CONGRUENCE DISTRIBUTIVITY AND MODULARITY PERMIT TOLERANCES

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Abstract. We prove that the distributive resp. modular law holds in congruence distributive resp. congruence modular varieties even for tolerance relations.

Dedicated to Béla Csákány on his seventieth birthday

Let $\operatorname{dist}(x,y,z)$ resp. $\operatorname{mod}(x,y,z)$ denote the distributive law

$$x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$$

resp. the modular law

$$x \wedge (y \vee (x \wedge z)) \leq (x \wedge y) \vee (x \wedge z).$$

For an algebra A, the *set* of tolerances and the *lattice* of congruences of A will be denoted by $\operatorname{Tol} A$ and $\operatorname{Con} A$, respectively. We say that dist(tol,tol,tol) holds in A if $\Gamma \wedge (\Phi \vee \Psi) \subseteq (\Gamma \wedge \Phi) \vee (\Gamma \wedge \Psi)$ is valid for any $\Gamma, \Phi, \Psi \in \operatorname{Tol} A$, where the meet resp. the join is the intersection resp. the transitive closure of the union. I.e., denoting the transitive closure by *, $\Phi \vee \Psi = (\Psi \cup \Psi)^* = \Psi^* \vee \Phi^*$ (the second join is from $\operatorname{Con} A$) for any tolerances Φ and Ψ *in the present paper throughout*. The meaning of mod(tol,tol,tol) is analogous.

Theorem 1. *If $\mathcal{V}$ is a congruence distributive resp. congruence modular variety then dist(tol,tol,tol) resp. mod(tol,tol,tol) holds in all algebras of $\mathcal{V}$.*

Proof. Suppose $\mathcal{V}$ is congruence distributive. Then we have Jónsson terms, cf. Jónsson [5], i.e. ternary $\mathcal{V}$-terms $t_0, \dots, t_n$ for some even $n \in \mathbf{N}_0 = \{0, 1, 2, \dots\}$ such that $\mathcal{V}$ satisfies the identities $t_0(x,y,z) = x$, $t_n(x,y,z) = z$, $t_i(x,x,y) = t_{i+1}(x,x,y)$ for i even, $t_i(x,y,y) =$

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$t_{i+1}(x, y, y)$ for i odd, and $t_i(x, y, x) = x$ for all i. Now let $A \in \mathcal{V}$, $\Gamma, \Phi, \Psi \in \operatorname{Tol} A$ and $(a, b) \in \Gamma \wedge (\Phi \vee \Psi)$. Then there is an even k and there are elements $c_0 = a, c_1, \ldots, c_{k-1}, c_k = b$ such that $(c_i, c_{i+1}) \in \Phi$ for i even, $(c_i, c_{i+1}) \in \Psi$ for i odd and $(a, b) = (c_0, c_k) \in \Gamma$. Since

$$t_i(a, u, b) = t_i(t_i(a, v, a), u, t_i(b, v, b)) \;\Gamma\; t_i(t_i(a, v, b), u, t_i(a, v, b)) = t_i(a, v, b),$$

for all i and any $u, v \in A$ we have

$$(t_i(a, u, b), t_i(a, v, b)) \in \Gamma. \tag{1}$$

Now we define a sequence from a to b as follows:

$$\begin{gathered} a = t_0(a, c_0, b) = t_1(a, c_0, b) \;\Phi\; t_1(a, c_1, b) \;\Psi\; t_1(a, c_2, b) \;\Phi\; t_1(a, c_3, b) \\ \Psi \;\ldots\; \Phi\; t_1(a, c_{k-1}, b) \;\Psi\; t_1(a, c_k, b) = t_1(a, b, b) = t_2(a, b, b) = \\ t_2(a, c_k, b) \;\Psi\; t_2(a, c_{k-1}, b) \;\Phi\; t_2(a, c_{k-2}, b) \;\Psi\; \ldots \;\Phi\; t_2(a, c_0, b) = \\ t_2(a, a, b) = t_3(a, a, b) \;\Phi\; t_3(a, c_1, b) \;\Psi\; t_3(a, c_2, b) \;\Phi\; \ldots \;\Psi \\ t_3(a, c_k, b) = t_4(a, c_k, b) \;\Psi\; t_4(a, c_{k-1}, b) \;\Phi \quad \ldots \quad \Phi \\ t_{n-1}(a, c_{k-1}, b) \;\Psi\; t_{n-1}(a, c_k, b) = t_{n-1}(a, b, b) = t_n(a, b, b) = b. \end{gathered}$$

It follows from (1) that any two consecutive members of this series are in $(\Gamma \cap \Phi) \cup (\Gamma \cap \Psi) \subseteq (\Gamma \wedge \Phi) \vee (\Gamma \cap \Psi)$. Thus $(a, b) \in (\Gamma \wedge \Phi) \vee (\Gamma \cap \Psi)$, whence dist(tol,tol,tol) holds in $\mathcal{V}$.

Now let $\mathcal{V}$ be congruence modular. Then we have Day terms, i.e. quaternary $\mathcal{V}$-terms $m_0, m_1, \ldots, m_k$ for some $0 < k \in \mathbf{N}_0$ such that $\mathcal{V}$ satisfies the identities

$$\begin{gathered} m_0(x, y, u, v) = x, \quad m_k(x, y, u, v) = y \\ m_i(x, y, x, y) = m_{i+1}(x, y, x, y) \text{ for } i \text{ even}, \\ m_i(x, y, z, z) = m_{i+1}(x, y, z, z) \text{ for } i \text{ odd, and} \\ m_i(x, x, y, y) = x \text{ for all } i, \end{gathered}$$

cf. Day [3]. First we show that, for any $A \in \mathcal{V}$ and $\Gamma, \Phi, \Psi \in \operatorname{Tol} A$,

$$\Gamma \cap (\Phi \circ (\Gamma \cap \Psi) \circ \Phi) \subseteq (\Gamma \cap \Phi) \vee (\Gamma \cap \Psi). \tag{2}$$

Let $(a, b) \in \Gamma \cap (\Phi \circ (\Gamma \cap \Psi) \circ \Phi)$. Then there are $c, d \in A$ with $(a, c), (d, b) \in \Phi$, $(c, d) \in \Gamma \cap \Psi$ and, of course, $(a, b) \in \Gamma$. Consider the elements $d_i = m_i(a, b, c, d)$ for $i = 0, 1, \ldots, k$, $e_i = m_i(a, b, c, c) = m_{i+1}(a, b, c, c)$ for i odd, and $e_i = m_i(a, b, a, b) = m_{i+1}(a, b, a, b)$ for i even. Then $d_0 = a$, $d_k = b$, and $(d_i, e_i), (e_i, d_{i+1}) \in \Gamma \cap \Psi$ for i odd.

For i even we have $(d_i, e_i), (e_i, d_{i+1}) \in \Phi$,

$$\begin{gathered} d_i = m_i(a, b, c, d) = m_i(m_i(a, b, c, d), m_i(a, b, c, d), a, a) \;\Gamma \\ m_i(m_i(a, a, c, c), m_i(b, b, d, d), a, b) = m_i(a, b, a, b) = e_i, \end{gathered}$$

i.e., $(d_i, e_i) \in \Gamma \cap \Phi$. Similarly, $(e_i, d_{i+1}) \in \Gamma \cap \Phi$.

Now $(a, b) \in (\Gamma \wedge \Phi) \vee (\Gamma \wedge \Psi)$ follows from transitivity and from the fact that all the (d_i, e_i) and (e_i, d_{i+1}) belong to $(\Gamma \wedge \Phi) \vee (\Gamma \wedge \Psi)$. This shows (2).

Now define $\Phi_0 = \Phi$ and $\Phi_{n+1} = \Phi_n \circ (\Gamma \cap \Psi) \circ \Phi_n$ for $n \geq 1$. Notice that all the Φ_n belong to $\operatorname{Tol} A$. We claim that, for all $n \in \mathbf{N}_0$,

$$\Gamma \cap \Phi_n \subseteq (\Gamma \cap \Phi) \vee (\Gamma \cap \Psi). \tag{3}$$

This is evident for $n = 0$. Assuming (3) for an arbitrary n and applying (2) we obtain $\Gamma \cap \Phi_{n+1} = \Gamma \cap (\Phi_n \circ (\Gamma \cap \Psi) \circ \Phi_n) \subseteq (\Gamma \cap \Phi_n) \vee (\Gamma \cap \Psi) \subseteq (\Gamma \cap \Phi) \vee (\Gamma \cap \Psi) \vee (\Gamma \cap \Psi) = (\Gamma \cap \Phi) \vee (\Gamma \cap \Psi)$, i.e. (3) holds for $n+1$. Thus (3) holds for all n and we obtain $\Gamma \wedge (\Phi \vee (\Gamma \wedge \Psi)) = \Gamma \cap \bigcup \{\Phi_n : n \in \mathbf{N}_0\} = \bigcup \{\Gamma \cap \Phi_n : n \in \mathbf{N}_0\} \subseteq (\Gamma \cap \Phi) \vee (\Gamma \cap \Psi)$. This proves Theorem 1. □

Corollary 1. *(Gumm [4]) If $\mathcal{V}$ is a congruence modular variety then it satisfies Gumm's Shifting Principle, i.e., for any $A \in \mathcal{V}$, $\alpha, \gamma \in \operatorname{Con} A$ and $\Phi \in \operatorname{Tol} A$ if $(x, y), (u, v) \in \alpha$, $(x, u), (y, v) \in \Phi$, $(u, v) \in \gamma$ and $\alpha \cap \Phi \subseteq \gamma$ then $(x, y) \in \gamma$.*

Proof. $(x, y) \in \alpha \cap (\Phi \vee (\alpha \wedge \gamma)) \subseteq (\alpha \wedge \Phi) \vee (\alpha \wedge \gamma) \subseteq \gamma \vee \gamma = \gamma$. □

Notice that Theorem 1 also implies the Triangular Principle and the Trapezoid Principle for congruence distributive varieties, cf. [1] and [2].

Now we give an example. Consider the monounary algebra $B = (\{0, 1, 2\}, -)$ where $-0 = 0$, $-1 = 2$ and $-2 = 1$. Then α with the associated partition $\{\{0\}, \{1, 2\}\}$ is the only nontrivial congruence of B, so $\operatorname{Con} B$ is distributive. Notice that

$$\Phi = \{(0, 1), (1, 0), (0, 2), (2, 0), (0, 0), (1, 1), (2, 2)\}$$

is a tolerance and $\alpha \cap \Phi^* \nsubseteq (\alpha \cap \Phi)^*$. Hence the following statement indicates that Theorem 1 cannot be extended for single algebras.

Proposition 1. *If mod(tol,tol,tol) or dist(tol,tol,tol) holds in an algebra A then $\Gamma \cap \Phi^* \subseteq (\Gamma \cap \Phi)^*$ for any $\Gamma, \Phi \in \operatorname{Tol} A$.*

Proof. Apply $\operatorname{mod}(\Gamma, \Phi, 0)$ or $\operatorname{dist}(\Gamma, \Phi, 0)$. □

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