

# CUT APPROACH TO ISLANDS IN RECTANGULAR FUZZY RELATIONS

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ABSTRACT. The paper investigates fuzzy relations on a finite domain in the cutworthy framework, dealing with a new property coming from the information theory. If the domain of a relation is considered to be a table, then a rectangular subset of the domain whose values under this relation are greater than the values of all neighboring fields is called an island. Consequently, the so called rectangular fuzzy relations are introduced; their cuts consist of rectangles as sub-relations of the corresponding characteristic functions. A characterization theorem for rectangular fuzzy relations is proved. We also prove that for every fuzzy relation on a finite domain, there is a rectangular fuzzy relation with the same islands, and an algorithm for a construction of such fuzzy relations is presented. In addition, using methods developed for fuzzy structures and their cuts, we prove that for every fuzzy relation there is a lattice and a lattice valued relation whose cuts are precisely the islands of this relation. A connection of the notion of an island with formal concept analysis is presented.

## 1. INTRODUCTION

It is well known that fuzzy relations in many cases can handle real life problems better than the crisp ones. Good examples are similarities and related fuzzy relations, than also pre-orderings and orders etc.; there are many results supporting this, see e.g., [2, 3, 4, 10, 16, 21], and references in these.

Our aim is to present a relational property that can be suitably applied as well, if adopted to fuzzy case. To explain our motivation, let us consider a finite crisp relation  $\rho$  on a domain  $A \times B$ ,  $A$  and  $B$  being linearly ordered:  $A = \{a_1, \dots, a_m\}$ ,  $B = \{b_1, \dots, b_n\}$ . The relation  $\rho$  can be given by the characteristic function organized as a table  $A \times B$ , with entries from the set

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*Date:* May 21, 2010.

*Key words and phrases.* Islands, Fuzzy relations.

This research of the first author was partially supported by the NFSR of Hungary (OTKA), grant no. T 049433.

This research of the second and third authors was partially supported by Serbian Ministry of Science and Environment, Grant No. 144011.

This research of all authors was partially supported by the Provincial Secretariat for Science and Technological Development, Autonomous Province of Vojvodina, grant "Lattice methods and applications".

$\{0, 1\}$ . We are interested in relations whose values form disjoint rectangles in the table, like the one given in Table 1.

$b_3$	1	1	0	1	1
$b_2$	0	0	0	1	1
$b_1$	0	0	0	1	1
	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$

Table 1

Similar notions (obtained by applying Galois connections to experimental data) appear for instance in modeling concepts and attributes (see e.g., the book [9] about Formal Concept Analysis). In fuzzy settings, analogous properties can be investigated for any fuzzy relation. Namely, we identify so-called islands of a fuzzy relation. These are rectangles in the domain with values greater than the values of all neighboring cells. Similar approach to fuzzy relations (but in different settings) exists in Fuzzy Formal Concept Analysis (e.g., [2, 18]). Apart from formal concepts, an interest in identifying islands in fuzzy relations can be found in pattern recognition, or in any visual identification of rectangular shapes when the data are not clear.

The notion of an island comes from the information theory. The characterization of the lexicographical length sequences of binary maximal instantaneous codes in [8] uses the notion of *full segments*, which are one-dimensional islands. Several generalizations of this notion gave interesting combinatorial problems. The maximum number of rectangular and square islands, as well as the upper and the lower bounds of such numbers are being determined. These combinatorial results (in one, two or several dimensions) can be found in [1, 5, 11, 12, 14, 15, 17]. We mention, that in [6] and in [7] the results about islands motivated lattice theoretical investigations and results.

Let us highlight the fact that fuzzy approach is very natural in the research of islands. Namely, up to now, in these investigations heights and water levels play crucial role; the heights here correspond to the values of the fuzzy relation and water level correspond to the p-cuts. In this context there are many open problems posed in the mentioned published papers about islands. We do believe that fuzzy approach gives a useful technique for dealing with these problems.

In the present paper, apart from using the notion of an island (adapted to the concept of fuzzy relation), we do not deal with the foregoing combinatorial problems. Since islands can be identified in any fuzzy relation, we investigate this new, relatively unknown property in the purely fuzzy, cutworthy settings. Our aim is to describe it in terms of fuzzy structures, to connect it to crisp relations, and to develop techniques by which this property can be applied.

First we prove that for every finite fuzzy relation, there is another fuzzy relation, called rectangular, whose cuts considered as characteristic functions consist only of distant rectangles which are associated to islands; these cuts

are relations described above and presented by an example in Table 1. We give an algorithm to construct a rectangular relation having the same islands as a given fuzzy relation. Further, we prove the characterization theorem for rectangular relations by metric properties of its domain. Finally, we were able to prove that for every fuzzy relation  $R : A \times B \rightarrow [0, 1]$  there is a lattice  $L$  and an  $L$ -valued relation  $\rho : A \times B \rightarrow L$  whose cuts are precisely the islands of  $R$ .

After preliminary section, there is a technical, but necessary part (Section 3) in which we analyze metric properties of the (ordered) domain of an arbitrary finite fuzzy relation. Our main results, briefly described above, are contained in Sections 4 and 5. In Section 6 we explain a connection between islands and the formal concepts. In Conclusion, we point to some applications and further research.

## 2. PRELIMINARIES

We advance some necessary notions and notations concerning fuzzy relations, more details could be found in e.g., [2, 13, 19, 20].

Let  $A$  and  $B$  be nonempty sets and  $L$  a complete lattice, in particular it can be the unit interval  $[0, 1]$  of real numbers under the natural order  $\leq$ . Then a **fuzzy relation**  $\rho$  is a mapping from  $A \times B$  to  $L$ ,  $\rho : A \times B \rightarrow L$ .

For every  $p \in L$ , the **cut relation**,  $p$ -**cut** of  $\rho$  is an ordinary relation  $\rho_p$  on  $A \times B$  defined by

$$(x, y) \in \rho_p \text{ if and only if } \rho(x, y) \geq p.$$

Obviously, for  $p \in L$ , a  $p$ -cut of  $\rho$  is the inverse image of the filter  $\uparrow p$  in  $L$ :

$$\rho_p = \rho^{-1}(\uparrow p).$$

Properties of cut relations:

- (1) If  $p \leq q$  then  $\rho_q \subseteq \rho_p$ .
- (2) for every pair  $(x, y) \in A \times B$

$$\rho(x, y) = \bigvee \{p \in L \mid (x, y) \in \rho_p\}.$$

- (3) Let  $K$  be a nonempty subset of  $L$ . Then

$$\bigcap \{\rho_p \mid p \in K\} = \rho_{\bigvee \{p \mid p \in K\}}.$$

Observe that by the definition of a  $p$ -cut, every fuzzy relation  $\rho : A \times B \rightarrow L$  determines a collection of cuts  $\{\rho_p \mid p \in L\}$  which is ordered by inclusion.

Conversely, property (2) enables a synthesis of a fuzzy relation by the corresponding collection of cut-relations.

In this paper we consider finite fuzzy relations  $A \times B \rightarrow [0, 1]$  and assume that  $A$  and  $B$  are linearly ordered, therefore we take  $A = \{1, \dots, m\}$ ,  $B = \{1, \dots, n\}$ , for arbitrary positive integers  $m, n$ . This assumption is kept throughout the paper. As usual, the co-domain is a lattice  $([0, 1], \leq)$  in which suprema and infima are max and min, respectively.

Particular fuzzy properties of this relation, being the main topic of this paper, are investigated in Section 4.

In Section 5 we switch to lattice valued relations, using properties (1) – (3) listed above.

Some notation and metric properties of the domain  $A \times B$  are presented in the next section.

### 3. DOMAIN OF THE FUZZY RELATION

In this part we deal with the domain of the foregoing fuzzy relation and we analyze its metric properties. These are rather simple and technical, but necessary for the investigation of our main topic - islands, corresponding fuzzy relations and their cuts.

In what follows, the set  $\{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ ,  $m, n \in \mathbb{N}$ , is called a **table of size**  $m \times n$ . Fields in the table are **cells**, which are denoted by the corresponding ordered pairs  $(i, j)$ ,  $i \in \{1, \dots, m\}$ ,  $j \in \{1, \dots, n\}$ .

For  $1 \leq \alpha \leq \beta \leq m$  and  $1 \leq \gamma \leq \delta \leq n$ , the set  $\{\alpha, \dots, \beta\} \times \{\gamma, \dots, \delta\}$  is called a **rectangle** in the table. In particular, a  $p \times p$  rectangle  $p \in \{1, \dots, \min\{m, n\}\}$  is a **square** in the table. A cell is a *trivial* square.

It is straightforward to show that the intersection of two rectangles (of a family of rectangles) is either the empty set or it is another rectangle.

The least rectangle that contains the subset  $X$  of table cells is denoted by  $R(X)$ . There is always such a rectangle, e.g., the table itself containing this subset. Obviously,  $R(X)$  is the intersection of all rectangles containing  $X$ .

If  $(i, j)$  and  $(k, l)$  are two cells then their **distance** is defined in a usual way by  $\sqrt{(i - k)^2 + (j - l)^2}$ .

Two different cells with distance at most  $\sqrt{2}$  are called **neighboring** cells. If  $(i, j)$  and  $(k, l)$  are neighboring cells, we also say that  $(i, j)$  is neighboring to  $(k, l)$  and we denote this by  $(i, j)N(k, l)$ . Observe that  $N$  is a symmetric relation on  $\{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ .

A cell  $(k, l)$  is neighboring to a rectangle  $R$  if  $(k, l) \notin R$  and there is a cell  $(i, j) \in R$  such that  $(i, j)N(k, l)$ .

We say that two rectangles  $\{\alpha, \dots, \beta\} \times \{\gamma, \dots, \delta\}$  and  $\{\alpha_1, \dots, \beta_1\} \times \{\gamma_1, \dots, \delta_1\}$  are **distant** if they are disjoint (in set theoretic sense) and for every two cells, namely  $(a, b)$  from the first rectangle and  $(c, d)$  from the second, we have  $(a - c)^2 + (b - d)^2 \geq 4$ .

We define a cell  $(\mu, \nu)$  to be **between**  $(\alpha, \gamma)$  and  $(\beta, \delta)$  if it is different from both of these cells and

- (1) If  $\alpha = \beta$ , and  $\gamma < \delta$ , then  $\mu = \alpha$  and  $\gamma < \nu < \delta$ . (If  $\alpha = \beta$ , and  $\delta < \gamma$ , then  $\delta < \nu < \gamma$ ).
- (2) If  $\alpha < \beta$ , then

$$\left[\mu - \frac{1}{2}, \nu - \frac{1}{2}\right] \times \left[\mu + \frac{1}{2}, \nu + \frac{1}{2}\right] \cap \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid \alpha < x < \beta \wedge$$

$$y = \frac{\gamma - \delta}{\alpha - \beta} \cdot x + \gamma - \alpha \cdot \frac{\gamma - \delta}{\alpha - \beta} \} \neq \emptyset$$

(in case  $\beta < \alpha$  it is defined analogously.)

**Lemma 1.** *If  $(\alpha, \gamma)$  and  $(\beta, \delta)$  are two cells, and if  $\alpha + 1 < \beta$ , then for all  $\alpha + 1, \dots, \alpha + k = \beta - 1$  there are  $y_1, \dots, y_k$  such that  $(\alpha + i, y_i)$ , for  $i = 1, \dots, k$ , are between  $(\alpha, \gamma)$  and  $(\beta, \delta)$ .*

*Proof.* Since  $\alpha \neq \beta$ , then we consider the set

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid \alpha < x < \beta \wedge y = \frac{\gamma - \delta}{\alpha - \beta} \cdot x + \gamma - \alpha \cdot \frac{\gamma - \delta}{\alpha - \beta}\},$$

where  $x$  is equal to  $\alpha + 1, \dots, \alpha + k = \beta - 1$ , respectively. The resulting set is

$$\{(\alpha + i, \gamma + \frac{\gamma - \delta}{\alpha - \beta} \cdot i) \mid i = 1, \dots, k\}.$$

If  $\lfloor \gamma + \frac{\gamma - \delta}{\alpha - \beta} \cdot i \rfloor = \lfloor \gamma + \frac{\gamma - \delta}{\alpha - \beta} \cdot i - 0.5 \rfloor$ , then  $y_i = \lfloor \gamma + \frac{\gamma - \delta}{\alpha - \beta} \cdot i \rfloor + 1$ .

If  $\lfloor \gamma + \frac{\gamma - \delta}{\alpha - \beta} \cdot i \rfloor > \lfloor \gamma + \frac{\gamma - \delta}{\alpha - \beta} \cdot i - 0.5 \rfloor$ , then  $y_i = \lfloor \gamma + \frac{\gamma - \delta}{\alpha - \beta} \cdot i \rfloor$ .  $\square$

A statement similar to Lemma 1 in which we consider second coordinates is also valid.

**Lemma 2.** *If two rectangles  $R_1$  and  $R_2$  on the table are distant, then there is either an  $f \in \{1, \dots, m\}$  such that the  $f$ -column is a rectangle disjoint with both,  $R_1$  and  $R_2$ , or there is a  $g \in \{1, \dots, n\}$  such that the  $g$ -row is a rectangle disjoint with both,  $R_1$  and  $R_2$ . In addition, this column (row) rectangle contains a cell between any two cells, the first being from  $R_1$  and the second from  $R_2$ .*

*Proof.*

- Case 1: There is an  $a \in \{1, \dots, m\}$  such that there are  $c, d \in \{1, \dots, n\}$ , with  $(a, c) \in R_1$  and  $(a, d) \in R_2$ .

- Case 2: There is a  $d \in \{1, \dots, n\}$  such that there are  $a, b \in \{1, \dots, m\}$  with  $(a, d) \in R_1$  and  $(b, d) \in R_2$ .

- Case 3: For all  $(a, b) \in R_1$  and  $(c, d) \in R_2$ ,  $a \neq c$  and  $b \neq d$ .

Proof in Case 1: Let  $c \leq d$  (similar proof is in case  $d \leq c$ ). Then, let  $c_1 = \max\{c \mid (a, c) \in R_1\}$  and let  $d_1 = \min\{d \mid (a, d) \in R_2\}$ . Then,  $c_1 < d_1$  and  $d_1 - c_1 \geq 2$  since the rectangles are distant. Then,  $c_1 + 1$  is the required element  $f$ .

Proof of Case 2 is similar to Case 1.

- Case 3: Let  $(a_1, b_1) \in R_1$  and  $(c_1, d_1) \in R_2$  be such elements for which  $(a - c)^2 + (b - d)^2$  reaches its minimum. Let  $a_1 < c_1$  and  $b_1 < d_1$ . Since  $(a_1 - c_1)^2 + (b_1 - d_1)^2 \geq 4$ , then the absolute value of some of the two differences is greater than 1. Let  $c_1 - a_1 \geq 2$ . Then  $a_1 + 1 < c_1$  and thus  $a_1 + 1$  is the required element  $f$ .  $\square$

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<sup>1</sup> $\lfloor x \rfloor$  denotes the lower integer part of  $x$ , or in other words the greatest integer less or equal to  $x$ .

**Lemma 3.** *Let  $R$  be a rectangle and let  $(\alpha, \gamma)$  and  $(\beta, \delta)$  be two cells, such that  $(\alpha, \gamma) \in R$  and  $(\beta, \delta) \notin R$ . Then, either  $(\beta, \delta)$  is neighboring to  $R$  or there is a cell  $(\mu, \nu)$  neighboring to  $R$  such that  $(\mu, \nu)$  is between  $(\alpha, \gamma)$  and  $(\beta, \delta)$ .*

*Proof.* Suppose that  $(\beta, \delta)$  is not neighboring to  $R$ . Suppose that  $\alpha + 1 < \beta$  (other three similar cases ( $\beta + 1 < \alpha$ ,  $\gamma + 1 < \delta$  and  $\delta + 1 < \gamma$ ) are treated analogously.) In case  $\gamma = \delta$ , the cell  $(\alpha + 1, \gamma)$  is neighboring to  $R$  and it is between  $(\alpha, \gamma)$  and  $(\beta, \delta)$ . Now let  $\gamma < \delta$  (the case  $\gamma > \delta$  is treated in a similar way). Since cells  $(\alpha, \gamma)$  and  $(\beta, \delta)$  are distant, we can apply Lemma 1 or a statement analogue to Lemma 1 in which we consider the second coordinates. In case  $\frac{\gamma - \delta}{\alpha - \beta} \leq 1$  we consider directly Lemma 1 and in case  $\frac{\gamma - \delta}{\alpha - \beta} > 1$ , we use the statement analogue to Lemma 1 in which the second coordinates are considered. Suppose that  $\frac{\gamma - \delta}{\alpha - \beta} \leq 1$ . Now, for all  $\alpha + 1, \dots, \alpha + k = \beta - 1$ , there are  $y_1, \dots, y_k$  such that  $(\alpha + i, y_i)$ , for  $i = 1, \dots, k$ , are between  $(\alpha, \gamma)$  and  $(\beta, \delta)$ . According to Lemma 1,  $y_i = \lfloor \gamma + \frac{\gamma - \delta}{\alpha - \beta} \cdot i + 0.5 \rfloor$ . Further, there is a  $j \in \{1, \dots, k\}$ , such that  $(\alpha + j, y_j) \in R$  and  $(\alpha + j + 1, y_{j+1}) \notin R$ . We prove that  $(\alpha + j, y_j)$  and  $(\alpha + j + 1, y_{j+1})$  are neighboring cells. Indeed,  $y_{j+1} - y_j = \lfloor \gamma + \frac{\gamma - \delta}{\alpha - \beta} \cdot (j + 1) + 0.5 \rfloor - \lfloor \gamma + \frac{\gamma - \delta}{\alpha - \beta} \cdot j + 0.5 \rfloor \leq 1$ , since  $\frac{\gamma - \delta}{\alpha - \beta} \leq 1$ .  $\square$

#### 4. FUZZY RELATION, ISLANDS AND CORRESPONDING CUTS

Let  $\Gamma : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow [0, 1]$  be a fuzzy relation. We say that the rectangle  $T = \{\alpha, \dots, \beta\} \times \{\gamma, \dots, \delta\}$  is an **island** of  $\Gamma$ , if for every cell  $(\mu, \nu)$  which does not belong to this rectangle but is neighboring to some cell of the rectangle, we have

$$\Gamma(\mu, \nu) < \min_{(x,y) \in T} \Gamma(x, y).$$

Throughout the paper we accept the convention that the table itself (the set  $\{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ ) is also an island, called also the trivial island.

**Lemma 4.** *If  $T$  is an island of a fuzzy relation  $\Gamma$ , then there is  $p \in [0, 1]$  and a cut relation  $\Gamma_p$ , such that  $T \subseteq \Gamma_p$  and no cell neighboring to  $T$  belongs to  $\Gamma_p$ .*

*Proof.* By the definition of a cut and of an island, it is straightforward that the cut  $\Gamma_p$ , with  $p = \min\{\Gamma(x, y) \mid (x, y) \in T\}$ , fulfills the requirements.  $\square$

The following are also properties of islands.

- (i) *Two different islands on the same table are either disjoint, or one is contained in the other.*
- (ii) *The greatest (under inclusion) island is the table itself.*
- (iii) *Every non-trivial island (different from the table itself), is contained in a maximal island. Maximal islands are non-trivial islands which are not properly contained in other non-trivial islands.*

The first two of the properties above are straightforward. In order to prove property (iii), we note that if there are non-trivial islands, then every island is either maximal or it is contained in another non-trivial island. Since we consider finite tables, every ascending chain of non-trivial islands has the greatest element and those greatest elements are maximal islands.

A fuzzy relation  $\Gamma$  is called **rectangular** if for every  $p \in [0, 1]$  such that the corresponding  $p$ -cut of  $\Gamma$  is not empty, this  $p$ -cut is a union of distant rectangles (distant rectangles are defined in the previous section). In the sequel we give the *characterization theorem* for the rectangular fuzzy relations.

**Theorem 1.** *A fuzzy relation  $\Gamma : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow [0, 1]$  is rectangular if and only if for all  $(\alpha, \gamma), (\beta, \delta) \in \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$  either*

- *these are not neighboring cells and there is a cell  $(\mu, \nu)$  between  $(\alpha, \gamma)$  and  $(\beta, \delta)$  such that  $\Gamma(\mu, \nu) < \min\{\Gamma(\alpha, \gamma), \Gamma(\beta, \delta)\}$ , or*
- *for all  $(\mu, \nu) \in [\min\{\alpha, \beta\}, \max\{\alpha, \beta\}] \times [\min\{\gamma, \delta\}, \max\{\gamma, \delta\}]$ ,*

$$(1) \quad \Gamma(\mu, \nu) \geq \min\{\Gamma(\alpha, \gamma), \Gamma(\beta, \delta)\}.$$

*Proof.* Suppose that for every  $p \in [0, 1]$ ,  $\Gamma_p$  is a union of distant rectangles. Now let  $(\alpha, \gamma)$  and  $(\beta, \delta)$  be any two cells in  $\{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ . Let  $\Gamma(\alpha, \gamma) = p_1$  and  $\Gamma(\beta, \delta) = p_2$ . Let  $q = \min\{p_1, p_2\}$ . Then,  $(\alpha, \gamma)$  and  $(\beta, \delta)$  belong to  $\Gamma_q$ . Since every cut is a union of distant rectangles, then  $(\alpha, \gamma)$  and  $(\beta, \delta)$  either belong to the same rectangle or to distant rectangles. If they belong to the same rectangle, then for all  $(\mu, \nu) \in [\min\{\alpha, \beta\}, \max\{\alpha, \beta\}] \times [\min\{\gamma, \delta\}, \max\{\gamma, \delta\}]$ ,

$$\Gamma(\mu, \nu) \geq q = \min\{\Gamma(\alpha, \gamma), \Gamma(\beta, \delta)\}.$$

If  $(\alpha, \gamma)$  and  $(\beta, \delta)$  belong to distant rectangles  $R_1$  and  $R_2$  respectively, then they are not neighboring cells. Since  $R_1$  and  $R_2$  are rectangles in  $\Gamma_q$ , this means that for all elements  $(\mu, \nu)$  from  $R_1$  or  $R_2$ ,  $\Gamma(\mu, \nu) \geq q$ , and that the value for all cells neighboring to  $R_1$  or to  $R_2$  is less than  $q$ . Now since  $(\alpha, \gamma)$  belongs to  $R_1$  and  $(\beta, \delta)$  does not belong to  $R_1$ , we use Lemma 3. First we note that  $(\beta, \delta)$  is not neighboring to  $R_1$  since  $R_1$  and  $R_2$  are distant. Hence, there is a cell  $(\mu, \nu)$  neighboring to  $R_1$  such that  $(\mu, \nu)$  is between  $(\alpha, \gamma)$  and  $(\beta, \delta)$ . Therefore,  $\Gamma(\mu, \nu) < q = \min\{\Gamma(\alpha, \gamma), \Gamma(\beta, \delta)\}$ .

To prove the converse, suppose that for all  $(\alpha, \gamma)$  and  $(\beta, \delta) \in \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ , either they are not neighboring cells and there is  $(\mu, \nu)$  between  $(\alpha, \gamma)$  and  $(\beta, \delta)$  such that  $\Gamma(\mu, \nu) < \min\{\Gamma(\alpha, \gamma), \Gamma(\beta, \delta)\}$ , or for every cell  $(\mu, \nu) \in [\min\{\alpha, \beta\}, \max\{\alpha, \beta\}] \times [\min\{\gamma, \delta\}, \max\{\gamma, \delta\}]$ , we have

$$\Gamma(\mu, \nu) \geq \min\{\Gamma(\alpha, \gamma), \Gamma(\beta, \delta)\}.$$

In addition, let  $p$  be an element from  $[0, 1]$ . We consider the cut  $\Gamma_p$ . If it is equal to the empty set, or to the whole board, then the statement is true. Suppose that  $(\alpha, \gamma) \in \Gamma_p$ . If it is the only cell in  $\Gamma_p$ , then the statement is true, this cell forms a trivial square. Suppose that there is more than

one cell in  $\Gamma_p$ . We prove that every cell from  $\Gamma_p$  belongs to a rectangle which is a subset of  $\Gamma_p$  and distant from all other rectangles in  $\Gamma_p$ . Let  $(\alpha, \gamma) \in \Gamma_p$ . Let  $X_1 = \{(\alpha, \gamma)\}$ . Let  $X_{i+1} = R(X_i \cup \{x\})$ , if there is a cell  $x \in \Gamma_p$ , such that  $x \notin X_i$  and  $xNy$  for some  $y \in X_i$ . Otherwise, let  $X_{i+1} = X_i$ .  $X_1$  consists of one cell and thus it is a rectangle. If  $X_i$  is a rectangle, then  $X_{i+1}$  is also a rectangle by the definition (since  $R$  is an operator that makes the minimal rectangle containing a subset). Hence,  $X_i$  is a rectangle for every  $i$ . Further, let

$$X = \bigcup_{i \geq 1} X_i.$$

Since there are finitely many cells in the table, then  $X = X_i$  for some  $i \in \mathbb{N}$  and thus  $X$  is a rectangle which is distant from all the other cells of  $\Gamma_p$ .

We prove by induction that rectangle  $X$  is a subset of  $\Gamma_p$ .

$X_1$  is a rectangle which is a subset of  $\Gamma_p$ .

We prove that if  $X_i \subseteq \Gamma_p$ , then also  $X_{i+1} \subseteq \Gamma_p$ . If  $X_i = X_{i+1}$ , then it is obviously true, so the only interesting case is when  $X_i \neq X_{i+1}$ . Suppose that  $X_i$  is a rectangle from  $\Gamma_p$ ,  $X_i = [\alpha_{\min}, \beta_{\min}] \times [\alpha_{\max}, \beta_{\max}]$ . Let  $x$  be a cell not belonging to  $X_i$ , such that  $xNy$  for some  $y \in X_i$ . Then  $x = (\gamma_1, \delta_1)$  and at least one of the following statements is true:

- (i)  $\gamma_1 = \alpha_{\min} - 1$
- (ii)  $\gamma_1 = \alpha_{\max} + 1$
- (iii)  $\delta_1 = \beta_{\min} - 1$
- (iv)  $\delta_1 = \beta_{\max} + 1$ .

Suppose that  $\gamma_1 = \alpha_{\min} - 1$  and  $\beta_{\min} \leq \delta_1 \leq \beta_{\max}$ .

Then we prove that all  $2 \times 2$  squares of the  $[\alpha_{\min} - 1, \alpha_{\min}] \times [\mu, \mu + 1]$  are included into  $\Gamma_p$ , for  $\mu = \beta_{\min}, \dots, \beta_{\max} - 1$ . We conclude first that it is true for the squares  $[\alpha_{\min} - 1, \alpha_{\min}] \times [\delta_1 - 1, \delta_1]$  (if  $\delta_1 - 1 \geq \beta_{\min}$ ) and  $[\alpha_{\min} - 1, \alpha_{\min}] \times [\delta_1, \delta_1 + 1]$  (if  $\delta_1 + 1 \leq \beta_{\max}$ ).

To prove that  $[\alpha_{\min} - 1, \alpha_{\min}] \times [\delta_1 - 1, \delta_1]$  is included in  $\Gamma_p$ , we notice that the cells  $(\alpha_{\min} - 1, \delta_1)$  and  $(\alpha_{\min}, \delta_1 - 1)$  are both in  $\Gamma_p$  and they are connected, thus applying condition (1) we prove that the square is in  $\Gamma_p$ . Using the same idea, we prove that the rectangle  $[\alpha_{\min} - 1, \alpha_{\max}] \times [\beta_{\min}, \beta_{\max}]$  is in  $\Gamma_p$ . This rectangle is the least one containing  $X_i \cup \{(\gamma_1, \delta_1)\}$ , so it is equal to  $X_{i+1}$ . Therefore,  $X_{i+1} \subseteq \Gamma_p$ .

Similarly we prove the cases when only one of the statement (ii), (iii) or (iv) is true.

A special case occurs when two of the statements (i), (ii), (iii) or (iv) are both true (it is not possible that more than two of them are true at the same time).

For example, suppose that (i) and (iii) are true, i.e., that  $\gamma_1 = \alpha_{\min} - 1$  and  $\delta_1 = \beta_{\min} - 1$  (all the other combinations can be proved similarly).

Then,  $R(X_i \cup \{(\gamma_1, \delta_1)\}) = [\alpha_{\min} - 1, \alpha_{\max}] \times [\beta_{\min} - 1, \beta_{\max}]$ . We prove that this rectangle is in  $\Gamma_p$ , first noting that  $2 \times 2$  square  $[\alpha_{\min} - 1, \alpha_{\min}] \times [\beta_{\min} - 1, \beta_{\min}]$  is in  $\Gamma_p$ . It is true by condition (1), since  $(\alpha_{\min} - 1, \beta_{\min} - 1)$



and  $(\alpha_{\min}, \beta_{\min})$  are in  $\Gamma_p$ . By the similar arguments, we prove that each of the following  $2 \times 2$  squares are also subsets of  $\Gamma_p$ :

$$[\alpha_{\min} - 1, \alpha_{\min}] \times [\mu, \mu + 1], \text{ for every } \mu = \beta_{\min}, \dots, \beta_{\max} - 1$$

and

$$[\nu, \nu + 1] \times [\beta_{\min} - 1, \beta_{\min}], \text{ for every } \nu = \alpha_{\min}, \dots, \alpha_{\max} - 1.$$

Thus, we proved that every cell from  $\Gamma_p$  belongs to a rectangle which is a subset of  $\Gamma_p$  such that it is distant from all the other rectangles, therefore every cut is a union of distant rectangles.  $\square$

The next proposition shows that, having islands in mind, we can consider only the rectangular fuzzy relations.

**Theorem 2.** *For every fuzzy relation  $\Gamma : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow [0, 1]$ , there is a rectangular fuzzy relation  $\Phi : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow [0, 1]$ , having the same islands.*

*Proof.* In case we have only one island in  $\Gamma$ , namely the whole table (which is by convention an island), then we can take  $\Phi$  to be a constant function, and it is a rectangular relation without non-trivial islands, and the theorem is fulfilled. Suppose there are non-trivial islands in  $\Gamma$ . Let  $\{a_1, a_2, \dots, a_h\}$  be the set of all different values of a fuzzy relation  $\Upsilon$ , such that  $0 \leq a_0 < a_1 < \dots < a_h \leq 1$ . Then, we can consider a fuzzy relation  $\Upsilon_1$  with the same islands having the set of values  $\{b_0, b_1, \dots, b_h\}$ , with  $0 < b_0 < b_1 < \dots < b_h \leq 1$ . It is easy to see that the fuzzy relation  $\Upsilon_1 : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow [0, 1]$ , defined by  $\Upsilon_1(x, y) = b_i$  if and only if  $\Upsilon(x, y) = a_i$ , has the same islands as  $\Upsilon$ . The purpose of transforming  $\Upsilon$  to  $\Upsilon_1$  is to ensure that none of the cells are mapped to 0, in order to give a proof by induction.

Now we prove the statement of the theorem by induction on the size of the table. For the table consisting of only one cell, the statement is obvious. We suppose that the statement is true for all tables with size  $(k, l)$ , where  $k \leq m$  and  $l \leq n$ , and  $(k, l) \neq (m, n)$ . Now, we consider the table of size  $(m, n)$  and the relation  $\Gamma : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow [0, 1]$ . By the assumption, there are non-trivial islands, and by the consideration (iii) after Lemma 4, every island is contained in a maximal island. Furthermore, every two maximal islands are distant. The induction hypothesis is valid for all the maximal islands. We define the rectangular fuzzy relation  $\Phi$  having the same islands as  $\Gamma$  by the values of the rectangular fuzzy relations of smaller sizes that correspond to the maximal islands of  $\Gamma$ . According to the consideration in the previous paragraph, we can take the values in the maximal islands of the relation  $\Phi$  to be greater than 0. Finally, if  $(\alpha, \beta)$  does not belong to any maximal island, then we define  $\Phi(\alpha, \beta) = 0$ .  $\square$

For practical reasons, we give an algorithm for a construction of a fuzzy rectangular relation having the same islands as a given fuzzy relation.

**Algorithm**

Let  $\Gamma : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow [0, 1]$  be a fuzzy relation containing a non-trivial island. To construct a fuzzy relation  $\Phi$  having the same islands as  $\Gamma$ , we proceed as follows. Let  $\{a_1, a_2, \dots, a_h\}$  be the set of different values of this fuzzy relation, such that  $0 \leq a_0 < a_1 < \dots < a_h \leq 1$ . ( $a_h$  is the maximum value of fuzzy relation  $\Gamma$ ).

1. FOR  $i = h$  TO 0
2. FOR  $y = 1$  TO  $n$
3. FOR  $x = 1$  TO  $m$
4. IF  $\Gamma(x, y) = a_i$  THEN
5.  $j := i$
6. WHILE there is no island of  $\Gamma$  which is a subset of  $\Gamma_{a_j}$  that contains  $(x, y)$  DO  $j := j - 1$
7. ENDWHILE
8. Let  $\Phi(x, y) := a_j$ .
9. ENDIF
10. NEXT  $x$
11. NEXT  $y$
12. NEXT  $i$
13. END.

**Theorem 3.** *For every fuzzy relation  $\Gamma$ , the algorithm constructs a fuzzy rectangular relation  $\Phi$  having the same islands as  $\Gamma$ .*

*Proof.* We prove the following:

- (1) The relation  $\Phi$  obtained by the algorithm is a rectangular relation.
- (2) The relation  $\Phi$  has the same islands as  $\Gamma$ .

(1) We have to prove that every cut of  $\Phi$  is equal to the empty set, or it is a union of islands. Since all the values of  $\Phi$  are equal to some of the values of  $\Gamma$ , we notice that every nonempty cut of  $\Phi$  is equal to some of the cuts  $\Phi_{a_k}$ , for  $0 \leq k \leq h^2$ . We prove that for every  $k$ ,  $\Phi_{a_k}$  is a union of distant rectangles. Consider the algorithm for  $i = k$ . By step 8,  $\Phi(x, y) = a_k$  only if there is an island  $I$  of  $\Gamma$ , that is also contained in  $\Gamma_{a_k}$ , that contains  $(x, y)$ . This means that for all  $(z, t) \in I$ ,  $\Gamma(z, t) \geq a_k$ . Considering all the elements from  $I$ , we can conclude that they also belongs to  $\Phi_{a_k}$ , and thus  $I$  is also an island in  $\Phi$ , since the value of  $\Phi$  for any element is always less or equal to the value of  $\Gamma$  for this element (that also concerns elements neighboring to  $I$ ).

So, every  $(x, y)$  that belongs to  $\Phi_{a_k}$  also belongs to an island, therefore  $\Phi_{a_k}$  is a union of islands.

(2) First we prove that every island in  $\Gamma$  is also an island in  $\Phi$ . As we already mentioned,  $\Phi \leq \Gamma$ . Hence, if  $I$  is an island of  $\Gamma$ , then it is also an island of  $\Phi$ , since for all cells  $(x, y) \in I$ ,  $\Gamma(x, y) = \Phi(x, y)$  and the cells that are neighboring to the cells of the island  $I$  have either the same value in  $\Phi$  as in  $\Gamma$ , or lower in  $\Phi$  than in  $\Gamma$ . This holds for every island belonging to every cut, from  $a_h$  to  $a_0$  (cut  $a_0$  is the table itself for  $\Gamma$  and also for  $\Phi$ ).

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<sup>2</sup>Here we use the notation from the algorithm.

Further, suppose that  $I$  is an island of  $\Phi$ . Let  $a_p = \min_{(x,y) \in I} \Phi(x,y)$ . If  $a_p = a_0$ , then  $I$  is the table itself, which is also an island in  $\Gamma$ . Let  $a_p > a_0$ . This means that  $I$  is a rectangle in  $\Phi_{a_p}$ , distant from all other rectangles. Since  $\Phi(x,y) \leq \Gamma(x,y)$ , we have  $I \subseteq \Gamma_{a_p}$ . Hence, the rectangle  $I$  is also a subset of the cut  $\Gamma_{a_p}$ . Since  $a_p$  is the minimum value for all cells in  $I$ ,  $I$  is not an island in the cut  $\Phi_{a_{p+1}}$ . Let  $(a,b) \in I$  be a cell for which  $\Phi(a,b) = a_p$ . Then, by the steps 6-8 of the algorithm, there is an island  $J$  of  $\Gamma$  containing  $(a,b)$ , which is a subset of  $\Gamma_{a_p}$ . Indeed, if such an island does not exist, then in WHILE loop,  $j$  would become strictly less than  $p$ , and then  $\Phi(a,b) \leq a_j < a_p$ . However,  $a_p$  is the minimum value of  $\Phi$  for cells in  $I$ . Hence, there is  $J$ , an island of  $\Gamma$  that contains  $(a,b)$ , and this island is also an island of  $\Phi$ , by the considerations above. Since,  $(a,b)$  is in the intersection of two islands  $I$  and  $J$ , one of them is a subset of another, and being subsets of the same cut, they coincide. Therefore,  $I$  is also an island of  $\Gamma$ .

Thus we proved that the islands of  $\Gamma$  and  $\Phi$  coincide. □

The following example illustrates the given algorithm.

**Example 1.** Consider the table  $\{1, 2, 3, 4, 5\} \times \{1, 2, 3\}$ , presented in Table 2 (as an example, the cell  $(4, 2)$  is indicated).

3					
2			(4,2)		
1					
	1	2	3	4	5

Table 2

A fuzzy relation  $\Gamma : \{1, 2, 3, 4, 5\} \times \{1, 2, 3\} \rightarrow [0, 1]$  is presented in Table 3. The values of  $\Gamma$  are given in the corresponding cells:  $\Gamma(1, 1) = 0.4$ ,  $\Gamma(1, 2) = 0.8$ , etc. The cuts of this fuzzy relation are as follows:

For  $p \in [0, 0.2]$ ,  $\Gamma_p = \{1, 2, 3, 4, 5\} \times \{1, 2, 3\}$ ,

for  $p \in (0.2, 0.4]$ ,  $\Gamma_p = \{1, 2, 3, 4, 5\} \times \{1, 2, 3\} \setminus \{(3, 1)\}$ ,

for  $p \in (0.4, 0.6]$ ,  $\Gamma_p = \{(1, 2), (1, 3), (2, 2), (2, 3), (3, 3), (4, 2), (4, 3), (5, 2), (5, 3)\}$ ,

for  $p \in (0.6, 0.8]$ ,  $\Gamma_p = \{(1, 2), (1, 3), (2, 2), (2, 3), (4, 2), (4, 3), (5, 2), (5, 3)\}$

and

for  $p \in (0.8, 0.9]$ ,  $\Gamma_p = \{(1, 3), (2, 3), (4, 3), (5, 3)\}$ .

All the other cuts are equal to the empty set.

3	0.9	0.9	0.6	0.9	0.9
2	0.8	0.8	0.4	0.8	0.8
1	0.4	0.4	0.2	0.4	0.4
	1	2	3	4	5

Table 3

There are five islands of this fuzzy relation:

trivial one (the table itself),  
 $\{(1, 2), (1, 3), (2, 2), (2, 3)\}$ ,  
 $\{(4, 2), (4, 3), (5, 2), (5, 3)\}$ ,  
 $\{(1, 3), (2, 3)\}$  and  
 $\{(4, 3), (5, 3)\}$ .

The cut  $\Gamma_{0.4}$  is not a union of distant rectangles, neither is  $\Gamma_{0.6}$ . Therefore,  $\Gamma$  is not a rectangular relation.

After applying the first iteration of the algorithm to  $\Gamma$ , the fuzzy relation presented in Table 4 is obtained.

3	0.9	0.9	0.4	0.9	0.9
2	0.8	0.8	0.4	0.8	0.8
1	0.4	0.4	0.2	0.4	0.4
	1	2	3	4	5

Table 4

The fuzzy rectangular relation constructed by the next iteration is given in Table 5.

3	0.9	0.9	0.2	0.9	0.9
2	0.8	0.8	0.2	0.8	0.8
1	0.2	0.2	0.2	0.2	0.2
	1	2	3	4	5

Table 5

## 5. LATTICE-VALUED REPRESENTATION

As defined above, cuts of a rectangular fuzzy relation are unions of distant rectangles. We have seen that for every finite fuzzy relation there is a rectangular one having the same islands. Here we prove more. Namely, for every rectangular fuzzy relation  $\Gamma$  there is a lattice  $L$  and an  $L$ -valued relation whose cuts are precisely the islands of  $\Gamma$ : each cut is an island and vice versa.

Observe that for any finite fuzzy relation  $\Gamma : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow [0, 1]$  its collection of islands  $\mathcal{I}(\Gamma)$  is a poset under inclusion (each island is a subset of the domain). Due to the properties of islands listed at the beginning of Section 4, it is easy to describe this poset. We recall that a **tree** is a poset with the top element in which every principal filter is a chain. Now the following statement follows directly from the mentioned properties of islands. We mention that the tree appearing in this proposition have already been used in paper [1], though within a different framework.

**Proposition 1.** *For any fuzzy relation  $\Gamma : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow [0, 1]$ , the poset  $\mathcal{I}(\Gamma)$  is a tree under inclusion.*

Let  $\mathcal{I}_0(\Gamma) := \mathcal{I}(\Gamma) \cup \emptyset$ . Obviously, by adding the empty set to the collection of islands, we obtain the lattice under inclusion.

The following theorem can be proved by techniques developed for cut approach to lattice valued fuzzy sets, see e.g. [20]. Due to the specific notation and properties of islands and fuzzy relations we formulate the proof for the readers' convenience.

**Theorem 4.** *Let  $\Gamma : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow [0, 1]$  be a rectangular fuzzy relation. Then there is a lattice  $L$  and an  $L$ -valued relation  $\Phi$ , such that the cuts of  $\Phi$  are precisely all islands of  $\Gamma$ .*

*Proof.* Let  $\Gamma : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow [0, 1]$  be a rectangular fuzzy relation and let  $L := (\mathcal{I}_0(\Gamma), \supseteq)$ . We define an  $L$ -valued fuzzy relation

$$\Phi : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow L,$$

by

$$\Phi((i, j)) := \bigcap_{(i, j) \in I} I.$$

Now we prove that for any element  $J$  of lattice  $L$  (which is an island or the empty set), we have that  $\Phi_J = J$ .

Indeed,

$$(i, j) \in \Phi_J \text{ iff } \Phi(i, j) \subseteq J \text{ iff } \bigcap_{(i, j) \in I} I \subseteq J \text{ iff } (i, j) \in J.$$

□

Combining Theorems 2 and 4 we conclude that *for every finite fuzzy relation  $\Gamma$  there is a lattice valued one whose cuts are islands of  $\Gamma$ , as described above.*

**Example 2.** Let  $\Gamma : \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4\} \rightarrow [0, 1]$  be a fuzzy relation presented in Table 6.

4	0.9	0.8	0.7	0.1	0.5
3	0.8	0.8	0.7	0.1	0.4
2	0.7	0.7	0.7	0.1	0.5
1	0.2	0.2	0.2	0.1	0.6
	1	2	3	4	5

Table 6

$\Gamma$  is a rectangular fuzzy relation. Its islands are

$$I_1 = \{(1, 4)\},$$

$$I_2 = \{(1, 3), (1, 4), (2, 3), (2, 4)\},$$

$$I_3 = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\},$$

$$I_4 = \{(5, 1)\},$$

$$I_5 = \{(5, 1), (5, 2)\},$$

$$I_6 = \{(5, 4)\},$$

$$I_7 = \{(5, 1), (5, 2), (5, 3), (5, 4)\},$$

$$I_8 = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4), (1, 1), (2, 1), (3, 1)\},$$

$$I_9 = \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4\}.$$

Its cut relations are:

- $\Gamma_1 = \emptyset$
- $\Gamma_{0.9} = I_1$  (one-element island)
- $\Gamma_{0.8} = I_2$  (four-element square island)
- $\Gamma_{0.7} = I_3$  (nine-element square island)
- $\Gamma_{0.6} = I_3 \cup I_4$  (this cut is a disjoint union of two islands)
- $\Gamma_{0.5} = I_3 \cup I_5 \cup I_6$  (union of three islands)
- $\Gamma_{0.4} = I_3 \cup I_7$  (union of two islands)
- $\Gamma_{0.2} = I_7 \cup I_8$  (union of two islands)
- $\Gamma_{0.1} = \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4\} = I_9$  (the whole domain).

As we did in the proof of Theorem 4, we take  $L$  to be the lattice dual to  $(\mathcal{I}_0(\Gamma), \subseteq)$ , its diagram is given in Figure 1.

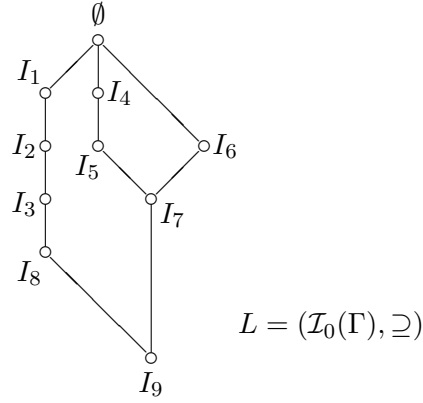


Figure 1

The corresponding  $L$ -valued relation  $\Phi$  is given by Table 7. Its cut relations are one by one single islands of  $\Gamma$ . E.g., for the cut  $\Phi_{I_6} = \{(5, 4)\} = I_6$ , and so on.

4	$I_1$	$I_2$	$I_3$	$I_9$	$I_6$
3	$I_2$	$I_2$	$I_3$	$I_9$	$I_7$
2	$I_3$	$I_3$	$I_3$	$I_9$	$I_5$
1	$I_8$	$I_8$	$I_8$	$I_9$	$I_4$
	1	2	3	4	5

Table 7

□

## 6. A COMMENT ON ISLANDS AND FORMAL CONCEPTS

As mentioned in Introduction, neither every island is a fuzzy formal concept, nor every fuzzy formal concept is an island. In the present section

we explain briefly connection among these. Our approach by the classical formal concepts should point out the complexity of the problem in fuzzy case. Therefore, since this section is not more than an illustration, we omit strict propositions, and remain at typical examples.

Like in Introduction, we consider a finite relation  $\rho$  on a domain  $A \times B$ ,  $A$  and  $B$  being linearly ordered:  $A = \{a_1, \dots, a_m\}$ ,  $B = \{b_1, \dots, b_n\}$  (the natural order of indices deduces the order on sets  $A$  and  $B$ ). The relation  $\rho$  can be given by the characteristic function organized as a table  $A \times B$ , with entries from the set  $\{0, 1\}$ .

Let  $\Gamma : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow \{0, 1\}$  be a relation. The rectangle  $T = \{\alpha, \dots, \beta\} \times \{\gamma, \dots, \delta\}$  is an **island** of  $\Gamma$ , if for every cell  $(\mu, \nu) \in T$ ,  $\Gamma(\mu, \nu) = 1$  and for every cell  $(x, y)$  which does not belong to this rectangle but is neighboring to some cell of the rectangle, we have  $\Gamma(x, y) = 0$ .

A **formal context**  $\mathbb{K} := (A, B, I)$  is an ordered triple consisting of two sets  $A$  and  $B$  and the relation  $I \subseteq A \times B$ .  $A$  is called the set of objects and  $B$  is called the set of attributes.

It is possible to consider linearly ordered sets  $(A, \leq)$  and  $(B, \leq)$  and thus to consider a formal context as some table, like e.g. Table 1.

For a set  $C \subseteq A$ ,  $C'$  is the set of all the attributes common to the objects in  $C$ ,

$$C' = \{y \in B \mid (x, y) \in I; \text{ for all } x \in C\}.$$

Analogously, for a set  $D \subseteq B$ ,  $D'$  is the set of all the objects common to the attributes in  $D$ ,

$$D' = \{x \in A \mid (x, y) \in I; \text{ for all } y \in D\}.$$

A **formal concept** of the context  $\mathbb{K} := (A, B, I)$  is a pair  $(C, D)$  with  $C \subseteq A$ ,  $D \subseteq B$ ,  $C' = D$  and  $D' = C$ .  $C$  is called the extent and  $D$  the intent of formal concept  $(C, D)$ .

It is obvious that if  $(C, D)$  is a concept of the context  $\mathbb{K} := (A, B, I)$ , then there are linear orderings  $\leq_1$  and  $\leq_2$  on  $A$  and  $B$ , respectively, such that  $C \times D$  is a rectangle in the table  $(A, \leq_1) \times (B, \leq_2)$ .

Although a formal concept makes always a rectangle in the table, it is not always an island of  $I$ , i.e., there are examples of concepts and contexts so that there are no orderings such that the concept makes an island. To illustrate this, we consider the context  $(\{a_1, a_2, a_3, a_4, a_5\}, \{b_1, b_2, b_3\}, I)$  given in Table 8.  $(\{a_1, a_2, a_3, a_4\}, \{b_1, b_2\})$  is a concept of this context. However, there are no linear orderings on sets  $\{a_1, a_2, a_3, a_4, a_5\}$  and  $\{b_1, b_2, b_3\}$  such that  $\{a_1, a_2, a_3, a_4\} \times \{b_1, b_2\}$  is an island of relation  $I$ .

$b_3$	0	1	0	1	0
$b_2$	1	1	1	1	0
$b_1$	1	1	1	1	1
	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$

Table 8

Next, if we start from an island  $A_1 \times B_1$  in a relation  $I \subseteq (A, \leq_1) \times (B, \leq_2)$ , again  $(A_1, B_1)$  is not necessarily a concept of the context  $(A, B, I)$ , which is illustrated by the following example.

$A = \{a_1, a_2, a_3, a_4, a_5\}$ ,  $B = \{b_1, b_2, b_3\}$  and  $I$  is given in Table 9.

$b_3$	0	0	0	0	0
$b_2$	1	1	1	0	1
$b_1$	1	1	1	0	1
	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$

Table 9

Obviously,  $\{a_1, a_2, a_3\} \times \{b_1, b_2\}$  and  $\{a_5\} \times \{b_1, b_2\}$  are two islands, but neither is a concept of the related context.

In the fuzzy framework, connections among contexts and rectangular relations is more complex, though analogue to the foregoing ones.

## 7. CONCLUSION

In this paper a new property of finite fuzzy relations is investigated. We deal with islands, which are particular subsets of the domain. In cutworthy settings they represent disjoint rectangles of the ordered domain of the fuzzy relation.

There are many combinatorial properties of islands which have been and could be investigated. For instance, using the results from [5], we are able to determine the cardinality of the lattice of islands appearing in Section 5. Since our present investigation is purely fuzzy oriented, we do not deal with these combinatorial aspects; the corresponding results will appear elsewhere.

On the other hand, is there an interest for islands and rectangular relations in fuzzy disciplines? Ordered domain represented as a board is a frequent object in many fields (e.g., a picture is such a board). Corresponding fuzzy relations in which islands are identified (rectangles as cut relations) could be a useful tool for investigations of such objects (e.g., in pattern recognition).

Also, an interesting task would be to highlight the role of islands in connecting concepts and attributes, i.e., to find real applications in fuzzy knowledge processing (similarly to the Fuzzy Formal Concept Analysis). Ordered domain is not a restriction - a finite domain can always be suitably ordered. Concepts and islands are similar mathematical objects, though they not coincide. We believe that this field is worth investigating.

**Acknowledgment.** *The referees' comments are appreciated. They were helpful and improved the paper.*

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