Invariance groups of threshold functions

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Permutations of variables leaving a given Boolean function $f(x_1, \ldots, x_n)$ invariant form a group, which we call the *invariance group* G of the function. We obtain that for threshold functions G is isomorphic to a direct product of symmetric groups.

A *threshold function* is a Boolean function, i.e. a mapping $\{0,1\}^n \to \{0,1\}$ with the following property: There exist real numbers $w_1, ..., w_n$, t such that

$$f(x_1, \ldots, x_n) = 1 \text{ iff } \sum_{i=1}^{n} w_i x_i \geq t,$$

where w_i is called the *weight* of x_i for $i = 1, 2, \ldots, n$, and t is a constant called the *threshold value.*We can suppose without loss of generality that

$$w_1 < w_2 < \ldots < w_n. \qquad [1], [2]$$

Throughout this paper, we use the notation: $(X) = (x_1, \ldots, x_n)$; $W = (w_1, \ldots, w_n)$; $W(X) = \sum_{i=1}^{n} w_i x_i$. Let X stand for the set consisting of the symbols $x_1, \ldots, x_n$. We define an ordering on the set X in the following way: $x_i < x_j$ iff $w_i < w_j$. For any permutation π of X, the *moving set* of π, denoted by $M(\pi)$, consists of all elements x of X satisfying $\pi(x) \neq x$. Denote by S_X the group of all permutations of the set X, and by S_k the symmetric group of degree k. If $P = (p_1, \ldots, p_n) \in \{0,1\}^n$ and $\sigma \in S_X$, then let $\sigma(P) = (\sigma(p_1), \ldots, \sigma(p_n))$ and $\sigma(X) = (\sigma(x_1), \ldots, \sigma(x_n))$.

Let $(X; \leq)$ be an ordered set. Consider a partition C of X. As usual, we shall denote the class of C that contains $x \in X$ by $\bar{x}$. We call C *convex* if $x_i \leq x_j \leq x_k$ and $\bar{x}_i = \bar{x}_k$ together imply $\bar{x}_i = \bar{x}_j$. For any convex partition C of X, the ordering of X induces an ordering of the set of blocks of C in a natural way: $\bar{x}_i \leq \bar{x}_j$ iff $x_i \leq x_j$.

Theorem 1 *For every n-ary threshold function f there exists a partition C_f of X such that the invariance group G of f consists of exactly those permutations of S_X which preserve each block of C_f .*

Conversely, for every partition C of X there exists a threshold function f_C such that the invariance group G of f_C consists of exactly those permutations of S_X that preserve each block of C.

Proof. First, consider an arbitrary n-ary threshold function f. Let us define the relation $\sim$ on the set X as follows: $i \sim j$ iff $i = j$ or f is invariant under the transposition $(x_i x_j)$. Clearly, this relation is reflexive, and symmetric. Moreover, it is transitive because

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$$(x_ix_j)(x_jx_k)(x_ix_j) = (x_ix_k).$$

Hence $\sim$ is an equivalence relation.

Claim 1. The partition C_f defined by $\sim$ is convex.

Proof. If it is not so then there exist a Boolean vector $D = (d_1, \ldots, d_n) \in \{0,1\}^n$ and $1 \le i \le j \le k \le n$ with $x_i \sim x_k$ such that

$$d + w_id_j + w_jd_i + w_kd_k < t, \tag{1}$$

$$d + w_id_i + w_jd_j + w_kd_k \ge t, \tag{2}$$

if $d = \sum_{q \neq i,j,k} c_qd_q$. Now (1) and (2) imply $d_i = 0$, $d_j = 1$. Since $x_i \sim x_k$, from (1) and (2) we infer:

$$d + w_id_k + w_jd_i + w_kd_j < t, \tag{3}$$

$$d + w_id_k + w_jd_j + w_kd_i \ge t. \tag{4}$$

Assume $d_k = 0$. Then $d + w_k < t \le d + w_j$ by (3) and (2), whence $w_k < w_j$, which is a contradiction. On the other hand, suppose $d_k = 1$. Then because of (1) and (4), $d + w_i + w_k < t \le d + w_i + w_j$, which is also a contradiction.

For the reason of convexity, the blocks of $\sim$ may be given this way:

$$\begin{aligned} C_1 &= \{x_1, \ldots, x_{i_1}\}, \\ C_2 &= \{x_{i_1+1}, \ldots, x_{i_1+i_2}\} \vdots \\ C_l &= \{x_{i_1+i_2+\ldots+i_{l-1}+1}, \ldots, x_{i_1+\ldots+i_l}\}. \end{aligned} \tag{5}$$

Every permutation that is a product of some "permitted" transpositions preserves the blocks of C_f, and belongs to G. We show that if a permutation does not preserve each blocks of C_f defined by $\sim$, then it cannot belong to G.

Lemma 1 *Let $\gamma = (x_{j_1}x_{j_2}\ldots x_{j_{k-1}}yx_{j_k}\ldots x_{j_m}) \in S_X$ be a cycle of length $m+1$ with $x_{j_s} \in C_p$, $1 \le s \le m$, $y \in C_q$, $p \neq q$. Then $\gamma \notin G$.*

Proof. Let us confine our attention to the following:

$$(yx_{j_{k-1}})(x_{j_1}x_{j_2}\ldots x_{j_{k-1}}yx_{j_k}\ldots x_{j_m}) = (x_{j_1}x_{j_2}\ldots x_{j_m})(y),$$

so

$$(yx_{j_{k-1}}) = (x_{j_1}x_{j_2}\ldots x_{j_m})(x_{j_1}x_{j_2}\ldots x_{j_{k-1}}yx_{j_k}\ldots x_{j_m})^{-1}.$$

If γ were an element of G, then $(yx_{j_{k-1}})$ would be also an element of G, which contradicts the definition of $\sim$.

Claim 1. If a cycle $\beta \in S_X$ has entries from at least two blocks of C_f, then $\beta \notin G$.

Proof. Given the convex partition C_f of $(X;\leq)$, for any cycle β of length k we construct a sequence of cycles of increasing length, called the *downward sequence* of β, as follows: Let $\bar{x}_p$, $\bar{x}_q$ ($\bar{x}_p > \bar{x}_q$) the two greatest blocks of C_f for which x_p, x_q are entries of β. We cancel some entries of β in such a way that we keep *all* entries in $\bar{x}_p$ *and* the greatest entry in $\bar{x}_q$, and we delete all the remaining entries of β. This results in the initial cycle of the downward sequence $\beta_{(r)}$ of length r; $r \geq 2$. We do not need to define members of the downward sequence with subscripts less then r. If we have constructed $\beta_{(i)}$, we obtain the next member $\beta_{(i+1)}$ of the downward sequence by taking back the greatest cancelled (and not restored yet) entry of β in its original place. Thus, the final member of the downward sequence is $\beta_{(k)} = \beta$. Let us denote by $x^{[i]}$ $(i > r)$, the "new" entry of $\beta_{(i)}$. If $i \leq r$, then we do not have to define $x^{[i]}$. As an illustration take the following:

$$X = \{x_1, \ldots, x_8\},$$

$$\begin{aligned} C_1 &= \{x_1, x_2\}, \\ C_2 &= \{x_3, x_4\}, \\ C_3 &= \{x_5, x_6, x_7\}, \\ C_4 &= \{x_8\}, \end{aligned}$$

and

$$\beta = (x_4x_5x_1x_7x_3) = (x_1x_7x_3x_4x_5).$$

The downward sequence is:

$$\beta_{(3)} = (x_7x_4x_5),$$

$$\beta_{(4)} = (x_7x_3x_4x_5), \qquad x^{[4]} = x_3,$$

$$\beta_{(5)}(= \beta) = (x_1x_7x_3x_4x_5), \qquad x^{[5]} = x_1.$$

It is obvious from the construction of the downward sequence that the weight of an arbitrary variable occuring in $\beta_{(i)}$ is not smaller than the weight of $x^{[i+1]}$. By Lemma 1, the initial cycle of the downward sequence (in our example $\beta_{(3)}$) is not in G. In order to prove that $\beta \notin G$, we show that if there exist $A_{(i)} = (a_{(i),1}, \ldots, a_{(i),n})$ and $B_{(i)} = (b_{(i),1}, \ldots, b_{(i),n})$ with $A_{(i)}, B_{(i)} \in \{0,1\}^n$ such that $f(A_{(i)}) = 0$ and $f(B_{(i)}) = 1$ and $\beta_{(i)}(A_{(i)}) = B_{(i)}$, then we are able to construct $A_{(i+1)} = (a_{(i+1),1}, \ldots, a_{(i+1),n})$ and $B_{(i+1)} = (b_{(i+1),1}, \ldots, b_{(i+1),n})$ with $A_{(i+1)}$, $B_{(i+1)} \in \{0,1\}^n$ satisfying $f(A_{(i+1)}) = 0$ and $f(B_{(i+1)}) = 1$ and $\beta_{(i+1)}(A_{(i+1)}) = B_{(i+1)}$. Let us denote with superscripts $[l(j)]$, and $[r(j)]$ the left, and the right neighbour of $x^{[j]}$ in the cycle $\beta_{(j)}$, respectively. In our example: $x^{[l(5)]} = x_5$, $x^{[r(5)]} = x_7$ because $x^{[5]} = x_1$. (For the sake of clarity: $[r([l(j)])] = [l([r(j)])] = j$; moreover, $x^{[r(j)]}$ and $x^{[j]}$ are the images of $x^{[j]}$ and $x^{[l(j)]}$, respectively.) We shall use this notation for the corresponding components of a concrete Boolean vector as well, i.e. for example: $a_{(i)}^{[l(j)]}$ and $a_{(i)}^{[r(j)]}$. We have four possibilities for $A_{(i)}$:

Case 1. $a_{(i)}^{[i+1]} = 0$, $a_{(i)}^{[r(i+1)]} = 0$.

Case 2. $a_{(i)}^{[i+1]} = 1$, $a_{(i)}^{[r(i+1)]} = 1$.

Case 3. $a_{(i)}^{[i+1]} = 1$, $a_{(i)}^{[r(i+1)]} = 0$.

Case 4. $a_{(i)}^{[i+1]} = 0$, $a_{(i)}^{[r(i+1)]} = 1$.

We show that in the first three cases A_i is appropriate for A_{i+1}. In Case 4 the only thing we have to do is to transpose two components of A_i in order to get a suitable A_{i+1}.

Case 1. $a_{(i)}^{[i+1]} = 0$, $a_{(i)}^{[r(i+1)]} = 0$.

Even though β_{i+1} bypasses $x^{[i+1]}$, $\beta_{(i+1)}(A_{(i)}) = \beta_{(i)}(A_{(i)})$ holds because $a_{(i)}^{[i+1]} = a_{(i)}^{[r(i+1)]}$. If $A_{(i+1)} = A_{(i)}$, then $\beta_{(i+1)}(A_{(i+1)}) = \beta_{(i)}(A_{(i)}) = B_{(i)}$. So let us choose $B_{(i+1)} = B_{(i)}$. Thus $f(A_{(i+1)}) = 0$, $f(B_{(i+1)}) = 1$, and $\beta_{(i+1)}(A_{(i+1)}) = B_{(i+1)}$ are satisfied.

	$x^{[l(i+1)]}$	$x^{[i+1]}$	$x^{[r(i+1)]}$
$A_{(i)}$	$a_{(i)}^{[l(i+1)]}$	0	0
$B_{(i)}$	0	0	$b_{(i)}^{[r(i+1)]}$
$A_{(i+1)}$	$a_{(i+1)}^{[l(i+1)]}$	0	0
$B_{(i+1)}$	0	0	$b_{(i+1)}^{[r(i+1)]}$

Case 2. $a_{(i)}^{[i+1]} = 1$, $a_{(i)}^{[r(i+1)]} = 1$.

The situation is the same as in Case 1: $a_{(i)}^{[i+1]} = a_{(i)}^{[r(i+1)]}$. Let $A_{(i+1)} = A_{(i)}$. Then $\beta_{(i+1)}(A_{(i+1)}) = \beta_{(i)}(A_{(i)}) = B_{(i)}$, hence let us choose $B_{(i+1)} = B_{(i)}$. Thus $f(A_{(i+1)}) = 0$, $f(B_{(i+1)}) = 1$, and $\beta_{(i+1)}(A_{(i+1)}) = B_{(i+1)}$ are satisfied for the reason as in Case 1.

	$x^{[l(i+1)]}$	$x^{[i+1]}$	$x^{[r(i+1)]}$
$A_{(i)}$	$a_{(i)}^{[l(i+1)]}$	1	1
$B_{(i)}$	1	1	$b_{(i)}^{[r(i+1)]}$
$A_{(i+1)}$	$a_{(i+1)}^{[l(i+1)]}$	1	1
$B_{(i+1)}$	1	1	$b_{(i+1)}^{[r(i+1)]}$

Case 3. $a_{(i)}^{[i+1]} = 1$, $a_{(i)}^{[r(i+1)]} = 0$.

Now, $A_{(i)}$ is appropriate for $A_{(i+1)}$ but we cannot guarantee the same for $B_{(i)}$ and $B_{(i+1)}$. Let $A_{(i+1)} = A_{(i)}$, and $B_{(i+1)} = \beta_{(i+1)}(A_{(i+1)})$. We can get the Boolean vector $B_{(i+1)}$ from $B_{(i)}$ if we transpose $b_{(i)}^{[i+1]}$ and $b_{(i)}^{[l(i+1)]}$, i.e.:

$$b_{(i+1)}^{[l(i+1)]} = 1, \text{ and } b_{(i+1)}^{[i+1]} = 0,$$

while

$$b_{(i)}^{[l(i+1)]} = 0, \text{ and } b_{(i)}^{[i+1]} = 1;$$

furthermore, all the other components of $B_{(i+1)}$ and $B_{(i)}$ are identical. Since $x^{[i+1]}$ has the smallest weight in $\beta_{(i+1)}$, we get

$$\sum_{j=1}^{n} w_j b_{(i),j} \leq \sum_{j=1}^{n} w_j b_{(i+1),j},$$

which means that $f(B_{(i+1)}) = 1$. Moreover, $f(A_{(i+1)}) = 0$, and $\beta_{(i+1)}(A_{(i+1)}) = B_{(i+1)}$ are satisfied.

	$x^{[l(i+1)]}$	$x^{[i+1]}$	$x^{[r(i+1)]}$
$A_{(i)}$	$a_{(i)}^{[l(i+1)]}$	1	0
$B_{(i)}$	0	1	$b_{(i)}^{[r(i+1)]}$
$A_{(i+1)}$	$a_{(i+1)}^{[l(i+1)]}$	1	0
$B_{(i+1)}$	1	0	$b_{(i+1)}^{[r(i+1)]}$

Case 4. $a_{(i)}^{[i+1]} = 0$, $a_{(i)}^{[r(i+1)]} = 1$.

Let us construct $A_{(i+1)}$ from $A_{(i)}$ as follows: Put $a_{(i+1)}^{[i+1]} = 1$, $a_{(i+1)}^{[r(i+1)]} = 0$, $a_{(i+1),j} = a_{(i),j}$ if $a_{(i+1),j} \neq a_{(i+1)}^{[i+1]}$ or $a_{(i+1),j} \neq a_{(i+1)}^{[r(i+1)]}$. (Transpose $a_{(i)}^{[i+1]}$ and $a_{(i)}^{[r(i+1)]}$ in the Boolean vector $A_{(i)}$ (and keep all the other components of it unchanged) to get $A_{(i+1)}$.) Since $x^{[i+1]}$ has the smallest weight in $\beta_{(i+1)}$, we get

$$\sum_{j=1}^{n} w_j a_{(i+1),j} \leq \sum_{j=1}^{n} w_j a_{(i),j};$$

hence $f(A_{(i+1)}) = 0$. Let $B_{(i+1)} = \beta_{(i+1)}(A_{(i+1)})$. With this choice $B_{(i+1)} = B_i$, hence $f(B_{(i+1)}) = 1$.

	$x^{[l(i+1)]}$	$x^{[i+1]}$	$x^{[r(i+1)]}$
$A_{(i)}$	$a_{(i)}^{[l(i+1)]}$	0	1
$B_{(i)}$	1	0	$b_{(i)}^{[r(i+1)]}$
$A_{(i+1)}$	$a_{(i+1)}^{[l(i+1)]}$	1	0
$B_{(i+1)}$	1	0	$b_{(i+1)}^{[r(i+1)]}$

Claim 2 is proved.

Every permutation that is a product of disjoint cycles such that any of them preserves each blocks of C_f belongs to the invariance group G of f. We have to show, that if not all of the factors have this property, then the permutation does not leave the threshold function f invariant.

Lemma 2 *Let $\pi \in S_X$ of the form $\pi = \pi_2\pi_1$, where $\pi_1, \pi_2 \in S_X$, with $M(\pi_1) \cap M(\pi_2) = \emptyset$ and $\pi_1 \notin G$. Then $\pi \notin G$.*

Proof. Suppose that it is not so, i.e. $\pi \in G$. Now $\pi_1 \notin G$ means that there exist $X_0, X_1 \in \{0,1\}^n$ with $f(X_0) = 0, f(X_1) = 1$, and $\pi_1(X_0) = X_1$. Let $X_2 = \pi_2(X_1)$, i.e. $X_2 = \pi(X_0)$. Since $f(X_2) = 1$ contradicts the assumption $\pi \in G$, we infer $f(X_2) = 0$. Let $X_3 = \pi_1(X_2)$. As $M(\pi_1) \cap M(\pi_2) = \emptyset$, we have $\pi_1\pi_2 = \pi_2\pi_1$. Therefore $X_3 = \pi(X_1)$. The assumption $\pi \in G$ implies $f(X_3) = 1$. Looking at the infinite series of Boolean vectors

$$X_0, X_1, \ldots, X_n, \ldots$$

we can establish in the same way that if $i = 2k$, $k \in \mathbf{N}$, then $f(X_i) = 0$, while if $i = 2k+1$ then $f(X_i) = 1$. On the other hand,

$$W(X) = S(X)^{[1]} + S(X)^{[2]} + S(X)^{[3]},$$

where $S(X)^{[1]} = \sum_{x_j \in M(\pi_1)} w_j x_j$, $S(X)^{[2]} = \sum_{x_j \in M(\pi_2)} w_j x_j$, $S(X)^{[3]} = \sum_{x_j \notin M(\pi)} w_j x_j$. With this notation: $S(X_0)^{[1]} < S(X_1)^{[1]}$, $S(X_0)^{[2]} = S(X_1)^{[2]}$, $S(X_0)^{[3]} = S(X_1)^{[3]}$. For the series of $S(X_i)^{[1]}$:

$$S(X_0)^{[1]} < S(X_1)^{[1]} = S(X_2)^{[1]} < S(X_3)^{[1]} = S(X_4)^{[1]} < \ldots, \tag{6}$$

as applying π_2 changes only $S(X_i)^{[2]}$; moreover, $f(X_{2k}) = 0$ and $f(X_{2k+1}) = 1$ imply $W(X_{2k}) < W(X_{2k+1})$, hence $S(X_{2k})^{[1]} < S(X_{2k+1})^{[1]}$. On the other hand, if z is the order of π_1, then $S(X_0)^{[1]} = S(X_{2z})^{[1]}$, which contradicts (6).

Claim 3. For $\pi \in S_X$, let $\pi = \gamma_1 \ldots \gamma_r$ where γ_i are disjoint cycles. If there exists a γ_j with $1 \le j \le r$ and $\gamma_j \notin G$, then $\pi \notin G$.

Proof. It is easy to see if there is only one such γ_j. If there is more, then $\pi \notin G$ is an immediate consequence of Lemma 2.

Claim 1, Claim 2, and Claim 3 together provide a proof of the first part of the Theorem.

For proving the converse of the theorem, we show first that for any n there exist a n-ary threshold function which is *rigid* in the sense that its invariance group has only one element (the identity permutation).

Suppose n is odd. With $n = 2k+1$, consider the following weights:

w_1	w_2	$\ldots$	w_k	w_{k+1}	w_{k+2}	$\ldots$	w_{2k}	w_{2k+1}
$-k$	$-k+1$	$\ldots$	-1	0	1	$\ldots$	$k-1$	k

(7)

Let $t = 0$. We prove that for any transposition τ of form $(x_j x_{j-1})$ where $2 \le j \le n$ there exists a Boolean vector $U = (u_1, \ldots, u_n) \in \{0,1\}^n$ such that $f(U) = 1$ and

$f(\tau(U)) = 0$. For a fixed j let $u_j = 1$, $u_{n+1-j} = 1$, $u_i = 0$ if $i \neq j$, $i \neq n+1-j$. It is obvious that $f(U) = 1$; however, $f(\tau(U)) = 0$. Hence f is rigid.

If $n = 2k$, then the weights can be chosen as

w_1	w_2	...	w_{k-1}	w_k	w_{k+1}	w_{k+2}	...	w_{2k-1}	w_{2k}
$-k$	$-k+1$	...	-2	-1	1	2	...	$k-1$	k

(8)

Let $t = 0$. The method is almost the same as before, i.e. consider the following $U = (u_1, \ldots, u_n)$.: If $j \neq k+1$ then let $u_j = 1$, $u_{n+1-j} = 1$, $u_i = 0$ if $i \neq j, -j$. If $j = k+1$ then let $u_{k+1} = 1$ and $u_i = 0$ if $i \neq k+1$. If $\tau = (x_j x_{j-1})$, where $2 \leq j \leq n$, then $f(U) = 1$ while $f(\tau(U)) = 0$.

Now, we construct a threshold function g_C for an arbitrary partition C of an arbitrary ordered set X of variables. Denote now by $\sim^*$ the equivalence relation on X defined by C. First, suppose that C is convex. Let $i_1, \ldots, i_l$ denote the number of elements of the blocks of C, respectively. Consider the rigid function f of l variables that is defined in (7) or (8), depending on the parity of l. Take the weight w_1 i_1 times, the weight w_2 i_2 times and so on in order to define a threshold function g of $n = i_1 + i_2 + \ldots + i_l$ variables. Variables of g with the same weight are permutable. However, transpositions σ of form $(x_j x_{j-1})$, where $2 \leq j \leq n$ and $j \not\sim^* j-1$, are "forbidden" for g because if we consider the corresponding U and construct a Boolean vector V of dimension n from U by rewriting it in the following way: instead of u_m $(m = 1, \ldots, l)$, write 0 i_m times, whenever $u_m = 0$; and write 1 (once) then 0 $i_m - 1$ times otherwise; then we shall get a Boolean vector V of dimension n, for which $g(V) = 1$ while $g(\sigma(V) = 0$. If C is not convex, the only thing we have to do is to reindex the variables in order to get a convex partition. After constructing a threshold function for the rearranged variables with the procedure described above, put the original indexes back and the desired threshold function is ready. Theorem is proved.

The invariance group G_B of an arbitrary Boolean function is not necessarily of the form

$$G_B \cong S_{i_1} \times \ldots \times S_{i_l}. \tag{9}$$

For example, let h be the following: $h(x_1, \ldots, x_n) = 1$ iff there exists i such that $x_i = 1$, $x_{i\oplus 1} = 1$, $x_j = 0$ if $j \neq i, i+1$ where $\oplus$ means addition *mod* n. The invariance group of h contains the cycle $(x_1, \ldots, x_n)$ and its powers but it does not contain transpositions of form $(x_i x_{i+1})$.

However, there exist Boolean functions with invariance groups of the form (9), which are not threshold functions.

Permutable variables of a threshold function does not mean equal weights. Here is an example: $h(x) = x_1 x_2 x_4 \vee x_3 x_4$. This is a threshold function with the following weights, and threshold value:

w_1	w_2	w_3	w_4	t
1	2	3	4	7

The transposition $(x_1 x_2)$ is "permitted" but the others are not.

But the weights can always be chosen to be identical for variables belonging to the same equivalence class. If the j-th class $C_j = \{x_{i_1+i_2+\ldots+i_{j-1}+1}, \ldots, x_{i_1+\ldots+i_j}\}$ by te notation of (5), then let $w_{[j]} = \frac{w_{i_1+i_2+\ldots+i_{j-1}+1} + \ldots + w_{i_1+\ldots+i_j}}{i_j}$. Replace

$w_{i_1+i_2+\ldots+i_{j-1}+1}, \ldots, w_{i_1+\ldots+i_j}$, by $w_{[j]}$. Since $x_{i_1+i_2+\ldots+i_{j-1}+1}, \ldots, x_{i_1+\ldots+i_j}$ are from the same equivalence class, for fixed x_1, $\ldots$,$x_{i_1+\ldots i_{j-1}}$ and $x_{i_1+\ldots i_j+1}, \ldots x_{i_1+\ldots i_l}$, the fact that $W(X)$ exceeds t (or not) depends only on the number r of 1-s among the coordinates $x_{i_1+i_2+\ldots+i_{j-1}+1}, \ldots, x_{i_1+\ldots+i_j}$; moreover, $W(X)$ has a maximum (minimum) if we put all our 1-s to places with the greatest (smallest) weights possible. Obviously

$$\frac{w_{i_1+\ldots i_{r-1}+1} + \ldots + w_{i_1+\ldots i_{j-1}+1+r}}{r} \leq w_{[j]};$$

moreover,

$$w_{[j]} \leq \frac{w_{i_1+\ldots i_j-r} + \ldots + w_{i_1+\ldots i_j}}{r}.$$

Hence

$$w_{i_1+\ldots i_{r-1}+1} + \ldots + w_{i_1+\ldots i_{j-1}+1+r} \leq rw_{[j]} \leq w_{i_1+\ldots i_j-r} + \ldots + w_{i_1+\ldots i_j}.$$

Consequently, after replacing $w_{i_1+i_2+\ldots+i_{j-1}+1}, \ldots, w_{i_1+\ldots+i_j}$ by $w_{[j]}$, we still have the same threshold function.

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References

[1] Algebraic aspects of threshold logic Russian Kibernetika (Kiev), p. 26–30, 1980

[2] N. N. Aĭzenberg, A. A. Bovdi, È. I. Gergo , F. È. Geche English transl. in Cybernetics vol. 16, 1980 p. 188–193

[3] Béla Bódi, Elemér Hergő, Ferenc Gecseg A lower bound of the number of threshold functions Trans.IEEE,EC-14 6, 1965, p. 926–929

[4] S. Yajima and T. Ibaraki Threshold logic

[5] H.G. Booker and N. DeClaris Academic Press, London and New York Ching Lai Sheng 1969

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