LATTICES AND ISLANDS

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The talk will give an overview of the following topic:

A rectangular $m \times n$ board consisting of square cells is given, with a positive integer (height) associated to each cell. A rectangular island is a rectangle in the board such that the height of its cells are greater than the heights of all neighboring cells. The notion of an island comes from information theory. The characterization of the lexicographical length sequences of binary maximal instantaneous codes in [5] uses the notion of *full segments*, which are one-dimensional islands. Several generalizations of this notion gave interesting combinatorial problems. In two dimensions, Czédli [2] has determined the maximum number of rectangular islands; for the maximum number of rectangular islands on the rectangular board of size $m \times n$ he obtained $f(m, n) = \lfloor (mn+m+n-1)/2 \rfloor$. Pluhár [17] gave upper and lower bounds in higher dimensions. E.K. Horváth, Z. Németh and G. Pluhár determined upper and lower bounds for the maximum number of triangular islands on a triangular grid in [6]. Some further results on triangular islands can be found in [12]. The number of square islands is a similar problem to the triangular case and it is treated in [7] and in [13]. Some proving methods for the maximum number of islands as well as exact formulas for some further island-problems are summarized in [1]. The problem of minimum cardinality of maximal systems of rectangular islands is treated in [11]. The investigations on islands motivated further research on independence properties in lattices, see [3], [4] and [8]. Using the cut technics of the height function, it is shown in [10] that the minimum cardinality of maximal systems of rectangular islands – determined by Lengvárszky in [11] – is equal to the maximum number of different cuts of a standard rectangular height function. In addition, the realizing standard rectangular height functions have the same islands as those height functions that realize the minimum cardinality of maximal systems of rectangular islands. In [10] it is proved also that if the height function gives maximally many islands, then the number of essentially different cuts is at least $\lceil log_2(m+1) \rceil + \lceil log_2(n+1) \rceil - 1$ and it is at most $\left\lfloor \frac{(m+n+3)}{2} \right\rfloor$. In paper [8], CD-independent sets in an arbitrary poset $\mathbb{P} = (P, \leq)$ are defined, and it is shown that the CD bases of any poset $\mathbb{P}$ can be characterized as maximal chains in a related poset $\mathcal{D}(P)$. If $\mathbb{P}$ is a complete lattice, then $\mathcal{D}(P)$ is also a lattice having a weak distributive property. We also point out two known lattice classes where the CD-bases in finite lattices have the mentioned property: The first class is that one of graded, dp-distributive lattices, and the second class is obtained by generalizing the properties of the so-called interval lattices (having their origine in graph theory). Since these classes are generalizations of distributive lattices, our results also imply that the CD-bases in a finite distributive lattice have the same number of elements, settled originally in [3].

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