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THE NUMBER OF TRIANGULAR ISLANDS ON A TRIANGULAR GRID

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Abstract

The aim of the present paper is to carry on the research of Czédli in determining the maximum number of rectangular islands on a rectangular grid. We estimate the maximum of the number of triangular islands on a triangular grid.

1. Introduction

The following combinatorial problem was raised in connection with instantaneous (prefix-free) codes, see [5]. For every square of a rectangular grid a real number a_{ij} is given, its height. Czédli [1] considered a rectangular lake whose bottom is divided into $(m+2) \times (n+2)$ cells. In other words, we identify the bottom of the lake with the table $\{0, 1, \dots, m+1\} \times \{0, 1, \dots, n+1\}$. The height of the bottom (above see level) is constant on each cell but definitely less than the height of the lake shore. Now a rectangle R is called a rectangular island iff there is a possible water level such that R is an island of the lake in the usual sense. There are other examples requiring only $m \times n$ cells; for example, a_{ij} may mean a colour on a gray-scale (before we convert the picture to black and white), transparency (against X-rays), or melting temperature.

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Czédli [1] has determined the maximum number of rectangular islands. In his paper, he shows that their number is $\lfloor (mn + m + n - 1)/2 \rfloor$. Pluhár [8] gave upper and lower bounds in higher dimensions. In this paper, we investigate the maximum number of triangular islands on the triangular grid (see Figure 1). Our lower bound is obtained by a construction. The key of the proof for the upper estimate is a brilliant observation of Czédli: in the rectangular case the number of islands can be measured by the area of the rectangle. We will prove that the same holds for triangular islands as well.

Czédli gave a reasonable upper bound for the number of islands using lattice theory. His methods not only apply in our case, but give an upper bound for the number of islands of arbitrary shape. Czédli's bound is sharp and without his method there is no hope to obtain a similar upper bound for the number of islands of arbitrary shape. Not many lattice theorists contributed to pure combinatorics in the last years. Nowadays, we are experiencing a change in this situation. Indeed, Czédli, Maróti and Schmidt have recently produced [1], [2] and [4]. All these three papers share the property that lattice theoretical methods and theorems are used in the proofs, but the main results are purely combinatorial and do not even mention lattices.

Our main reference will be Czédli [1]. The methods and lemmas estimating the upper bound straightforwardly apply to our case. We present these statements without their proofs in Section 3. In Section 4, we deal with triangular islands and give a lower and upper bound for their maximal number.

2. Preliminaries

On the plane, given an equilateral triangle of side length n ($n \in \mathbb{N}$), we divide this triangle into small equilateral triangles of side lengths 1 as seen in Figure 1.

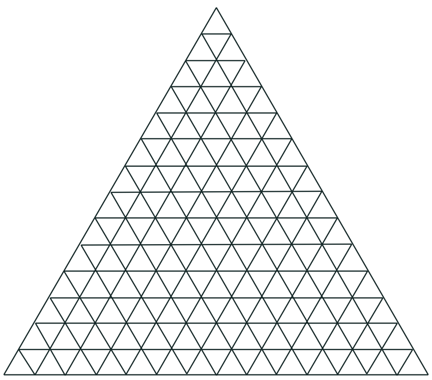

FIGURE 1

The triangles of side lengths 1 are called *cells* or *tringular units*. Denote the set of all cells by $\mathbf{T}_n$. Throughout the paper we deal with equilateral triangles consisting of triangular units from $\mathbf{T}_n$. The set of all subsets of $\mathbf{T}_n$, which consist of triangles on the plane, is denoted by $\mathbf{T}(n)$. Of course, $\mathbf{T}_n \in \mathbf{T}(n)$, but the empty set does not belong to $\mathbf{T}(n)$.

Given two distinct cells, we say that the two cells are *neighbouring*, if the distance of their center points is at most $2\sqrt{3}/3$. In other words two cells are neighbouring if they intersect by their sides or vertices. For any $T_1, T_2 \in \mathbf{T}(n)$, we say that T_1 and T_2 are *far from each other*, if they are disjoint and no cell of T_1 is neighbouring with any cell of T_2. Obviously, the role of T_1 and T_2 is symmetric in the definition.

Consider a mapping $A\colon \mathbf{T}_n \to \mathbb{R}$, $t \mapsto a_t$. For any $T \in \mathbf{T}(n)$, we call T a *triangular island of* A, if for each cell $\hat{t} \notin T$ neighbouring with a cell of T, the inequality $a_{\hat{t}} < \min\{a_t : t \in T\}$ holds. The set of triangular islands of A is denoted by $S_{\mathrm{tr}}(A)$.

3. Auxiliary statements

In this chapter we list all statements from Section 2 of [1]. These statements are valid on the triangular grid and the proofs are identical, hence we omit them.

LEMMA 1. *Let $\mathcal{H}$ be a subset of $\mathbf{T}(n)$. The following two conditions are equivalent:*

(i) *There exists a mapping A such that $\mathcal{H} = \mathcal{S}_{\mathrm{tr}}(A)$.*

(ii) *$\mathbf{T}_n \in \mathcal{H}$, and for any $T_1, T_2 \in \mathcal{H}$ either $T_1 \subseteq T_2$ or $T_2 \subseteq T_1$ or T_1 and T_2 are far from each other.*

In the sequel, subsets $\mathcal{H}$ of $\mathbf{T}(n)$ satisfying the (equivalent) conditions of Lemma 1 will be called *systems of triangular islands.*

The notion of a distributive lattice can be found e.g. in [6]. Let $L = (L; \vee, \wedge)$ be a finite distributive lattice. Following [3], a subset H of L is called *weakly independent*, if for any $k \in \mathbb{N}$ and $h, h_1, \dots, h_k \in H$ with $h \leq h_1 \vee \cdots \vee h_k$, there exists $i \in \{1, \dots, k\}$ such that $h \leq h_i$. A maximal weakly independent subset is called a *weak basis* of L. It is known from [3] that the set $J_0(L)$ of join-irreducible elements and all maximal chains are weak bases of L. The main theorem of [3] asserts that

LEMMA 2. *Any two weak bases of a finite distributive lattice have the same number of elements.*

The lattice of all subsets of $\mathbf{T}_n$ will be denoted by $\mathcal{P}(\mathbf{T}_n) = (\mathcal{P}(\mathbf{T}_n); \cup, \cap)$. Note that $\mathcal{P}(\mathbf{T}_n)$ is a finite distributive lattice.

Lemma 2 gives the following for a system of subsets of a set of size n.

LEMMA 3. *Let S be a set of size n and $\mathcal{H} \subset \mathcal{P}(S)$ such that the following property holds: for any $H_0, H_1, \dots, H_k \in \mathcal{H}$, if $H_0 \subseteq \bigcup_{i=1}^{k} H_i$ then there is a $j \in \{1, \dots, k\}$ such that $H_0 \subseteq H_j$. Then $|\mathcal{H}| \leq n$.*

This gives a straightforward upper bound for the number of triangular islands. There are n^2 unit triangles in $\mathbf{T}_n$, hence we obtain the following lemma.

LEMMA 4. *Let $\mathcal{H}$ be a system of triangular islands of $\mathbf{T}_n$. Then $\mathcal{H}$ is a weakly independent subset of $\mathcal{P}(\mathbf{T}_n)$. Consequently, $|\mathcal{H}| \leq n^2$.*

Note that Lemma 4 assumes nothing about the shape of the islands, they can be arbitrary sets of weakly independent subsets of the grid that are unions of unit triangles. This upper bound is sharp: we have n^2 islands of arbitrary shape if we assign distinct numbers to distinct triangular units. We cannot get the same upper bound if we think of a triangular island as the set of gridpoints it contains. Indeed, the gridpoints contained in smaller triangles can cover all gridpoints of a large one, thus the weak independence condition does not hold any more. We shall present a sharp upper bound for the number of triangular islands — using the geometry of islands.

4. Estimating the number of islands

Let $f(n)$ denote the maximum of the number of triangular islands on the triangular grid of side length n.

LEMMA 5. *Let $n = 2k + 2$, where $k \geq 0$, $k \in \mathbb{Z}$. Then $f(n) = f(2k + 2) \geq 3f(k) + f(k + 1) + 1$.*

PROOF. Now, we start with a triangle of side length $n = 2k + 2$. Into this triangle we draw three equilateral subtriangles of side lengths k to the vertices and one equilateral subtriangle of side length $k + 1$ into the center as shown in Figure 2. If we add 1 to the maximum number of islands corresponding to these four triangles (since $\mathbf{T}_n$ itself is also a triangular island), then this sum is clearly not greater than the maximum of the number of triangular islands of $\mathbf{T}_n$. □

LEMMA 6. *Let $n = 2k + 1$, where $k \geq 1$, $k \in \mathbb{Z}$. Then $f(n) = f(2k + 1) \geq 3f(k) + f(k - 1) + 1$.*

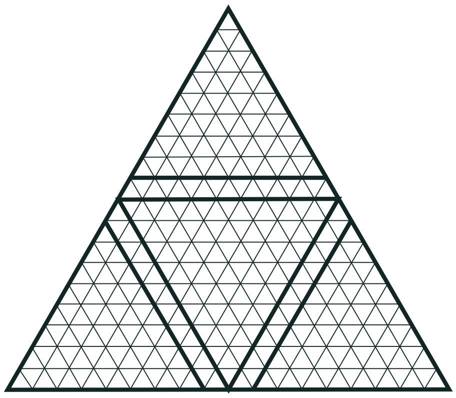

FIGURE 2

PROOF. Now, we start with a triangle of side length $n = 2k + 1$. Into this triangle, we draw three subtriangles of side lengths k to the vertices and one triangle of side length $k - 1$ into the center, as shown in Figure 3. If we add 1 to the maximum number of islands corresponding to these four triangles (since $\mathbf{T}_n$ itself is also a triangular island), then this sum is clearly not greater than the maximum of the number of triangular islands of $\mathbf{T}_n$.

□

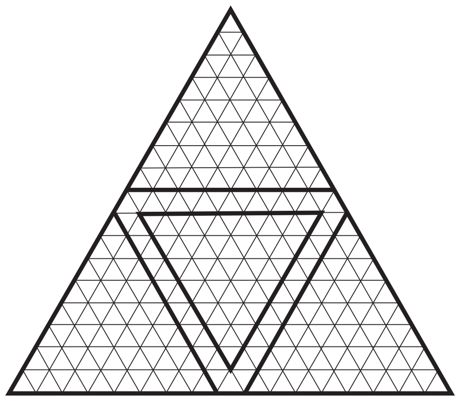

FIGURE 3

There are many other natural ways to estimate $f(n)$. For example, if we draw a triangle of side length $n - 3$ into the triangle of side length n, then it is easy to prove that $f(n) \geq f(n-3) + n + 1$. By solving this recurrence formula and examining the small cases we get that $f(n) \geq \frac{1}{6}n^2 + \frac{5}{6}n - \frac{1}{3}$.

From Lemmas 5 and 6 it is natural to examine the following sequence. Let

$s(n)$ denote the sequence defined by the following relations:

$$\begin{aligned}
&s(2n+1) = 3s(n) + s(n-1) + 1,\\
&s(2n+2) = 3s(n) + s(n+1) + 1,\\
&s(1) = 1, \quad s(2) = 2, \quad s(3) = 4.
\end{aligned}$$

As $s(i) = f(i)$ for $i = 1, 2, 3$, the sequence $s(n)$ is a lower bound for $f(n)$, that is, $s(n) \leq f(n)$ holds. The first few values of the sequence are $1, 2, 4, 6, 8, 11, 15, 19, 23, 27, \cdots$. For an arbitrary sequence $g(n)$ let Δg denote the sequence $\Delta g(n) = g(n+1) - g(n)$. For $\Delta s(n)$ we have

$$\begin{aligned}
&\Delta s\,(2n+1) = 2(s(n) - s(n-1)) = 2\Delta s(n-1),\\
&\Delta s\,(2n+2) = s(n+1) - s(n-1) = \Delta s(n) + \Delta s(n-1),\\
&\Delta s\,(1) = 1, \quad \Delta s(2) = 2, \quad \Delta s(3) = 2.
\end{aligned}$$

The first few values of this sequence are $1, 2, 2, 2, 3, 4, 4, 4, 4, 4, 5, 6, \ldots$.

For $\Delta\Delta s(n) = \Delta^2 s(n)$ we have

$$\begin{aligned}
&\Delta^2 s\,(2n+1) = \Delta^2 s(n-1),\\
&\Delta^2 s\,(2n+2) = \Delta^2 s(n),\\
&\Delta^2 s\,(1) = 1, \quad \Delta^2 s(2) = 0, \quad \Delta^2 s(3) = 0.
\end{aligned} \tag{1}$$

The first few values of the sequence $\Delta^2 s$ are

$$1,\ 0, 0, 1, 1,\ 0, 0, 0, 0, 1, 1, 1, 1,\ 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, \ldots\ .$$

The following pattern of $\Delta^2 s$ is easy to see and prove by induction: introducing $\Delta^2 s(0) = 0$, we can observe that segments of 0-s and 1-s alter in the sequence $\Delta^2 s$. The n-th segment contains 2^n 0-s and 1-s, respectively. Thus Δs can be visualized as an alternating sequence of constant functions and arithmetic progressions. The length of each sequences is an appropriate power of 2, and the difference of the arithmetic progressions is 1. From here, $s(n)$ is an alternating sequence of linear and quadratic functions. Obviously, $f(n) \geq s(n)$ holds. Hence, $f(n)$ is bounded from below by $s(n)$. By the above-mentioned arguments one might hope for quadratic lower and upper bounds for $s(n)$. After a thorough analysis of $s(n)$ we obtain the following inequalities:

$$\frac{n^2 + 3n}{5} \leq s(n) \leq \frac{3n^2 + 9n + 2}{14}. \tag{2}$$

Moreover, the lower bound is attained at the numbers of the form $n = \lfloor \frac{10}{3} 2^k - \frac{4}{3} \rfloor$ and the upper bound is attained at $n = \lfloor \frac{7}{3} 2^k - \frac{4}{3} \rfloor$. We prove that the lower bound and the upper bound are sharp at infinitely many places. This obviously implies that $\frac{n^2+3n}{5} \leq f(n)$. Later we prove that $f(n) \leq \frac{3n^2+9n+2}{14}$. These statements together imply that the quadratic upper bound in (2) is the best possible.

THEOREM 7. *We have*

$$s(n) \geq \frac{n^2+3n}{5}\,.$$

PROOF. We prove this statement via induction on n. When n is 1, 2 and 3, then $s(n)$ is at least 1, 2 and 4, respectively. This settles the small cases, $n \leq 3$.

The function $s(n)$ is integer valued, so the statement is equivalent to $s(n) \geq \lceil (n^2+3n)/5 \rceil$. Hence, if $\frac{n^2+3n}{5}$ is not an integer, then $s(n) \geq \frac{n^2+3n}{5} + \frac{1}{5}$. For $n \in \mathbb{Z}$, we have $\frac{n^2+3n}{5} \in \mathbb{Z}$ if $n \equiv 0$ or $n \equiv 2 \bmod 5$. Thus for any integer n

$$s(n) \geq \frac{n^2+3n}{5} + \frac{1}{5} \quad \text{or} \quad s(n-1) \geq \frac{(n-1)^2+3(n-1)}{5} + \frac{1}{5}$$

(or both) holds. Using Lemmas 5, 6 and the induction hypothesis for $2n+1, 2n+2 \geq 4$, we obtain the statement as follows:

$$\begin{aligned} s(2n+1) &\geq 3s(n) + s(n-1) + 1 \\ &\geq 3\frac{n^2+3n}{5} + \frac{(n-1)^2+3(n-1)}{5} + \frac{1}{5} + 1 \\ &= \frac{4n^2+10n+4}{5} = \frac{(2n+1)^2+3(2n+1)}{5}\,. \end{aligned}$$

Similarly,

$$\begin{aligned} s(2n+2) &\geq 3s(n) + s(n+1) + 1 \\ &\geq 3\frac{n^2+3n}{5} + \frac{(n+1)^2+3(n+1)}{5} + \frac{1}{5} + 1 \\ &= \frac{4n^2+14n+10}{5} = \frac{(2n+2)^2+3(2n+2)}{5}\,. \end{aligned}$$

□

As $s(n)$ is a lower bound for $f(n)$, we obtain the following corollary.

COROLLARY 8. *We have*

$$f(n) \geq \frac{n^2+3n}{5}\,.$$

THEOREM 9. *If* $n_k = \lfloor \frac{7}{3}2^k - \frac{4}{3} \rfloor$ *then*

$$s(n_k) = \frac{3n_k^2+9n_k+2}{14}\,.$$

Proof. Let n_k can be the sequence obtained by the following recurrence formulas:

$$\begin{aligned} n_k &= 2n_{k-1}+1 \ \text{ if } k \text{ is odd}, \\ n_k &= 2n_{k-1}+2 \ \text{ if } k \text{ is even}. \end{aligned} \tag{3}$$

One can easily find the following formula for n_k using the standard methods for linear recurrence formulas:

$$n_k = \left\lfloor \frac{7}{3}2^k - \frac{4}{3} \right\rfloor .$$

We prove by induction that for $n_k = \lfloor \frac{7}{3}2^k - \frac{4}{3} \rfloor$ we have

$$s(n_k) = \frac{3n_k^2 + 9n_k + 2}{14}$$

and

$$s(n_k \pm 1) = \begin{cases} \dfrac{3(n_k \pm 1)^2 + 9(n_k \pm 1) + 2}{14} - \dfrac{2}{7} & \text{if } k \text{ is even and } k \neq 0 \ (n_k \text{ even}), \\ \dfrac{3(n_k \pm 1)^2 + 9(n_k \pm 1) + 2}{14} - \dfrac{1}{7} & \text{if } k \text{ is odd } (n_k \text{ odd}). \end{cases}$$

The statement above holds for $k = 0, 1, 2$. Now, let us assume that the statement holds for $i \leq k$ and we prove it for $k+1$. We prove the statement for k even, the odd case can be handled similarly.

$$\begin{aligned} s(n_{k+1}) &= s(2n_k+1) = 3s(n_k) + s(n_k-1) + 1 \\ &= 3\frac{3n_k^2 + 9n_k + 2}{14} + \frac{3(n_k-1)^2 + 9(n_k-1) + 2}{14} - \frac{2}{7} + 1 \\ &= \frac{3n_{k+1}^2 + 9n_{k+1} + 2}{14}, \\ s(n_{k+1}+1) &= s(2n_k+2) = 3s(n_k) + s(n_k+1) + 1 \\ &= 3\frac{3n_k^2 + 9n_k + 2}{14} + \frac{3(n_k+1)^2 + 9(n_k+1) + 2}{14} - \frac{2}{7} + 1 \\ &= \frac{3(2n_k+2)^2 + 9(2n_k+2) + 2}{14} - \frac{1}{7} \\ &= \frac{3(n_{k+1}+1)^2 + 9(n_{k+1}+1) + 2}{14} - \frac{1}{7}, \\ s(n_{k+1}-1) &= s(2n_k) = 3s(n_k-1) + s(n_k) + 1 \\ &= 3\frac{3(n_k-1)^2 + 9(n_k-1) + 2}{14} + \frac{3n_k^2 + 9n_k + 2}{14} - \frac{2}{7} + 1 \\ &= \frac{3(2n_k)^2 + 9(2n_k) + 2}{14} - \frac{1}{7} \\ &= \frac{3(n_{k+1}-1)^2 + 9(n_{k+1}-1) + 2}{14} - \frac{1}{7}. \end{aligned}$$

□

Now we have arrived at the proof of our upper bound. This proof is due to the referee of the paper and it is an essential simplification of our original proof.

THEOREM 10. *We have*

$$f(n) \leq \frac{3}{14}(n+1)(n+2) - \frac{2}{7}\,.$$

PROOF. Let $\mathcal{G}_n$ be the set of grid points in Figure 1 or the set of the vertices of the triangles in $\mathbf{T}_n$ (obviously $|G_n| = (n+1)(n+2)/2$). We measure the size of a triangle T by counting the grid points covered by T. Actually, we define $\mu(T)$ as twice the number of the covered gridpoints, i.e. $\mu(T) = (l+1)(l+2)$, where l is the side length of T.

Let $\mathcal{H}$ be an arbitrary system of islands and $\max(\mathcal{H})$ is the set of maximal islands in $\mathcal{H} - \mathbf{T}_n$, $\max(\mathcal{H}) = \{R_1, \ldots, R_t\}$. By Lemma 1 the islands in $\max(\mathcal{H})$ are disjoint (far from each other) and any island is in a member of $\max(\mathcal{H})$. So we count the islands in our system as $1 + \sum_{\mathcal{R}_i \in \max \mathcal{H}} f(\mathcal{R}_i)$. To prove the theorem we need to show that for any triangle T we have $f(T) \leq 3\mu(T)/14 - 2/7$.

We use induction on n. The case $n \leq 3$ can be easily checked. If $\mathcal{H} = \{\mathbf{T}_n\}$, the claim is obvious. Otherwise, we count the islands as we did above and apply the induction hypothesis:

$$\begin{aligned} f(\mathcal{H}) &= 1 + \sum_{\mathcal{R}_i \in \max \mathcal{H}} f(\mathcal{R}_i) \\ &\leq 1 + \sum_{\mathcal{R}_i \in \max \mathcal{H}} \Big(\frac{3}{14}\mu(\mathcal{R}_i) - \frac{2}{7}\Big) = \frac{3}{14} \sum_{\mathcal{R}_i \in \max \mathcal{H}} \mu(\mathcal{R}_i) + \Big(1 - |\max(\mathcal{H})| \cdot \frac{2}{7}\Big). \end{aligned}$$

The first term can be bounded above by $\mu(\mathbf{T}_n) = (n+1)(n+2)$, the second term can bounded above by $-1/7$ assuming that $|\max(\mathcal{H})| \geq 4$. In this case we obtain (by an elementary number theoretical case analysis)

$$f(\mathcal{H}) \leq \Big\lfloor \frac{3}{14}(n+1)(n+2) - \frac{1}{7} \Big\rfloor \leq \frac{3}{14}(n+1)(n+2) - \frac{2}{7}\,.$$

In the case of $|\max(\mathcal{H})| \leq 3$ one can easily prove $\sum_{\mathcal{R}_i \in \max \mathcal{H}} \mu(\mathcal{R}_i) \leq \mu(\mathbf{T}_n)$ to get the desired bound.

□

COROLLARY 11. *For the maximum number $f(n)$ of triangular islands on the triangular grid of side length n, we have*

$$\frac{n^2+3n}{5} \leq f(n) \leq \frac{3n^2+9n+2}{14}.$$

The upper bound is the best possible quadratic upper bound.

PROOF. From Corollary 8 and Theorem 10 we have the claimed lower and upper bound for $f(n)$. Furthermore, as Theorem 9 and Theorem 10 show $s(n) \leq f(n) \leq \frac{3n^2+9n+2}{14}$ and $s(n) = \frac{3n^2+9n+2}{14}$ at infinitely many places, we get that $f(n) = \frac{3n^2+9n+2}{14}$ infinitely many times. $\square$

One might be interested in the exact value of $f(n)$. We presented this value only for a few cases, when $f(n)$ attains its maximum. Both the construction in Lemmas 5, 6 and the proof of Theorem 10 suggest that $f(n) = s(n)$.

PROBLEM 12. *Is it true that $f(n) = s(n)$ for every n?*

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