INTRODUCING CZÉDLI-TYPE ISLANDS, EXTENDED VERSION (WITH PROOFS)

1. Introduction

What age is the best to start researching mathematics? Of course as early as possible. However in most cases it is very difficult to give real "living" research topics and open problems to students because understanding most research areas need higher level of knowledge. In this paper we would like to introduce a topic that *needs no specific pre-knowledge,* but since it is a new area, it has many possibilities for raising new open problems. Like in all research fields, it needs creativity to continue the investigation, however we feel that in this topic the investigation can proceed in elementary level as well, the appearing problems can be solved by elementary (sometimes tricky and sometimes easy) way; we believe that with the help of teachers' guidance, the students can become beginner researchers and miniature mathematicians in this topic by creating and/or solving easy exercises as well as challenging problems that are possible to post even at competitions. Moreover, we think teachers should encourage students to write computer programs for conjecturing or answering some island problems.

Many students being interested in informatics, in their early years of studies become attracted by binary and other codes, and by techniques suitable for handling codewords. Instantaneous codes are easy to understand and useful in applications. The topic we are dealing with offers an interesting approach to these codes, though in a particular setting of one-dimensional islands, as indicated in the sequel.

It was the notion of Czédli-type islands that caught attention of several mathematicians. The notion comes from a recent result from the paper

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[6]. This paper contains a necessary and sufficient condition for a code to be instantaneous. This condition uses the notion of full segments of vectors. Now these full segments are precisely one-dimensional islands. It is important to know the maximum number of one dimensional islands because it shows – by the mentioned necessary and sufficient condition – the number of equations one should check in order to decide whether the given code is instantaneous.

Several generalizations of the notion of full segments gave rise to interesting combinatorial problems. In two dimensions, Gábor Czédli [2] has determined the maximum number of rectangular islands: on the $m \times n$ size rectangular board, for the maximum number of rectangular islands $f(m, n)$ he obtained

$$f(m,n) = \left\lfloor \frac{(mn + m + n - 1)}{2} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the lower integer part of x or in other words the greatest integer in x. His proof is based on a result in lattice theory ([4]), but now by [1] two elementary ways are also known to prove the same result. We mention, that the exact result i.e. exact formula for the maximum number of triangular islands on the triangular grid is not yet known, only upper and lower bounds are given in [7]. The situation is the same with the square islands on a square grid, details are in [8]. The topic of islands is still developing, already many branches of mathematics are involved. Investigation of island-problems has already lead to further results in lattice theory, see e.g. [3] and [5]. The reader might find further details in the References, but for understanding the present paper, *the reader need not read anything in advance.*

Now we would like to introduce the topic to the reader in such a way that we present only one challenging – but elementary – problem in details, with its solution. We give some easier exercises as well, the solution of which the reader can figure out after reading the 4th section or even earlier – but it is useful if we want to introduce the topic to students step by step. We hope that the reader can continue the topic with his/her ideas by creating other exercises, problems or even research results.

If we want to introduce the topic to a group of students, then we recommend to show some well-chosen easy exercises first, turn to our detailed problem next, and finally a brainstorming can come in order to find new, solvable but not trivial problems in lucky cases (as usual in research).

We believe that every teacher knows his/her students best. So we do not want to give strict rules for the presentation of the material. The topic is also a challange for teachers, but we strongly hope that after reading our paper teachers will write papers about how they introduced the topic to students, how students reacted and which questions appeared during discussion...

Islands are very important objects in our life, see Figure 1.

Figure 1.

Up to recent times, the "story" on the picture was not a topic of classroom mathematics. However we encourage math teachers to discuss some digitalized versions of it with students (of all levels) — just like some researchers are doing nowadays.

2. The definition of a one-dimensional island

Let n square cells be given, this will be our so-called "board". We write nonnegative integers into each cell. We call these nonnegative numbers *heights*. We say that some consecutive cells constitute an *island*, if the integers in them are all greater than the integers in the neighboring cells. See also Figures 2-6. We can observe that an island can be included in another island. We indicate "higher" islands with darker grey coloring in our Figures.

0	0	1	1	1	0	0	0

Figure 2, here we see one island.

0	1	2	2	1	0	1	2	1	2

Figure 3, here we see 5 islands.

0	1	2	2	1	0	1	2	1	2	3	3

Figure 4, here we see 6 islands.

2	1	2	0	2	1	2	0	2	1

Figure 5, here we see 8 islands.

3	2	3	1	3	2	3	0	3	2	3	1

Figure 6, here we see 11 islands.

3. The problem

Determine the maximum number of islands in the one-dimensional board of length n, if the integers in the cells are elements of the set $\{0, 1, 2, \ldots, h\}$. (Notice, that the brute force method is not effective, i.e. if we want to list all the cases one by one and count the islands for all cases, then e.g. for $n = 20$ and $h = 2$ there are approximately 3.48×10^9 cases to investigate, see also Exercise 6.) We suppose that the board is bordered with cells filled with 0 at the ends, i.e. the 0th and the $n + 1$-th cells are filled with 0. Therefore, if the board does not contain 0, then the board itself is considered to be an island.

4. The solution of the problem

The answer is: the maximum number of islands $I(n, h) = n - \left\lfloor \frac{n}{2^h} \right\rfloor$, where $\lfloor x \rfloor$ denotes the lower integer part of x, i.e. in other words the greatest integer in x. More usual notation for the lower integer part is $[x]$, but in other papers in this topic the upper integer part $\lceil x \rceil$ is also used.

We show first that we can create *at least* this number of islands in such a way that each second cell has the height h starting with the first. We proceed by induction on h. If $h = 1$, then obviously we can realize at least $n - \left\lfloor \frac{n}{2} \right\rfloor$ islands, putting 1 to each second cell, starting with the first, see also Figures 7-8.

1	0	1	0	1	0	1	0

Figure 7, the even case.

1	0	1	0	1	0	1	0	1

Figure 8, the odd case.

If $h > 1$, then let $n = 4k$ first. Then by the induction hypothesis, on $2k$ cells with height $h-1$ we can realize at least

$$2k - \left\lfloor \frac{2k}{2^{h-1}} \right\rfloor$$

islands, and each second cell has height $h-1$ starting with the first; see also Figure 9 where $h = 3$.

2	1	2	0	2	1

Figure 9.

Now we replace each cell with maximum height $h-1$ by three cells with the heights h, $h-1$, h respectively. Then we obtain $n = 4k$ cells with

$$2k - \left\lfloor \frac{2k}{2^{h-1}} \right\rfloor + 2k = 4k - \left\lfloor \frac{4k}{2^{h}} \right\rfloor$$

islands. See also Figure 10 where still $h = 3$, we replaced all second cells that were filled with 2 by three cells with heights 3, 2, 3 respectively.

3	2	3	1	3	2	3	0	3	2	3	1

Figure 10.

If $n = 4k + 1$, then, because of an additional cell, we have

$$4k + 1 - \left\lfloor \frac{4k}{2^{h}} \right\rfloor$$

islands, however

$$4k + 1 - \left\lfloor \frac{4k}{2^{h}} \right\rfloor \geq 4k + 1 - \left\lfloor \frac{4k+1}{2^{h}} \right\rfloor .$$

See also Figure 11.

3	2	3	1	3	2	3	0	3	2	3	1	3

Figure 11.

If $n = 4k+2$, then, because of two additional cells, we have

$$4k + 2 - \left\lfloor \frac{4k}{2^h} \right\rfloor$$

islands, however

$$4k + 2 - \left\lfloor \frac{4k}{2^h} \right\rfloor \geq 4k + 2 - \left\lfloor \frac{4k+2}{2^h} \right\rfloor .$$

See also Figure 12.

3	2	3	1	3	2	3	0	3	2	3	1	3	2

Figure 12.

If $n = 4k+3$, then, because of three additional cells, we have

$$4k + 3 - \left\lfloor \frac{4k}{2^h} \right\rfloor$$

islands, however

$$4k + 3 - \left\lfloor \frac{4k}{2^h} \right\rfloor \geq 4k + 3 - \left\lfloor \frac{4k+3}{2^h} \right\rfloor .$$

See also Figure 13.

3	2	3	1	3	2	3	0	3	2	3	1	3	2	3

Figure 13.

So, we proved that

$$I(n,h) \geq n - \left\lfloor \frac{n}{2^h} \right\rfloor$$

holds.

The other part is to show that we cannot create more islands, i.e.

$$I(n,h) \leq n - \left\lfloor \frac{n}{2^h} \right\rfloor ,$$

we show this by induction on n.

If $n = 1$, then the statement is true.

Let $n > 1$. The induction hypothesis is that for all $n' < n$,

$$I(n', h) = n' - \left\lfloor \frac{n'}{2^h} \right\rfloor .$$

We first suppose that we use height 0. The cell containing this 0 divide the board into two parts, we denote the lengths of these parts with k and l, obviously $k + l + 1 = n$, where $k, l \geq 0$. If the number of the islands on our board now is $|I|$, then

$$|I| \leq k - \left\lfloor \frac{k}{2^h} \right\rfloor + l - \left\lfloor \frac{l}{2^h} \right\rfloor = k + l + 1 - \left(\left\lfloor \frac{k}{2^h} \right\rfloor + \left\lfloor \frac{l}{2^h} \right\rfloor + 1 \right).$$

We prove the following inequality first:

$$\left\lfloor \frac{k}{2^h} \right\rfloor + \left\lfloor \frac{l}{2^h} \right\rfloor + 1 \geq \left\lfloor \frac{k+l+1}{2^h} \right\rfloor .$$

To see this, we obviously have

$$\left\lfloor \frac{k+l+1}{2^h} \right\rfloor \leq \frac{k}{2^h} + \frac{l}{2^h} + \frac{1}{2^h} \leq \left\lfloor \frac{k}{2^h} \right\rfloor + \frac{2^h - 1}{2^h} + \left\lfloor \frac{l}{2^h} \right\rfloor + \frac{2^h - 1}{2^h} + \frac{1}{2^h} =$$
$$= \left\lfloor \frac{k}{2^h} \right\rfloor + \left\lfloor \frac{l}{2^h} \right\rfloor + \frac{2^{h+1} - 1}{2^h}.$$

Taking the lower integer part of both sides of the inequality

$$\left\lfloor \frac{k+l+1}{2^h} \right\rfloor \leq \left\lfloor \frac{k}{2^h} \right\rfloor + \left\lfloor \frac{l}{2^h} \right\rfloor + \frac{2^{h+1} - 1}{2^h}$$

we obtain

$$\left\lfloor \frac{k+l+1}{2^h} \right\rfloor \leq \left\lfloor \frac{k}{2^h} \right\rfloor + \left\lfloor \frac{l}{2^h} \right\rfloor + 1.$$

With the last inequality, we have

$$|I| \leq k+l+1 - \left(\left\lfloor \frac{k}{2^h} \right\rfloor + \left\lfloor \frac{l}{2^h} \right\rfloor + 1 \right) \leq k+l+1 - \left\lfloor \frac{k+l+1}{2^h} \right\rfloor = n - \left\lfloor \frac{n}{2^h} \right\rfloor .$$

If we do not use the height 0, then we denote by m the minimum of the heights on the board. Then, the whole board is an island. The other islands on this board are the same as those on the new board obtained by subtracting m from each cell. Since this new board has a zero and the maximum value of $h-m$, it has at most $n-\left\lfloor\frac{n}{2^{h-m}}\right\rfloor$ islands. Adding one for the whole board island, gives an upper bound of

$$n-\left\lfloor\frac{n}{2^{h-m}}\right\rfloor+1.$$

Now, since $\left\lfloor\frac{n}{2^{h-m}}\right\rfloor \geq \left\lfloor\frac{n}{2^{h-1}}\right\rfloor$, we have

$$n-\left\lfloor\frac{n}{2^{h-m}}\right\rfloor+1 \leq n-\left\lfloor\frac{n}{2^{h-1}}\right\rfloor+1.$$

Now

$$n-\left\lfloor\frac{n}{2^{h-1}}\right\rfloor+1 = n-\left\lfloor\frac{2n-2^h}{2^h}\right\rfloor.$$

If $n \geq 2^h$, then $2n-2^h \geq n$, so we again have

$$n-\left\lfloor\frac{2n-2^h}{2^h}\right\rfloor \leq n-\left\lfloor\frac{n}{2^h}\right\rfloor$$

as an upper bound.

If $n < 2^h$, then $\left\lfloor\frac{n}{2^h}\right\rfloor = 0$, so it is enough to prove that the number of islands on an n-length board cannot exceed n (independently of h). We proceed by induction on n. If $n=1$, then we can have at most one island. Let $n>1$. The induction hypothesis is, that if $n'<n$, then on a board of length n' the number of islands cannot exceed n'. Now, a cell with minimal height divides the n-length board into subboards of length k and $n-k-1$, where $k \geq 0$. However, the whole board might be an island now, so, after using the induction hypothesis, the number of islands cannot exceed $k+n-k-1+1=n$.

* * *

The students might like that we proved the lower estimate by induction on h, while the upper estimate by induction on n, and they "luckily" reached each other.

5. Some didactic remarks

For younger students it might not be appropriate to present the above solution to them immediately. In this case we suggest to introduce the topic in the following way.

At the beginning, we should fill in the board somehow, fix the water level and count the number of islands at fixed water level. Then count the islands with all water levels and add them together.

After discussing some easy exercises, we can turn to our proposed problem in Sections 3-4 in the following way: first let $h = 1$, draw the following Figures 13-14:

1	0	1	0	1	0	1	0

Figure 13, the even case.

1	0	1	0	1	0	1	0	1

Figure 14, the odd case.

Then the maximum number of islands is obviously equal to $n - \left\lfloor \frac{n}{2} \right\rfloor$.

Then let $h = 2$, draw the following Figures 15-18:

2	1	2	0	2	1	2	0	2	1	2	0

Figure 15, $n = 4k$.

2	1	2	0	2	1	2	0	2	1	2	0	2

Figure 16, $n = 4k + 1$.

2	1	2	0	2	1	2	0	2	1	2	0	2	1

Figure 17, $n = 4k + 2$.

2	1	2	0	2	1	2	0	2	1	2	0	2	1	2

Figure 18, $n = 4k + 3$.

By Figures 15-18 we conjecture that $I(n, 2) = n - \left\lfloor \frac{n}{4} \right\rfloor$. Let us prove this statement. Then, make conjecture for arbitrary finite h and try to prove the conjecture.

6. Exercises

1. For given n and h, put numbers into the cells of the one dimensional board in such a way that we obtain exactly:

(a) 0, (b) 1, (c) 2, (d) 3, (e) $\left\lfloor \frac{n}{2} \right\rfloor - 1$, (f) $\left\lfloor \frac{n}{2} \right\rfloor$, (g) $\left\lfloor \frac{n}{2} \right\rfloor + 1$, (h) $n - 2$, (i) $n - 1$, (j) n, (k) $n + 1$, (l) h, (m) $h + 1$, (n) $2h$, (o) h^2

islands, or show that it is not possible.

2. Prove that if we write arbitrary natural numbers into the cells of our one dimensional board of length n, then the maximum number of islands is equal to n, if we consider the whole board as the biggest island (because it is above its surroundings). (Hint: let us put the increasing numbers $1, 2, 3, \ldots$ in the cells. This configuration obviously gives a lower estimate for the maximum number of possible islands. Then prove by induction on the number of cells that n as an upper estimate is also valid.)

3. Prove that if for the length n of the one dimensional board we have $n \leq 2^h - 1$, then if we put elements of the set $\{0, 1, 2, \ldots, h\}$ in the cells, then we can have n islands, but if $n \geq 2^h$, then the maximum number of islands is less than n. (Hint: use the solved problem of this paper.)

4. Prove that if we put the numbers $\{1, 2, \ldots, n\}$ into the cells of our one dimensional board of length n in *any* order, then we will have always *exactly* n islands.

5. Show, that for arbitrary h the number of islands cannot be maximal if we have height h in (at least two) neighbouring cells.

6. In the problem of Section 3 show that if we want to list all the cases one by one and count – or compare – the islands for all cases, then e.g. for $n = 20$ and $h = 2$ there are approximately 3.48×10^9 cases to investigate.

7. Increasing the water level

Our last picture shows that if we increase water level, then the number of islands could increase, moreover, people are closer to discuss math problems, to find new open problems and to solve them ...

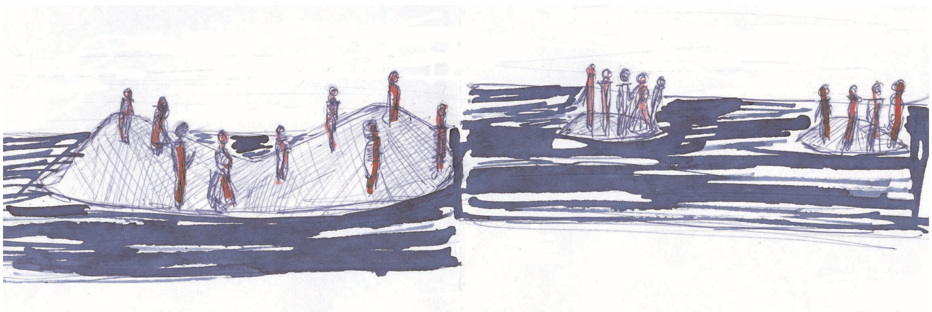

Figure 19.

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