CARDINALITY OF HEIGHT FUNCTION'S RANGE IN CASE OF MAXIMALLY MANY RECTANGULAR ISLANDS – COMPUTED BY CUTS

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Abstract. We deal with rectangular $m \times n$ boards of square cells, using the cut technics of the height function. We investigate combinatorial properties of this function, and in particular we give lower and upper bounds for the number of essentially different cuts. This number turns out to be the cardinality of the height function's range, in case the height function has maximally many rectangular islands.

1. Introduction

1.1. **Historical background.** Let a rectangular $m \times n$ board be given, consisting of square cells. A positive integer is associated to each cell of the board, its height. A rectangle in the board is called a rectangular island, if the heights of its cells are greater than the heights of the neighboring cells. The notion of an island comes from information theory. The characterization of the lexicographical length sequences of binary maximal instantaneous codes in [5] uses the notion of *full segments*, which are one-dimensional islands. Several generalizations of this notion gave interesting combinatorial problems. In two dimensions, Czédli [2] has determined the maximum number of rectangular islands; for the maximum number of rectangular islands on the rectangular board of size $m \times n$ he obtained $f(m,n) = \lfloor (mn + m + n - 1)/2 \rfloor$. Pluhár [18] gave upper and lower bounds in higher dimensions. Horváth, Németh and Pluhár determined upper and lower bounds for the maximum number of triangular islands on a triangular grid in [7]. Some further interesting investigations and nice results on islands

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appear in [12, 14] and [17]. The number of square islands is a similar problem to the triangular case and it is treated in [8] and in [13]. Some proving methods for the maximum number of islands are summarized in [1] and in [16]. Exact formulas for some further island-problems are summarized in [1]. The problem of minimum cardinality of maximal systems of rectangular islands is treated in [11]. The investigations on islands motivated further research on independence properties in lattices, see [3] and [4].

1.2. **Motivation.** The paper [9] solves the problem of the maximum number of islands on a one-dimensional board in case of finitely many heights; the two-dimensional generalization of this problem is still an open problem. In [15] there are partial results: the authors obtained that if the heights are natural numbers that are at least 1 and at most h, then the maximum number of rectangular islands on the $1\times n$ board is $I_h(n) = n + 1 - \left[\frac{n}{2^{h-1}}\right]^+$ while if $h \geq 3$, then on the $2\times n$ board it is $\left[\frac{3n+1}{2}\right] + 1 - \left[\frac{n}{2^{h-2}}\right]^+$ and on the $3\times n$ board it is $2n + 2 - \left[\frac{n}{2^{h-2}}\right]^+$ where $[.]^+ = \max\{1, [.]\}$. In addition, our investigation starting in [10] is related to this work; we started to investigate rectangular islands by cuts which we continue in the present paper; this approach might get closer to the solution of the mentioned open problem and also gives interconnection of the lattice method and tree-graph method of [1].

1.3. **Outline.** We investigate height functions which map an $m \times n$ board into $\mathbb{N}$. Our main notion is the p-cut, $p \in \mathbb{N}$ (originating in fuzzy set theory). For a height function h and $p \in \mathbb{N}$, the p-cut is the inverse image of $\{x \in \mathbb{N} \mid p \leq x\}$, represented by its characteristic function (thus a cut corresponds to a same size board with values in $\{0, 1\}$).The collection of cuts uniquely determines the corresponding height function and vice versa. Hence, analyzing cuts one can get information about height functions and thus reveal combinatorial properties of rectangular islands on the board.

Results from [10] for the co-domain $[0, 1]$ (real interval) remain analogously valid; e.g, dealing with islands, we can use rectangular height functions only.

We show that for every rectangular height function there is a so-called standard rectangular height function, as follows: it has the same rectangular islands as the starting function, and for each rectangular island of the first function there exists exactly one $p \in \mathbb{N}$ such that the rectangular island appears in that cut. So the cuts can be used for identification of particular islands. Whence we prove that the minimum cardinality of maximal systems of rectangular islands (given in [11]) is equal to the maximum number of different cuts of rectangular height functions. Also, the standard rectangular height functions with maximally many cuts have the same rectangular islands as the height functions with the minimum cardinality of maximal systems of rectangular islands ([11]). If the height function gives maximally many rectangular islands, i.e., if $f(m, n) = \left\lfloor \frac{mn+m+n-1}{2} \right\rfloor$, then the number of different cuts is at least $\lceil \log_2(m+1) \rceil + \lceil \log_2(n+1) \rceil - 1$ and it is at most $\left\lfloor \frac{(m+n+3)}{2} \right\rfloor$.

2. Height function and cuts

The set $\{1,2,...,m\} \times \{1,2,...,n\}$, $m,n \in \mathbb{N}$ is called *a table of size* $m \times n$ (according to [2]) or *a board of size* $m \times n$ (see [1]). Then a *height function* h is a mapping from $\{1,2,...,m\} \times \{1,2,...,n\}$ to $\mathbb{N}$, $h : \{1,2,...,m\} \times \{1,2,...,n\} \to \mathbb{N}$. Here $\mathbb{N}$ is the set of positive integers, but we also consider non-negative integers or the two-element set $\{0,1\}$.

For every $p \in \mathbb{N}$, the *cut of the height function*, the p*-cut* of h is a relation h_p on $\{1,2,...,m\} \times \{1,2,...,n\}$ defined by

$$(x,y) \in h_p \text{ if and only if } h(x,y) \geq p.$$

In order to consider cuts as boards of the same size $(m \times n)$ as the height function h, *we identify* h_p *with its characteristic function*:

$$h_p(x,y) = \begin{cases} 1 & \text{if } h(x,y) \geq p \\ 0 & \text{else} \end{cases} . \tag{1}$$

We mention that the notion of a p*-cut* comes from the theory of fuzzy sets. More details could be found e.g. in [6, 10, 19, 20].

Let $h : \{1,2,...,m\} \times \{1,2,...,n\} \to \mathbb{N}$ be a height function.

Fields on the board are called *cells*, which we denote by the corresponding ordered pairs $(i,j), i \in \{1,...,m\}, j \in \{1,...,n\}$.

If (i,j) and (k,l) are two cells then their *distance* is defined in the usual way by $\sqrt{(i-k)^2 + (j-l)^2}$. Two different cells with distance at most $\sqrt{2}$ are called *neighboring* cells.

For $1 \leq \alpha \leq \beta \leq m$ and $1 \leq \gamma \leq \delta \leq n$, the set $\{\alpha,...,\beta\} \times \{\gamma,...,\delta\}$ is called a *rectangle* in the table. The set of all rectangles of the table of size $m \times n$ is by $\mathcal{R}(m \times n)$.

We say that the rectangle $T = \{\alpha,...,\beta\} \times \{\gamma,...,\delta\}$ is a *rectangular island* of h if for every cell (μ,ν) which does not belong to this rectangle but is neighboring to some cell of the rectangle, we have

$$h(\mu,\nu) < \min_{(x,y) \in T} h(x,y).$$

We take the whole board to be a rectangular island (of size $m \times n$). We say that a rectangular island is *maximal,* if the only rectangular island that properly contains it is the rectangular island of size $m \times n$. We say that a rectangular island is *minimal,* if there is no rectangular island that is properly contained in it.

As it is mentioned in Introduction, for a height function there is a family of cuts; on the other hand, these cuts determine values of the height function, as follows.

Let $h : \{1,2,...,m\} \times \{1,2,...,n\} \to \mathbb{N}$ be a height function, and

$$p_g := \max\{h(x,y) \mid (x,y) \in \{1,2,...,m\} \times \{1,2,...,n\}\}.$$

Then

$$H := \{h_p \mid p \in \{1, 2, \dots, p_g\}\}$$

is the family of cuts determined by the values of h (observe that some elements of the family might be equal). The family H can be ordered naturally, componentwise: for $p, q \in \{1, 2, \dots, p_g\}$,

$$h_p \leq h_q \text{ if for all } (x, y),\ \ h_p(x, y) \leq h_q(x, y)\ .$$

Under this order, the family H is a chain, and its connection to the order $\leq$ for numbers is as follows: for $p, q \in \{1, 2, \dots, p_g\} = G$,

$$p \leq q \ \text{ implies } \ h_q \leq h_p.$$

In the following, for $p \in \mathbb{N}$, we use the product $p \cdot h_p(x, y)$, which is either p or 0, since $h_p(x, y) \in \{0, 1\}$.

Proposition 1. *Let $h : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$ be a height function. Then for every pair $(x, y) \in \{1, 2, ..., m\} \times \{1, 2, ..., n\}$,*

$$h(x, y) = \max_{p \in G}(p \cdot h_p(x, y)).$$

Proof. It is a straightforward computation, by the definition of a cut. □

The next lemma shows how cuts actually identify particular islands.

Lemma 1. *If T is a rectangular island, then there is $p \in \mathbb{N}$ and a cut relation h_p, such that $T \subseteq h_p$ and no cell neighboring to T belongs to h_p.*

Proof. By the definition of a cut and of a rectangular island, it is straightforward that the cut h_p, with $p = \min\{h(x, y) \mid (x, y) \in T\}$, fulfills the requirements. □

In the next section, among other, we deal with a construction of a height function by means of simple boards being its cuts.

3. Rectangular height function and cuts

Let us denote by $\mathcal{I}_{rect}(h)$ the set of all rectangular islands of the height function h. The poset $(\mathcal{I}_{rect}(h), \subseteq)$ is a tree, where $\subseteq$ means set inclusion.

A subset $\mathcal{H}$ of $\mathcal{R}(m \times n)$ is called *a system of rectangular islands* if there is a height function $h : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \to \mathbb{N}$, such that $\mathcal{H} = \mathcal{I}_{rect}(h)$.

We say that two rectangles $\{\alpha, ..., \beta\} \times \{\gamma, ..., \delta\}$ and $\{\alpha_1, ..., \beta_1\} \times \{\gamma_1, ..., \delta_1\}$ are *distant* if they are disjoint and for every two cells, one cell from the first rectangle and the other cell from the second, their distance is at least 2. The height function h is called *rectangular height function* if for every $p \in \mathbb{N}$, every nonempty p-cut of h is a union of distant rectangles. There is a characterization Theorem in [10] for rectangular height functions with co-domain $[0, 1]$; the analogous Theorem is valid for rectangular height functions with co-domain $\mathbb{N}$ (and the proof is similar).

It is proved in [10] that for every height function $h : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \to \mathbb{N}$, there is a rectangular height function $h^* : \{1, 2, ..., n\} \times$

$\{1,2,...,m\} \to \mathbb{N}$, such that $\mathcal{I}_{rect}(h) = \mathcal{I}_{rect}(h^*)$. In [10] an algorithm is presented for constructing rectangular height function having the same rectangular islands as the given height function.

The next statement goes one step further:

Lemma 2. *For every rectangular height function*

$$h^* : \{1,2,...,n\} \times \{1,2,...,m\} \to \mathbb{N},$$

there is a rectangular height function

$$h^{**} : \{1,2,...,n\} \times \{1,2,...,m\} \to \mathbb{N},$$

such that $\mathcal{I}_{rect}(h^*) = \mathcal{I}_{rect}(h^{**})$ and in h^{**} for every rectangular island, there exists exactly one $p \in \mathbb{N}$, such that the rectangular island appears in h^{**}_p.

Proof. We prove the statement by induction on the size of the rectangular table (board). For $n = m = 1$ the statement is obvious. If $m > 1$ or $n > 1$, then by the induction hypothesis, we can replace the value of the mapping h^* in the cells of maximal rectangular islands of h^* according to such rectangular height functions of smaller sizes that have the same rectangular islands and every rectangular island appears exactly in one cut. In this way we define the values of h^{**} inside of the maximal rectangular islands. Then, we increase the values inside of the maximal rectangular islands by 1; now the smallest value of the mapping h^{**} inside of the maximal rectangular islands is 2. Finally, if (α, β) does not belong to any maximal rectangular island, then we define $h^{**}(\alpha, \beta) = 1$. □

Next we deal with the following problem. We start with a collection of same size boards each having rectangular islands whose fields are filled with 1, and all other fields with 0. Each board consists of one or more such rectangular islands which are pairwise distant as defined above. Our aim is to construct a height function for the same size board, so that its cuts are precisely the boards we started with.

Observe that by (1), every cut of a height function is a height function on the same board, with values in $\{0,1\}$. Therefore, in the above problem we start with a collection of rectangular height functions with the co-domain $\{0,1\}$, and the resulting height function is supposed to be also rectangular, with co-domain $\mathbb{N}$.

Proposition 2. *Let $F = \{f_1, \dots, f_p\}$ be a finite collection of rectangular height functions $\{1,2,...,n\} \times \{1,2,...,m\} \to \{0,1\}$ linearly ordered componentwise $f_1 > f_2 > \dots > f_p$, such that f_1 is a constant function 1. Then there is a rectangular height function $h : \{1,2,...,n\} \times \{1,2,...,m\} \to \mathbb{N}$, so that its cuts are precisely functions from F.*

Proof. Let us define $h : \{1,2,...,n\} \times \{1,2,...,m\} \to \mathbb{N}$ as follows:

$$h(x,y) := \max\{r \mid f_r(x,y) = 1\}.$$

We show that the cuts of h are functions from F. Indeed, for $q \in \{1,2,\dots,p\}$ and any $(x,y) \in \{1,2,...,m\} \times \{1,2,...,n\}$,

$h_q(x, y) = 1$ if and only if $h(x, y) \geq q$ if and only if $r_0 = \max\{r \mid f_r(x, y) = 1\} \geq q$ if and only if $1 = f_{r_0}(x, y) \leq f_q(x, y)$. Hence, $h_q = f_q$.

We have proved that the cuts of h coincide with functions in F, and since these are rectangular, then also h is a rectangular height function. □

4. Combinatorial properties

If a rectangular height function has the property described in Lemma 2 for h^{**}, i.e., that each rectangular island appears exactly in one cut, then we call it *standard rectangular height function.* We denote by $\Lambda_{max}(m, n)$ the maximum number of different nonempty p-cuts of a standard rectangular height function on the rectangular table of size $m \times n$. Similarly, if R is a rectangle, we denote by $\Lambda_{max}(R)$ the maximum number of different nonempty p-cuts in a standard rectangular height function in the rectangle R.

Theorem 1. $\Lambda_{max}(m, n) = m + n - 1$.

Proof. $\Lambda_{max}(m, n) \geq m + n - 1$ because of the construction in Figure 1: if we put the numbers $1, 2, \ldots, k, k + 1, \ldots, m + n - 1$ for the values of the mapping Ψ as it is shown in Figure 1, then we have $m + n - 1$ different nonempty p-cuts and Ψ is obviously a standard rectangular height function.

Now we prove $\Lambda_{max}(m, n) \leq m + n - 1$. Obviously, for rectangles R_1 and R_2, $R_1 \subseteq R_2$ implies $\Lambda_{max}(R_1) \leq \Lambda_{max}(R_2)$. Suppose H is a system of rectangular islands of the standard rectangular height function $\widehat{\Psi}$ having maximally many different nonempty p-cuts. Then clearly

$$\Lambda_{max}(m, n) = 1 + \max_{I \in H} \Lambda_{max}(I).$$

This means that for some maximal rectangular island $I_{\max}$ of $\widehat{\Psi}$,

$$\Lambda_{max}(m, n) = 1 + \Lambda_{max}(I_{\max})$$

holds. But for some rectangle R^* of size $(m - 1) \times n$ or $m \times (n - 1)$,

$$I_{\max} \subseteq R^*$$

holds, which implies

$$\Lambda_{max}(m, n) \leq 1 + \Lambda_{max}(R^*).$$

Figure 1

Now we can proceed by induction. For $m = 1$ and $n = 1$ we have $\Lambda_{max}(1,1) = 1$. By induction hypothesis $\Lambda_{max}(R^*) \leq m + n - 2$. Consequently, using the last inequality, we obtain $\Lambda_{max}(m,n) \leq m + n - 1$.

□

Let I_R denote the ordered set of systems of rectangular islands on the rectangular board R of size $m \times n$, and let $\max(I_R)$ denote the collection of maximal elements of I_R.

We have a straightforward corollary of the above theorem, if we use the following Proposition from [11]:

Proposition 3. *For $H \in \max(I_R)$, if $|H| = m+n-1$ then H is a sequence of rectangles each contained in the next except possibly for the first $m-1$ (or $n-1$), all contained in the m-th (or n-th) rectangle which is $m \times 1$(or $1 \times n$).*

Corollary 1. *The maximum number of different nonempty p-cuts of a standard rectangular height function is equal to the minimum cardinality of* *maximal systems of rectangular islands. Moreover, the rectangular island-systems for both situations are the same, i.e. a maximal system of rectangular islands realize the minimum cardinality of maximal systems of rectangular islands if and only if it is the island system of a standard rectangular height function with maximally many different cuts.*

The next proposition is an obvious consequence of the former statements:

Proposition 4. *For each integer k between 1 and $m+n-1$ there is a standard rectangular height function $h : \{1,2,...,m\} \times \{1,2,...,n\} \to \mathbb{N}$ such that its number of different nonempty p-cuts is k.*

Proof. One can easily check that it is possible to give the same value for as many neighboring levels as we wish, starting from the highest level.

□

The next lemma shows that in case the board contains maximally many rectangular islands, if $m \geq 3$ and $n \geq 3$, then the number of maximal rectangular islands on the board is exactly 2.

Lemma 3. *If $m \geq 3$ and $n \geq 3$ and the height function $h : \{1,2,...,m\} \times \{1,2,...,n\} \to \mathbb{N}$ has maximally many rectangular islands, then it has exactly two maximal rectangular islands.*

We present two proofs for this lemma. The reason is that Proof A is self-contained, Proof B is much shorter and based on [2].

Proof. Proof A:

We recall from [1] that rectangular islands constitute rooted tree under inclusion. Sometimes a rectangular island, which is a vertex in the tree, has only one son - successor (imagine that by the increase of the water level the rectangular island shrinks). We interpret the decline of the rectangular island as a division into a smaller rectangular island (its only son) and a so-called *dummy* rectangular island, i.e. if a rectangular island shrinks, then we associate a smaller rectangular island and a dummy rectangular island. More precisely: if T_1 is a rectangular island and T_2 is its only rectangular sub-island that is covered by T_1, then we call $D = T_2 \setminus T_1$ *dummy* rectangular island. Now, the dummy rectangular island will be the second son of the shrinking vertex. With dummy rectangular islands, the rooted tree of rectangular islands is at least binary (for more explanation the reader should consult [1]).

If we have maximally many rectangular islands, then we denote by s the number of minimal rectangular islands and by d the number of dummy rectangular islands. Minimal rectangular islands cover obviously at least 4 grid-points; moreover dummy rectangular islands "cover" at least 2 grid-points because if a rectangular island shrinks, then at least 2 gridpoints remain uncovered. We have

$$(m+1)(n+1) \geq 4s + 2d$$

since minimal rectangular islands cover 4 grid-points, dummy rectangular islands cover 2 grid-points and $(m+1)(n+1)$ is the number of all grid-points on the $m \times n$ table. Let T be the rooted tree of rectangular islands with ℓ being the number of leaves and V the number of vertices; T contains also the dummy islands, in other formulation, the number of "proper" islands is $V-d$. Then, $\ell = s+d$. Moreover, since we have $\frac{(m+1)(n+1)}{2} \geq 2s+d$, consequently

$$\left\lfloor \frac{mn+m+n-1}{2} \right\rfloor \geq 2s+d-1 = 2(\ell-d)+d-1 = (2\ell-1)-d \geq V-d;$$

here we use the fact that if the tree is at least binary (any non-leaf node has at least two sons), then $2\ell - 1 \geq V$ (see [1], Lemma 5). However now we have maximally many rectangular islands, so:

$$\left\lfloor \frac{mn+m+n-1}{2} \right\rfloor = V-d,$$

hence

$$V-d = (2\ell-1)-d,$$

consequently

$$V = 2\ell - 1. \tag{2}$$

Now we direct the edges of T from bigger to smaller rectangular islands. For the sum of in-degrees and out-degrees of the tree T we have

$$\sum D^+ = \sum D^-.$$

Now

$$\sum D^+ = V-1$$

because each vertex has one father, except the root.

By (2) we have

$$\sum D^+ = V-1 = 2\ell-2.$$

By $V = 2\ell-1$ we have $2V = 4\ell-2$, moreover $2V-2\ell = 2\ell-2$, so

$$\sum D^- = \sum D^+ = V-1 = 2\ell-2 = 2V-2\ell = 2(V-\ell).$$

Since the rooted tree-graph of rectangular islands is at least binary, we obtained that it is exactly binary, i.e. each non-leaf vertex has exactly two sons. Consequently the height function has at most two maximal rectangular islands. But we should not forget about the possibility of dummy rectangular islands, which might mean that we have one maximal rectangular island. However, if $m \geq 3$ and $n \geq 3$, then in case of one maximal rectangular island, for the maximum number of rectangular islands we have either

$$1 + \left\lfloor \frac{m(n-1)+m+(n-1)-1}{2} \right\rfloor = \left\lfloor \frac{m(n-1)+m+(n-1)-1+2}{2} \right\rfloor =$$

$$= \left\lfloor \frac{mn+m+n-m}{2} \right\rfloor < \left\lfloor \frac{mn+m+n-1}{2} \right\rfloor$$

or

$$1 + \left\lfloor \frac{(m-1)n+(m-1)+n-1}{2} \right\rfloor = \left\lfloor \frac{(m-1)n+(m-1)+n-1+2}{2} \right\rfloor =$$

$$= \left\lfloor \frac{mn+m+n-n}{2} \right\rfloor < \left\lfloor \frac{mn+m+n-1}{2} \right\rfloor .$$

Thus it is proved that the number of maximal rectangular islands is exactly two.

Proof B:

We use some results and notions from paper [2]. Namely, let $\mathcal{H} = \mathcal{I}_{rect}(h)$ be a system of rectangular islands corresponding to h. Then, *deficiency* of $\mathcal{H}$ is defined by

$$d(\mathcal{H}) = mn - |\mathcal{H}|.$$

In case h has maximally many rectangular islands,

$$d(\mathcal{H}) = \left\lceil \frac{mn+1-m-n}{2} \right\rceil .$$

Further, let k be the number of maximal rectangular islands in $\mathcal{I}_{rect}(h)$, and let e be the number of grid points not covered by any of maximal rectangular islands.

In [2], page 8, (10), it is proved that

$$e - \left\lceil \frac{e}{2} \right\rceil + k - 2 \geq 0.$$

It is also proved that

$$d(\mathcal{H}) \geq e - \left\lceil \frac{e}{2} \right\rceil + k - 2 + \left\lceil \frac{mn+1-m-n}{2} \right\rceil ,$$

and hence in case of maximally many rectangular islands,

$$e - \left\lceil \frac{e}{2} \right\rceil + k - 2 = 0.$$

Since $e - \lceil \frac{e}{2} \rceil \geq 0$, we have $k \leq 2$. However, if $k = 0$, then we have only one rectangular island (the whole board), which means that the number of rectangular islands cannot be maximal; if $k = 1$, then since $m \geq 3$ and $n \geq 3$, we have $e \geq 4$, which means $e - \lceil \frac{e}{2} \rceil \geq 2$, consequently

$$e - \left\lceil \frac{e}{2} \right\rceil + k - 2 \geq 1.$$

So, the only possibility for $m \geq 3$ and $n \geq 3$ is $k = 2$, i.e., we have exactly two maximal rectangular islands. □

Remark 1. It can be easily shown that if $m = 2$, then for even n, in case of maximally many rectangular islands, the number of maximal rectangular islands can be one or two (and the same is true for $n = 2$ and for even m). Moreover, if $m = 2$ and n is odd, then the number of maximal islands can be only two (and the same is true for $n = 2$ and for odd m). In addition it is trivial that if $m = 1$ or $n = 1$, then in case of maximally many rectangular islands, the number of maximal rectangular islands is 1 or 2.

Proposition 5. *If $m \geq 3$ and $n \geq 3$ and a standard rectangular height function $h : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$ has maximally many rectangular islands, then the number of different nonempty cuts is strictly less than $\Lambda_{max}(m, n)$.*

Proof. If the standard rectangular height function h has maximally many rectangular islands, then by Lemma 3, it has exactly two maximal rectangular islands. Therefore, neither of the maximal rectangular islands is bigger then $m \times (n-2)$ or $(m-2) \times n$. By Theorem 1, in these maximal rectangular islands we can have at most $m+n-3$ different nonempty cuts. So, h cannot have more than $m + n - 2$ different nonempty cuts. □

Remark 3. The last proposition is not true in case the number of maximal rectangular islands is 1.

Next we denote by $\Lambda_h^{cz}(m, n)$ the number of different nonempty cuts of a standard rectangular height function h in case h has maximally many rectangular islands, i.e., when the number of rectangular islands is

$$f(m, n) = \left\lfloor \frac{mn + m + n - 1}{2} \right\rfloor .$$

Theorem 2. *Let $h : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$ be a standard rectangular height function having maximally many rectangular islands $f(m, n)$. Then, $\Lambda_h^{cz}(m, n) \geq \lceil \log_2(m + 1) \rceil + \lceil \log_2(n + 1) \rceil - 1$.*

Proof. We prove the statement by induction on n and m. If $m \leq 2$ and $n \leq 2$, then the statement is easy to check.

Let $n = 1$, we have to prove that $\Lambda_h^{cz}(m, 1) \geq \lceil \log_2(m + 1) \rceil$. Suppose that for all $m' < m$, $\Lambda_h^{cz}(m', 1) \geq \lceil \log_2(m' + 1) \rceil$. Concerning the number of different nonempty cuts: $\Lambda_h^{cz}(m, 1) \geq \Lambda_h^{cz}(m', 1) + 1$.

Since h has maximally many rectangular islands, there can be one maximal rectangular island, or two maximal rectangular islands.

In case of one maximal rectangular island we can choose $m' = m - 1$,
$\Lambda_h^{cz}(m, 1) = \Lambda_h^{cz}(m - 1, 1) + 1 \geq \lceil \log_2 m \rceil + 1 \geq \lceil \log_2(m + 1) \rceil$.

In case of two maximal rectangular islands, if m is odd, then the bigger maximal rectangular island has side-length at least $m' = \frac{m-1}{2}$.

By $\Lambda_h^{cz}(m, 1) \geq \Lambda_h^{cz}(m', 1) + 1$, we have

$$\Lambda_h^{cz}(m, 1) \geq \left\lceil \log_2 \left(\frac{m - 1}{2} + 1 \right) \right\rceil + 1 = \left\lceil \log_2 \frac{m + 1}{2} \right\rceil + 1 =$$

$$\lceil \log_2(m+1) - \log_2 2 \rceil + 1 = \lceil \log_2(m+1) \rceil .$$

In case m is even, then the bigger maximal rectangular island has side-length at least $m' = \frac{m}{2}$. Similarly to the case m is odd, we can put

$$\Lambda_h^{cz}(m,1) \geq \left\lceil \log_2 \left(\frac{m}{2} + 1 \right) \right\rceil + 1 = \left\lceil \log_2 \frac{m+2}{2} \right\rceil + 1 =$$

$$= \lceil \log_2 (m+2) \rceil - \log_2 2 + 1 = \lceil log_2(m+2) \rceil = \lceil \log_2(m+1) \rceil .$$

In the last equality we used the fact that m is an even number, hence $\log_2(m+1)$ is not an integer.

Let $n = 2$. Again there can be one maximal rectangular island or two maximal rectangular islands. In case there is one maximal rectangular island, if $m \geq 3$, this maximal rectangular island cannot be of size $m \times 1$. Indeed, for the board of this type, the maximum number of rectangular islands would be $f(m,2) = \left\lfloor \frac{3m+1}{2} \right\rfloor$. In case we have one maximal rectangular island of the size $m \times 1$, there will be $1 + m$ rectangular islands, which is not equal to $f(m,2) = \left\lfloor \frac{3m+1}{2} \right\rfloor$ for $m \geq 3$.

Hence, we have one maximal rectangular island of size $(m-1) \times 2$, or two maximal rectangular islands of sizes $k \times 2$, $l \times 2$, respectively, where $k + l + 1 \leq m$. In both cases, the proof of $\Lambda_h^{cz}(m,2) \geq \lceil \log_2(m+1) \rceil + 1$ is similar to the proof for $n = 1$.

Let $m \geq 3$ and $n \geq 3$. By Lemma 3 there are exactly two maximal rectangular islands. Without loss of generality we can suppose that both maximal rectangular islands reach the side of the board of length m. If m is odd, then the biggest maximal rectangular island has side-length at least $\frac{m-1}{2}$. Indeed, due to maximally many rectangular islands, there is only a one-cell row between the two maximal rectangular islands. Again we use the fact that if $m > m'$, then for any fixed h, $\Lambda_h^{cz}(m,n) > \Lambda_h^{cz}(m',n)$ holds, consequently:

$$\Lambda_h^{cz}(m,n) \geq \left\lceil \log_2 \left(\frac{m-1}{2} + 1 \right) \right\rceil + \lceil \log_2(n+1) \rceil - 1 + 1 =$$

$$= \left\lceil \log_2 \frac{m+1}{2} \right\rceil + \lceil \log_2(n+1) \rceil =$$

$$= \lceil \log_2(m+1) - \log_2 2 \rceil + \lceil \log_2(n+1) \rceil =$$

$$= \lceil \log_2(m+1) \rceil + \lceil \log_2(n+1) \rceil - 1.$$

If m is even, then the biggest maximal rectangular island has side-length at least $\frac{m}{2}$ because again for having maximally many rectangular islands, there is only a one-cell row between the two maximal rectangular islands. Now

$$\Lambda_h^{cz}(m,n) \geq \left\lceil \log_2 (\frac{m}{2} + 1) \right\rceil + \lceil \log_2(n+1) \rceil - 1 + 1 =$$

$$= \left\lceil \log_2 \frac{m+2}{2} \right\rceil + \lceil \log_2(n+1) \rceil =$$
$$= \lceil \log_2 (m+2) \rceil - \log_2 2 + \lceil \log_2(n+1) \rceil =$$
$$= \lceil \log_2(m+1) \rceil + \lceil \log_2(n+1) \rceil - 1.$$

Here we again use the fact that if m is even, then $m+1$ is odd and $m+2$ is even, which means that $\log_2(m+1)$ is not an integer; so $\lceil \log_2(m+2) \rceil = \lceil \log_2(m+1) \rceil$.

□

Lemma 4. *If $m \geq 3$ or $n \geq 3$, then for any odd number $t = 2k+1$ with $1 \leq t \leq max\{m-2, n-2\}$, there is a standard rectangular height function $h : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$ having the maximum number of rectangular islands f(m,n), such that one of the side-lengths of one of the maximal rectangular islands is equal to t.*

Proof. Let $m \geq 3$. We construct a standard rectangular height function $h : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$, as follows. We embed standard rectangular height functions of sizes $(2k+1) \times n$ and $(m-2k-2) \times n$ with maximally many rectangular islands (adding 1 to each value of the functions), into the board of size $m \times n$, leaving a $1 \times n$ column between them, for which we assign value 1. In this way we obtain the value $\left\lfloor \frac{mn+m+n-1}{2} \right\rfloor$ as a lower bound of the maximum number of rectangular islands as follows:

$$f(m,n) \geq f(2k+1, n) + f(m-2k-2, n) + 1$$
$$= \left\lfloor \frac{(2k+1)n + 2k+1+n-1}{2} \right\rfloor + \left\lfloor \frac{(m-2k-2)n + m-2k-2+n-1}{2} \right\rfloor + 1$$
$$= \left\lfloor \frac{mn+m+n-1}{2} \right\rfloor .$$

Here we used that $(2k+1)n + 2k+1+n-1$ is even.

Similar proof is valid if we put maximal rectangular islands of sizes $m \times (2k+1)$ and $m \times (n-2k-2)$ into the board of size $m \times n$.

□

Remark 4. The statement of the last lemma is not true for even side-lengths as t, one can construct counterexample easily!

Our lower bound in the Theorem 2 is sharp as shown in the following.

Proposition 6. *There exists a standard rectangular height function with maximally many rectangular islands and with the number of different non-empty cuts exactly $\lceil \log_2(m+1) \rceil + \lceil \log_2(n+1) \rceil - 1$.*

Proof. We prove the statement by induction on the side lengths n and m. Throughout the proof we use Lemma 3, and also the fact that the number of different nonempty p-cuts in a standard rectangular height function is equal to the number of different nonempty p-cuts in the bigger maximal island plus 1.

If $m \leq 2$ and $n \leq 2$, then the statement is easy to check. Let $m > 2$. If m is odd and $\frac{m-1}{2}$ is also odd, then by Lemma 4 the induction step applies, and we can define a standard rectangular height function realizing the required number of different nonempty cuts using the induction hypothesis and height functions on maximal islands. If h is the obtained standard rectangular height function, then

$$\Lambda_h^{cz}(m,n) = \left\lceil \log_2\left(\frac{m-1}{2}+1\right)\right\rceil + \lceil \log_2(n+1)\rceil - 1 + 1 =$$

$$= \lceil \log_2(m+1)\rceil + \lceil \log_2(n+1)\rceil - 1.$$

If m is odd and $\frac{m-1}{2}$ is even, then $\frac{m+1}{2}$ is odd,

$$\left\lceil \log_2\left(\frac{m+1}{2}+1\right)\right\rceil = \left\lceil \log_2\left(\frac{m-1}{2}+1\right)\right\rceil,$$

so by the last lemma again the induction step applies,

$$\Lambda_h^{cz}(m,n) = \left\lceil \log_2\left(\frac{m+1}{2}+1\right)\right\rceil + \lceil \log_2(n+1)\rceil - 1 + 1 =$$

$$= \left\lceil \log_2\left(\frac{m-1}{2}+1\right)\right\rceil + \lceil \log_2(n+1)\rceil - 1 + 1 =$$

$$= \lceil \log_2(m+1)\rceil + \lceil \log_2(n+1)\rceil - 1.$$

If m is even and $\frac{m}{2}$ is odd, then by the last lemma the induction step also applies,

$$\Lambda_h^{cz}(m,n) = \left\lceil \log_2(\frac{m}{2}+1)\right\rceil + \lceil \log_2(n+1)\rceil - 1 + 1 =$$

$$= \lceil \log_2(m+1)\rceil + \lceil \log_2(n+1)\rceil - 1.$$

If m is even and $\frac{m}{2}$ is even, then

$$\left\lceil \log_2\left(\frac{m}{2}+1\right)\right\rceil = \left\lceil \log_2\left(\frac{m+2}{2}+1\right)\right\rceil,$$

so by the last lemma again the induction step applies,

$$\Lambda_h^{cz}(m,n) = \left\lceil \log_2\left(\frac{m+2}{2}+1\right)\right\rceil + \lceil \log_2(n+1)\rceil - 1 + 1 =$$

$$= \left\lceil \log_2(\frac{m}{2}+1)\right\rceil + \lceil \log_2(n+1)\rceil - 1 + 1 =$$

$$= \lceil \log_2(m+1)\rceil + \lceil \log_2(n+1)\rceil - 1.$$

□

Proposition 7. *If $m, n \geq 2$, then $\Lambda_h^{cz}(m,n) \leq \left\lfloor \frac{(m+n+3)}{2} \right\rfloor$. In addition, for $m, n \geq 3$ this bound is sharp.*

Proof. We prove the statement by induction on the size of the rectangle. In case $m = n = 2$ the statement is easy to check. Suppose that $n = 2$, and $m \geq 2$. We prove our statement by induction on m.

Having in mind that the rectangular height function has maximum number of islands, if we have maximally many cuts, then in case m is even there must be a maximum island $k \times 2$, where $k \leq m - 1$. If m is odd, then $k \leq m - 2$.

Then, in case m is even,

$$\Lambda_h^{cz}(m, 2) \leq \left\lfloor \frac{(m - 1 + 2 + 3)}{2} \right\rfloor + 1 = \left\lfloor \frac{(m + 2 + 3)}{2} \right\rfloor .$$

In case m is odd,

$$\Lambda_h^{cz}(m, 2) \leq \left\lfloor \frac{(m - 2 + 2 + 3)}{2} \right\rfloor + 1 = \left\lfloor \frac{(m + 2 + 3)}{2} \right\rfloor .$$

For $m = 3$ and $n = 3$ the statement is easy to check.

We recall that for rectangles R_1 and R_2, $R_1 \subseteq R_2$ implies $\Lambda_{max}(R_1) \leq \Lambda_{max}(R_2)$. By Lemma 4, the bigger maximal island has sidelength $m - 2$ or $n - 2$ if we have maximally many rectangular islands and want to realize maximally many different nonempty cuts. Now

$$\Lambda_h^{cz}(m, n) \leq \left\lfloor \frac{(m + n - 2 + 3)}{2} \right\rfloor + 1 = \left\lfloor \frac{(m + n + 3)}{2} \right\rfloor .$$

This induction argument gives concrete representation for $m, n \geq 3$, we prove by induction that there exists h such that $\Lambda_h^{cz}(m, n) = \left\lfloor \frac{(m+n+3)}{2} \right\rfloor$. □

Remark 5. If $n = 1$, then $\Lambda_h^{cz}(m, n) \leq m$ (similarly for $m = 1$).

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