Finite segments of the harmonic series

Donald Silberger
2012 May 19
State University of New York at New Paltz

For $\mathcal{F}$ the family of all finite nonempty sets of positive integers we define the function $\sigma : \mathcal{F} \to \mathbf{Q}^+$ by

$$\sigma : X \mapsto \sigma X := \sum_{x \in X} \frac{1}{x}.$$

Let $\mathcal{I} \subset \mathcal{F}$ be the set of all finite nonempty intervals $[m, n]$ of positive integers. That is to say, $[m, n] := \{m, m+1, m+2, \ldots, n-1, n\}$ for $m \leq n$.

Theorem. The function $\sigma \restriction \mathcal{I}$ is injective.

The talk sketches a proof of this fact, relying in part upon the famous theorem of J. J. Sylvester which states that, unless $m = n = 1$, there is a prime $p \geq n-m+1$ which divides an element in $[m, n]$. The paper presenting our proof in detail will eventually appear in ArXiv, and be announced. (There is a flawed and incomplete predecessor already under the same title in ArXiv, and we expect that the new version will come to replace it.)

The talk also offers an idea for extending Sylvester's Theorem, as follows: Call a prime power p^v *sylvester* for the set $X \in \mathcal{F}$ when both p^v is the largest power of p dividing any element in X and $p^v > \max X - \min X$.

Conjecture. If $1 \leq m \leq n \leq m' < n'$ with $n - m \leq n' - m'$ then there is a p^v which is sylvester for $[m', n']$ but which divides no element in $[m, n]$.

A salient consequence of this conjecture is that, if $1 < m \leq n \leq m' < n'$ with $n - m \leq n' - m'$, then $\delta \neq \delta'$ where δ and δ' are the denominators of the respective reduced fractions $\sigma[m, n]$ and $\sigma[m', n']$.

Call a subfamily $\mathcal{G} \subset \mathcal{F}$ *sylvester distinguished*, (SD), iff whenever X and Y are distinct elements in $\mathcal{G}$ there is a prime power which is sylvester for one of those sets while dividing no element in the other set.

Problem: Characterize the maximal sylvester subfamilies of $\mathcal{F}$.

The family $\mathcal{I}$ is not SD. Is $\mathcal{I}_6 := \{[m, n] : m \geq 6\}$ SD?