (2.1), it suffices to deal with the converse inclusion. So let $z \in \bar{x}$. By (2.3), $\mu(z) \geq \mu(x)$. We also have $\mu(z) \geq \mu(y)$ by the same reason and since $z \in \bar{x} \subseteq \bar{y}$. Hence, $\mu(z) \geq$ $\mu(x) \vee \mu(y)=\mu(x \vee y)$ and (2.3) give $z \in \overline{x \vee y}$. Thus, (i) holds for $\mathcal{F}$.

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