All congruence lattice identities implying modularity have Mal'tsev conditions

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Dedicated to the memory of Alan Day

Abstract. For an arbitrary lattice identity implying modularity (or at least congruence modularity) a Mal'tsev condition is given such that the identity holds in congruence lattices of algebras of a variety if and only if the variety satisfies the corresponding Mal'tsev condition.

It is an old problem if all congruence lattice identities are equivalent to Mal'tsev (=Mal'cev) conditions. In other words, we say that a lattice identity λ can be characterized by a Mal'tsev condition if there exists a Mal'tsev condition M such that, for any variety $\mathcal{V}$, λ holds in congruence lattices of all algebras in $\mathcal{V}$ if and only if M holds in $\mathcal{V}$; and the problem is if all lattice identities can be characterized this way. This problem was raised first in Grätzer [15], where the notion of a Mal'tsev condition was defined. A *strong Mal'tsev condition* for varieties is a condition of the form "there exist terms $h_0, \dots, h_k$ satisfying a set Σ of identities" where k is fixed and the form of Σ is independent of the type of algebras considered. By a *Mal'tsev condition* we mean a condition of the form "there exists a natural number n such that P_n holds" where the P_n are strong Mal'tsev conditions and P_n implies P_{n+1} for every n. The condition "P_n implies P_{n+1}" is usually expressed by saying that a Mal'tsev condition must be weakening in its parameter. (For a more precise definition of Mal'tsev conditions cf. Taylor [23].) The problem was repeatedly asked by several authors, including Taylor [23], Jónsson [13] and Freese and McKenzie [11].

Certain lattice identities have known characterizations by Mal'tsev conditions. The first two results of this kind are Jónsson's characterization of (congruence) distributivity by the existence of Jónsson terms, cf. Jónsson [12], and Day's characterization of (congruence) modularity by the existence of Day terms, cf. Day [8]. Since Day's result will be needed in the sequel, we formulate it now. For $n \geq 2$ let $(\mathbf{D}_n)$ denote the strong Mal'tsev condition "there are quaternary terms $m_0, \dots, m_n$

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satisfying the identities

$$\begin{gathered}
m_0(x,y,z,u)=x, \qquad m_n(x,y,z,u)=u,\\
m_i(x,y,y,x)=x \quad \text{for } i=0,1,\dots,n,\\
m_i(x,x,y,y)=m_{i+1}(x,x,y,y) \quad \text{for } i=0,1,\dots,n, \quad i \text{ even},\\
m_i(x,y,y,z)=m_{i+1}(x,y,y,z) \quad \text{for } i=0,1,\dots,n, \quad i \text{ odd}".
\end{gathered}$$

Now Day's celebrated result says that a variety $\mathcal{V}$ is congruence modular iff the Mal'tsev condition "$(\exists n)(\mathbf{D}_n)$" holds in $\mathcal{V}$.

Jónsson terms and Day terms were soon followed by some similar characterizations for other lattice identities, given for example by Gedeonová [14] and Mederly [19], but Nation [20] and Day [9] showed that these Mal'tsev conditions are equivalent to the existence of Day terms or Jónsson terms; the reader is referred to Jónsson [13] and Chapter XIII in Freese and McKenzie [11] for more details.

The next milestone is Chapter XIII in Freese and McKenzie's book [11]. Let us call a lattice identity λ in n^2 variables a *frame identity* if λ implies modularity and λ holds in a modular lattice iff it holds for the elements of every (von Neumann) n-frame of the lattice. Freese and McKenzie showed that frame identities can be characterized by Mal'tsev conditions. Although that time there was a hope that their method combined with [17] gives a Mal'tsev condition for each λ that implies modularity, cf. p. 155 in [11], Pálfy and Szabó [21] destroyed this expectation.

The goal of the present paper is to prove that each lattice identity implying modularity is equivalent to a Mal'tsev condition. Moreover, this Mal'tsev condition is very easy to construct. In order to formulate a slightly stronger statement, some definitions come first.

A lattice identity λ is said to imply modularity in congruence varieties, in notation $\lambda \models_c \text{mod}$, if for any variety V if all the congruence lattices $\mathrm{Con}(A)$, $A \in V$, satisfy λ then all these lattices are modular. If λ implies modularity in the usual lattice theoretic sense then of course $\lambda \models_c \text{mod}$ as well. However, it was a great surprise by Nation [20] that $\lambda \models_c \text{mod}$ is possible even when λ does not imply modularity in the usual sense. Jónsson [13] gives an overview of similar results. We mention that there is an algorithm to test if $\lambda \models_c \text{mod}$, cf. [5], which is based on Day and Freese [10].

Given a lattice term p and $k \geq 2$, we define $p^{(k)}$ via induction as follows. If p is a variable then let $p^{(k)} = p$. If $p = r \wedge s$ then let $p^{(k)} = r^{(k)} \cap s^{(k)}$. Finally, if $p = r \vee s$ then let $p^{(k)} = r^{(k)} \circ s^{(k)} \circ r^{(k)} \circ s^{(k)} \circ \dots$ with k factors on the right. When congruences or, more generally, reflexive compatible relations are substituted for the variables of $p^{(k)}$ then the operations $\cap$ and $\circ$ will be interpreted as intersection and relational product, respectively. Now and in the sequel by a *lattice identity* λ we mean an *inequality* $p \leq q$ where p and q are lattice terms. This does not hurt generality, for each $p \leq q$ is equivalent to an appropriate identity $r = s$ modulo lattice theory and vice versa. If $\lambda : p \leq q$ is a lattice identity and $m, n \geq 2$ then we can consider the *inclusion* $p^{(m)} \subseteq q^{(n)}$. If A is an algebra then $p^{(m)}$ and $q^{(n)}$ do not give congruences in general when their variables are substituted by congruences of A. However, it makes sense to say that $p^{(m)} \subseteq q^{(n)}$ holds or fails for congruences of A. Now Wille [24] and Pixley [22] give an easy algorithm to construct a strong Mal'tsev condition $M(p^{(m)} \subseteq q^{(n)})$ such that, for any variety $\mathcal{V}$, $p^{(m)} \subseteq q^{(n)}$ holds for congruences of all algebras in $\mathcal{V}$ if and only if $M(p^{(m)} \subseteq q^{(n)})$ holds in $\mathcal{V}$. (Notice that the construction of $M(p^{(m)} \subseteq q^{(n)})$ is outlined in Freese and McKenzie [11], and, with the notation $U(G_m(p) \leq G_n(q))$, it is detailed in [4].) Wille and Pixley showed also that $p^{(m)} \subseteq q$ holds for congruences of algebras in $\mathcal{V}$ if and only if $\mathcal{V}$ satisfies the Mal'tsev condition "there is an n such that $M(p^{(m)} \subseteq q^{(n)})$ holds"; this will be needed in our proof. Now we can formulate the main result.

Theorem 1. *Let $\lambda : p \leq q$ be a lattice identity such that $\lambda \models_c$ modularity. Then for any variety $\mathcal{V}$ the following two conditions are equivalent.*

(a) For all $A \in \mathcal{V}$, λ holds in the congruence lattice of A.

(b) $\mathcal{V}$ satisfies the Mal'tsev condition "there is an $n \geq 2$ such that $M(p^{(3)} \subseteq q^{(n)})$ and ($\mathbf{D}_n$) hold".

This paper will not detail the construction of $M(p^{(3)} \subseteq q^{(n)})$, but we mention that if we consider $\lambda : x \wedge (y \vee (x \wedge z)) \leq (x \wedge y) \vee (x \wedge z)$, the modular law, then Day's characterization of congruence modularity becomes a particular case of Theorem 1.

Before proving Theorem 1 we give some definitions and remarks. Reflexive symmetric compatible relations of an algebra are called *tolerances*, cf. Chajda [1] for an overview. The set of tolerances of A will be denoted by $\operatorname{Tol} A$. The *transitive closure* of a tolerance $\Phi \in \operatorname{Tol} A$ will be denoted by

$$\Phi^* = \bigcup_{n=1}^{\infty} (\Phi \circ \Phi \circ \Phi \circ \ldots) \quad (n \text{ factors}).$$

Note that Φ^* always belongs to $\operatorname{Con} A$, the congruence lattice of A, and

$$\alpha \vee \beta = (\alpha \cup \beta)^* \tag{1}$$

holds for any $\alpha, \beta \in \operatorname{Con} A$. Our interest in tolerances started with generalizing the Shifting Principle from Gumm [16] for congruence distributive varieties, cf. [2] and [3]. It appeared soon that formulas give stronger generalizations than diagrams both for the congruence distributive and for the congruence modular case, and we proved in [6] that if $\mathcal{V}$ is a congruence modular variety, $A \in \mathcal{V}$ and $\Gamma, \Phi, \Psi \in \operatorname{Tol} A$ then

$$\Gamma \cap (\Phi \cup (\Gamma \cap \Psi))^* \subseteq ((\Gamma \cap \Phi) \cup (\Gamma \cap \Psi))^*. \tag{2}$$

Notice that formally, according to (1), (2) is a variant of the modular law. Substituting 0 for Ψ in (2) we obtained, cf. Proposition 1 in [6], that

$$\Gamma \cap \Phi^* \subseteq (\Gamma \cap \Phi)^*. \tag{3}$$

Notice that it is essential to consider varieties here, for [6] presents a single algebra with modular congruence lattice, a tolerance Φ and a congruence Γ of this algebra such that $\Gamma \cap \Phi^* \subseteq (\Gamma \cap \Phi)^*$ fails. As the next step towards Theorem 1, Radeleczki [7] and later, independently, Kearnes [18] noticed that (3) trivially implies a more useful statement: if A belongs to a congruence modular variety and $\Gamma, \Phi \in \operatorname{Tol} A$ then

$$\Gamma^* \cap \Phi^* = (\Gamma \cap \Phi)^*. \tag{4}$$

Indeed, applying (3) for Γ^* and Φ, and then for Φ and Γ we obtain the nontrivial inclusion part of (4). To make the present paper self-contained, we will give a direct proof of (3), which is of course a special (and therefore a bit shorter) case of the proof of (2).

Proof. In order to prove Theorem 1 first we prove (3). Let V be a congruence modular variety with Day-terms $m_0, \dots, m_n$. Let Γ and Φ be tolerances of an algebra A in V. First we show that

$$\Gamma \cap (\Phi \circ \Phi) \subseteq (\Gamma \cap \Phi)^*. \tag{5}$$

Suppose $(a,b) \in \Gamma \cap (\Phi \circ \Phi)$. Then there is an element $c \in A$ with $(a,c),(c,b) \in \Phi$, and of course, $(a,b) \in \Gamma$. Now we define further elements. Let $d_i = m_i(a,c,c,b)$ for $i = 0, \dots, n$ and let $e_i = m_i(a,a,b,b)$ for i even, $i = 0, \dots, n$. Notice that $d_i = d_{i+1}$ for i odd. Let j denote an arbitrary even index. Then $(d_j, e_j) \in \Phi$ is clear. Since

$$\begin{aligned} d_j &= m_j(m_j(a,c,c,b),a,a,m_j(a,c,c,b)) \ \Gamma \ m_j(m_j(a,c,c,a),a,b,m_j(b,c,c,b)) \\ &= m_j(a,a,b,b) = e_j, \end{aligned}$$

we obtain $(d_j, e_j) \in \Gamma \cap \Phi$. Since $e_j = m_j(a,a,b,b) = m_{j+1}(a,a,b,b)$, we conclude $(d_{j+1}, e_j) \in \Gamma \cap \Phi$ exactly the same way. Since any two neighbouring members of the sequence

$$a = d_0,\ e_0,\ d_1 = d_2,\ e_2,\ d_3 = d_4,\ e_4,\ d_5 = d_6,\ \dots,\ d_n = b$$

are in the relation $\Gamma \cap \Phi$, we infer $(a,b) \in (\Gamma \cap \Phi)^*$. This proves (5).

Now let $\Phi_0 = \Phi$ and $\Phi_{n+1} = \Phi_n \circ \Phi_n$, these are tolerances again. We claim that, for all n,

$$\Gamma \cap \Phi_n \subseteq (\Gamma \cap \Phi)^*. \tag{6}$$

This is evident for $n = 0$. If (6) holds for some n then, applying (5) for Γ and Φ_n and using the induction hypothesis, we have

$$\Gamma \cap \Phi_{n+1} = \Gamma \cap (\Phi_n \circ \Phi_n) \subseteq (\Gamma \cap \Phi_n)^* \subseteq ((\Gamma \cap \Phi)^*)^* = (\Gamma \cap \Phi)^*.$$

Hence (6) holds for all n. Therefore we obtain

$$\Gamma \cap \Phi^* = \Gamma \cap \bigcup_{n=0}^{\infty} \Phi_n = \bigcup_{n=0}^{\infty} (\Gamma \cap \Phi_n) \subseteq \bigcup_{n=0}^{\infty} (\Gamma \cap \Phi)^* = (\Gamma \cap \Phi)^*$$

This proves (3) for any tolerances Γ and Φ.

Applying (3) first for Γ^* and Φ and then for Φ and Γ we obtain

$$\Gamma^* \cap \Phi^* \subseteq (\Gamma^* \cap \Phi)^* = (\Phi \cap \Gamma^*)^* \subseteq ((\Phi \cap \Gamma)^*)^* = (\Gamma \cap \Phi)^*,$$

i.e., $\Gamma^* \cap \Phi^* \subseteq (\Gamma \cap \Phi)^*$. Since forming transitive closure is a monotone operation, the reverse inclusion is evident. This proves (4).

For tolerances Φ and Ψ it is easy to see that $\Phi \circ \Psi \circ \Phi$ is again a tolerance. It follows from reflexivity that

$$(\Phi \circ \Psi \circ \Phi)^* = \Phi^* \vee \Psi^*, \tag{7}$$

where the join is taken in $\operatorname{Con} A$. An easy induction shows that if $r = r(x_1, \dots, x_k)$ is a lattice term and $\Phi_1, \dots, \Phi_k$ are tolerances or, as a particular case, congruences of an algebra A then $r^{(3)}(\Phi_1, \dots, \Phi_k)$ is a tolerance again. Now let $\mathcal{V}$ be a variety and assume (a). Let p and q be, say, k-ary lattice terms. Since an easy induction shows that, for any $A \in \mathcal{V}$ and any congruences $\alpha_1, \dots, \alpha_k$ of A we have $p^{(3)}(\alpha_1, \dots, \alpha_k) \subseteq p(\alpha_1, \dots, \alpha_k)$, we conclude that $p^{(3)} \subseteq q$ holds for congruences of any $A \in \mathcal{V}$. Hence the afore-mentioned result of Wille and Pixley yields that $M(p^{(3)} \subseteq q^{(n_1)})$ holds in $\mathcal{V}$ for some n_1. Since $\lambda \models_c$ mod, there is an n_2 such that $\mathbf{D}_{n_2}$ holds in $\mathcal{V}$. Now let n be the maximum of n_1 and n_2. Since Mal'tsev conditions are weakening in their parameter, we obtain that $\mathcal{V}$ satisfies (b).

Now, to show the reverse implication, assume that (b) holds. By Day's result, $\mathcal{V}$ is congruence modular, whence (4) holds as well. The afore-mentioned result of Wille and Pixley gives that $p^{(3)} \subseteq q$ holds for congruences in V. So for any congruences $\alpha_1, \dots, \alpha_k$ of $A \in V$, we have $p^{(3)}(\alpha_1, \dots, \alpha_k) \subseteq q(\alpha_1, \dots, \alpha_k)$. Hence

$$p_3(\alpha_1, \dots, \alpha_k)^* \subseteq q(\alpha_1, \dots, \alpha_k)^*. \tag{8}$$

Since $q(\alpha_1, \dots, \alpha_k)$ is a congruence, it equals its transitive closure. On the other hand, a trivial induction based on (4) and (7) gives that

$$p_3(\alpha_1, \dots, \alpha_k)^* = p(\alpha_1^*, \dots, \alpha_k^*) = p(\alpha_1, \dots, \alpha_k).$$

This way (8) turns into

$$p(\alpha_1, \dots, \alpha_k) \subseteq q(\alpha_1, \dots, \alpha_k),$$

proving that λ holds in $\operatorname{Con}(A)$ for all $A \in V$. Thus (a) holds. □

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