NOTES ON CD-INDEPENDENT SUBSETS

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Abstract. It is proved in [8] that any two CD-bases in a finite distributive lattice have the same number of elements. We investigate CD-bases in posets, semilattices and lattices. It is shown that their CD-bases can be characterized as maximal chains in a related poset or lattice. We point out two known lattice classes characterized by some "0-conditions" whose CD-bases satisfy the mentioned property.

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Introduction

Several independence notions are already investigated in lattice theory, see e.g. [6, 8, 9, 10]. The main result in [9] about weak independence was successfully applied to a combinatorial problem, namely to the problem of determining the maximum number of rectangular islands, see [5] for details. The notion of an island appears first in [13] under the name of "full segment". It was observed that many subsets in island problems (see e.g. [1] or [14]) are in fact CD-independent. Furthermore, the notion of a classification tree can be also defined as a particular CD-independent set (see [20]).

In [8] the authors showed that the CD-bases in a finite distributive lattice have the same number of elements, and conversely, if all finite lattices in a lattice variety have this property, then the variety must coincide with the variety of distributive lattices.

In this paper we define CD-independent sets in an arbitrary poset $\mathbb{P} = (P, \leq)$, and we show that the CD bases of any poset $\mathbb{P}$ can be characterized as maximal chains in a related poset $\mathcal{D}(P)$. We prove that if $\mathbb{P}$ is a complete lattice, then $\mathcal{D}(P)$ is also a lattice having a weak distributive property. We also point out two known lattice classes where the CD-bases in finite lattices have the mentioned property: The first class is that one of graded, dp-distributive lattices, and the second class is obtained by generalizing the properties of the so-called interval lattices (having their origine in graph theory). None of these classes is a variety, however their existence can motivate the study of CD-bases in some particular lattice classes related to

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combinatorial problems. Since these classes are generalizations of distributive lattices, our results also imply that the CD-bases in a finite distributive lattice have the same number of elements, settled originally in [8] (see e.g. Corollary 3.6 or Corollary 3.11).

1. CD-independent subsets in posets

Let $\mathbb{P} = (P, \leq)$ be a partially ordered set and $a, b \in P$. The elements a and b are called *disjoint* and we write $a \perp b$ if

either $\mathbb{P}$ has least element $0 \in P$ and $\inf\{a, b\} = 0$,
or $\mathbb{P}$ is without 0 and the elements a and b have no common lowerbound.

Notice, that $a \perp b$ implies $x \perp y$ for all $x, y \in P$ with $x \leq a$ and $y \leq b$. (1)

A nonempty set $X \subseteq P$ is called *CD-independent* if for any $x, y \in X$, $x \leq y$ or $y \leq x$, or $x \perp y$ holds. Maximal CD-independent sets (with respect to $\subseteq$) are called *CD-bases* in $\mathbb{P}$. If $\mathbb{P}$ contains least element 0 (greatest element 1) and B is a CD-base, then obviously $0 \in B$ ($1 \in B$). A nonempty set D of nonzero elements of P is called a *disjoint set* in $\mathbb{P}$ if $x \perp y$ holds for all $x, y \in D$, $x \neq y$; if $\mathbb{P}$ has 0-element, then $\{0\}$ is considered to be a disjoint set, too. Observe, that D is a disjoint set if and only if it is a CD-independent antichain in $\mathbb{P}$. This characterization and the fact that any nonempty subset of a CD-independent set is also CD-independent yield:

Remark 1.1. (i) If D is a disjoint set in P, then $0 \in D \Leftrightarrow D = \{0\}$.
(ii) If X is a CD-independent set in P, then any antichain $A \subseteq X$ is a disjoint set in P.

We recall that any antichain $A = \{a_i \mid i \in I\}$ of a poset $\mathbb{P}$ determines a unique order-ideal $I(A)$ of $\mathbb{P}$, namely

$$I(A) = \bigcup_{i \in I} (a_i] = \{x \in P \mid x \leq a_i \text{ for some } i \in I\},$$

where $(a]$ stands for the principal ideal of an element $a \in P$. As the order-ideals of any poset form a (distributive) lattice with respect to $\subseteq$, the antichains of a poset can be ordered as follows: If A_1, A_2 are antichains in $\mathbb{P}$, then we say that A_1 *is dominated by* A_2, and we denote it by $A_1 \leqslant A_2$ if

$$I(A_1) \subseteq I(A_2).$$

It is well-known that $\leqslant$ is a partial order (see e.g. [4] or [11]), and it is easy to see that $A_1 \leqslant A_2$ is satisfied if and only if the following condition holds:

(A) For each $x \in A_1$ there exists a $y \in A_2$ with $x \leq y$.

Let $\mathcal{D}(P)$ denote the set of all disjoint sets of $\mathbb{P}$. As the disjoint sets of $\mathbb{P}$ are also antichains, restricting $\leqslant$ to $\mathcal{D}(P)$, we obtain a poset $(\mathcal{D}(P), \leqslant)$.

Clearly, if $\mathbb{P}$ has least element 0, then $\{0\}$ is the least element of $(\mathcal{D}(P), \leqslant)$. The following facts are immediate consequeces of this definition (and (1)):

Remark 1.2. (i) $I(A_1) \prec I(A_2) \Leftrightarrow A_1 \prec A_2$ for any antichains $A_1, A_2 \subseteq P$.

(ii) Let D_1 and D_2 be disjoint sets in P. Then $D_1 \subseteq D_2$ implies $D_1 \leqslant D_2$. Furthermore, if $D_1 \leqslant D_2$, then

$$\text{for all } (x_1, x_2) \in D_1 \times D_2,\ \ x_1 \leq x_2 \text{ or } x_1 \perp x_2. \tag{2}$$

(iii) Observe, that the poset $(P, \leq)$ can be order-embedded into $(\mathcal{D}(P), \leqslant)$: Indeed, for any $x \in P$ the set $\{x\}$ itself is a disjoint set, and clearly,

$$x \leq y \Leftrightarrow (x] \subseteq (y] \Leftrightarrow \{x\} \leqslant \{y\}$$

hold for any $x, y \in P$.

Now define a relation $\rho \subseteq P \times P$ as follows: For any $x, y \in P$

$$(x, y) \in \rho \Leftrightarrow x \leq y \text{ or } y \leq x \text{ or } x \perp y.$$

Then ρ is reflexive and symmetric by its definition, i.e. it is a tolerance relation on P. A *block* of a tolerance relation $\tau \subseteq A \times A$ is a subset $B \subseteq A$ maximal with respect to the property $B \times B \subseteq \tau$ (see e.g. [2]). It is easy to see that the CD-bases of $\mathbb{P}$ are exactly the tolerance blocks of ρ. As any tolerance relation has at least one tolerance block, and its blocks form a covering of the underlying set, we obtain:

Proposition 1.3. *Any poset $\mathbb{P} = (P, \leq)$ hast at least one CD-base, and the set P is covered by the CD-bases of $\mathbb{P}$.*

Proposition 1.4. *If B is a CD-base and $D \subseteq B$ is a disjoint set in the poset $(P, \leq)$, then $I(D) \cap B$ is also a CD-base in the subposet $(I(D), \leq)$.*

Proof. As $I(D) \cap B$ remains CD-independent in $(I(D), \leq)$, it is enough to show that for any $x \in I(D) \setminus B$ the set $(I(D) \cap B) \cup \{x\}$ is not CD-independent. Indeed, as B is a CD-base, $B \cup \{x\}$ is not CD-independent, and hence there exists a $b \in B$ such that b and x are not comparable and have a common lowerbound $u \neq 0$. Then $u \leq x \leq a$ for some $a \in D$, and $u \in I(D)$. Since $0 < u \leq a, b$ and $a, b \in B$, a and b must be comparable. Hence $b \leq a$, otherwise $a \leq b$ would imply $x \leq b$, a contradiction. Thus we get $b \in I(D) \cap B$, and hence $(B \cap I(D)) \cup \{x\}$ is not CD-independent. □

Given a set X, let $|X|$ denote its cardinality. The connection between CD-bases of a poset $\mathbb{P}$ and the poset $(\mathcal{D}(P), \leqslant)$ is shown by the next theorem:

Theorem 1.5. *Let B be a CD-base of a finite poset $(P, \leq)$, and* let $|B| = n$. *Then there exists a maximal chain $\{D_i\}_{1 \leq i \leq n}$ in $\mathcal{D}(P)$, such that $B = \bigcup_{i=1}^{n} D_i$. Moreover, for any maximal chain $\{D_i\}_{1 \leq i \leq m}$ in $\mathcal{D}(P)$ the set $D = \bigcup_{i=1}^{m} D_i$ is a CD-base in $(P, \leq)$ with $|D| = m$.*

First we prove two lemmas:

Lemma 1.6. *If $D_1 \prec D_2$ in $\mathcal{D}(P)$, then $D_2 = \{a\} \cup \big\{y \in D_1 \setminus \{0\} \mid y \perp a\big\}$ for some minimal element a of the set*

$$S = \big\{s \in P \setminus (D_1 \cup \{0\}) \mid y \perp s \text{ or } y < s \text{ for all } y \in D_1\big\}.$$

Moreover, $D_1 \prec \{a\} \cup \big\{y \in D_1 \setminus \{0\} \mid y \perp a\big\}$ holds for any minimal element a of the set S.

Proof. Define $T_s = \{s\} \cup \big\{y \in D_1 \setminus \{0\} \mid y \perp s\big\}$ for each $s \in S$. Then T_s is a disjoint set, $T_s \neq D_1$, and $y \in T_s$ or $y < s$ holds for all $y \in D_1$. Hence, in view of (A), we obtain

$$D_1 < T_s \text{ for all } s \in S. \tag{3}$$

Further, let $D_1 < D_2$. Then $D_2 \neq \{0\}$, and hence $0 \notin D_2$, by Remark 1.1(i). Since, in virtue of (2), for any $y \in D_1$ and $s \in D_2$, $y \perp s$, or $y < s$, or $y = s$ holds, we have $D_2 \setminus D_1 \subseteq S$. Clearly, $D_2 \setminus D_1 \neq \emptyset$, otherwise by Remark 1.2(ii) $D_2 \subseteq D_1$ would imply $D_2 \leqslant D_1$, a contradiction. Select an element $a \in D_2 \setminus D_1$. Then $a \in S$, and in virtue of (3), $T_a = \{a\} \cup \{y \in D_1 \setminus \{0\} \mid y \perp a\}$ satisfies $D_1 < T_a$. Observe that $T_a \leq D_2$ by (A) since $a \leq a \in D_2$ and for each $y \in D_1$ there is a $y' \in D_2$ with $y \leq y'$. So, $D_1 < D_2$ and $a \in D_2 \setminus D_1$ imply that $D_1 < T_a \leq D_2$.

Assume now $D_1 \prec D_2$. Notice at this point that if b is also in $D_2 \setminus D_1$, then $T_b = D_2 = T_a$, and $\{b\} = T_b \setminus D_1 = T_a \setminus D_1 = \{a\}$. Thus

$$\text{if } D_1 \prec D_2, \text{ then } |D_2 \setminus D_1| = 1. \tag{4}$$

Then also $D_2 = T_a = \big\{a\} \cup \{y \in D_1 \setminus \{0\} \mid y \perp a\big\}$, as it was desired. Suppose that $s < a$ for some $s \in S$. As $T_s \setminus \{s\} \subseteq D_1 < T_a$, for each $y \in T_s \setminus \{s\}$ there is a $t \in T_a$ with $y \leq t$ according to (A). Since $s < a$ and $s \notin T_a$, by (3) and (A) we get $D_1 < T_s < T_a = D_2$, a contradiction to $D_1 \prec D_2$. Thus a is a minimal element in S.

If $a \in S$ is minimal, then T_a is a disjoint set, and $D_1 < T_a$ by (3). In order to prove $D_1 \prec T_a$, assume that $D_1 < D_2 \leqslant T_a$ for some $D_2 \in \mathcal{D}(P)$. Then, in view of the first part of our proof, $0 \notin D_2$, $\emptyset \neq D_2 \setminus D_1 \subseteq S$, and for any $b \in D_2 \setminus D_1$, T_b is a disjoint set satisfying $D_1 < T_b \leqslant D_2 \leqslant T_a$. Clearly, we have $b \leq t$ for some $t \in T_a = \{a\} \cup \big\{y \in D_1 \setminus \{0\} \mid y \perp a\big\}$ according to (A). If $t \in D_1$, then $b \in S$ and $t \not< b$ imply $t \perp b$, hence we get $0 = \inf\{b, t\} = b$, a contradiction to $0 \notin S$. Thus $t = a$ and $b \leq a$. As a is minimal element of S, we get $b = a$, $T_b = T_a$, and hence $D_2 = T_a$. This proves $D_1 \prec T_a$. □

Now let $\max(X)$ stand for the set of maximal elements of the set $X \subseteq P$.

Lemma 1.7. *Assume that B is a CD-base with at least two elements in a finite poset $\mathbb{P} = (P, \leq)$, $M = \max(B)$, and $m \in M$. Then M and $N := \max(B \setminus \{m\})$ are disjoint sets. Moreover M is a maximal element in $\mathcal{D}(P)$, and $N \prec M$ holds in $\mathcal{D}(P)$.*

Proof. Since M and N are antichains in a CD-independent set, they are disjoint sets. Suppose $M \leqslant D$ for some $D \in \mathcal{D}(P)$. In virtue of (2), for all $m \in M$ and $d \in D$ we have $m \leq d$ or $m \perp d$. Then by (1), $b \leq d$ or $b \perp d$ holds for all $b \in B$ and $d \in D$. This means that $B \cup D$ is a CD-independent set. Since B is a CD-base, we deduce $D \subseteq B$. Then for each $d \in D$ there is an $m \in M$ with $d \leq m$. In view of (A), this implies $D \leqslant M$ in $\mathcal{D}(P)$. Thus we get $M = D$, proving that M is maximal in $\mathcal{D}(P)$.

In order to prove $N \prec M$ in $\mathcal{D}(P)$, consider the subposet $(B, \leq)$. For any antichain $A \subseteq B$, denote by $I_B(A)$ the order-ideal determined by A in $(B, \leq)$. Clearly, $I_B(M) = B$. Since $I_B(N) = B \setminus \{m\} = I_B(M) \setminus \{m\}$, we obtain $I_B(N) \prec I_B(M)$, and hence $N \prec M$ holds in $\mathcal{D}(B)$ according to Remark 1.2(i).

Now, in virtue of (2), $N < M$ yields $y < m$ or $y \perp m$ for each $y \in N$, since $m \in M \setminus N$. Moreover, $m \neq 0$ because $m \notin I_B(N)$. The last two facts imply that m belongs to the set

$$S = \big\{s \in P \setminus (N \cup \{0\}) \mid y \perp s \text{ or } y < s \text{ for all } y \in N\big\}.$$

We claim that m is a minimal element in S. Indeed, let $s \leq m$ for some $s \in S$. Since for any $b \in B \setminus \{m\}$, $b \leq y$ for some $y \in N$, by (1) we have $b \perp s$ or $b < s$ for all $b \in B \setminus \{m\}$. Then $B \cup \{s\}$ is a CD-independent set. As B is a CD-base, we get $s \in B$. Now $s \in B \setminus \{m\}$ would imply $s \perp s$ or $s < s$, a contradiction. Thus $s = m$, proving our claim. Then, in view of Lemma 1.6, $T_m = \{m\} \cup \big\{y \in N \setminus \{0\} \mid y \perp m\big\}$ is a disjoint set and $N \prec T_m$ in $\mathcal{D}(P)$. Hence, by showing $T_m = M$ our proof is completed. As $T_m \subseteq B$, any $t \in T_m$ is less than or equal to some element of $\max(B) = M$. Thus $N < T_m \leqslant M$ holds in $\mathcal{D}(B)$ by (A). Hence $N \prec M$ in $\mathcal{D}(B)$ implies $T_m = M$. □

Proof of Theorem 1.5. Any poset $(P, \leq)$ without least element becames a poset with 0 by adding a new element 0 to P. In this way both the number of the elements in the CD-bases of $\mathbb{P}$ and the length of the maximal chains in $\mathcal{D}(P)$ are increased by one. Therefore, without loss of generality we may assume that $\mathbb{P}$ contains 0 and $|P| \geq 2$.

To prove the first part of Theorem 1.5, assume that B is a CD-base in $\mathbb{P}$. Then clearly $0 \in B$ and $|B| \geq 2$. Let $D_1 = \max(B)$. Take any $m_1 \in D_1$ and form $D_2 = \max(B \setminus \{m_1\})$. Then, in view of Lemma 1.7, $D_1, D_2 \in \mathcal{D}(P)$, $D_1 \succ D_2$, and D_1 is a maximal element in $\mathcal{D}(P)$. Further, suppose that we already have a sequence (D_i, m_i), $1 \leq i \leq k$ $(k \geq 2)$ such that $m_i \in D_i$, $D_1 \succ ... \succ D_k$ in $\mathcal{D}(P)$ and

$$D_k = \max(B \setminus \{m_1, ..., m_{k-1}\}).$$

We show that for all $i \in \{1, ..., k-1\}$ and $d \in D_k$ we have $d \not\geq m_i$. (5)

This is clear for $i = 1$ since $m_1 \in \max(B)$ and $d \in B$, $d \neq m_1$. If $2 \leq i \leq k-1$, then $m_i \in \max(B \setminus \{m_1, ..., m_{i-1}\})$, and since $d \in B \setminus \{m_1, ..., m_{i-1}\}$, $d \geq m_i$ would imply $m_i = d \in B \setminus \{m_1, ..., m_i, ..., m_{k-1}\}$, a contradiction.

Further, if $|B \setminus \{m_1, ..., m_{k-1}\}| \geq 2$, then form the next set $D_{k+1} := \max(B \setminus \{m_1, ..., m_{k-1}, m_k\})$ and let $m_{k+1} \in D_{k+1}$. Since D_{k+1} is an antichain in the CD-base B, it is a disjoint set, and clearly $D_{k+1} \neq D_k$.

In order to prove $D_k \succ D_{k+1}$, consider the subposet $(I(D_k), \leq)$. By Proposition 1.4, $B_k := B \cap I(D_k)$ is a CD-base in $(I(D_k), \leq)$. We claim that

$$B_k = B \setminus \{m_1, ..., m_{k-1}\}.$$

Indeed, $D_k = \max(B \setminus \{m_1, ..., m_{k-1}\})$ implies $B \setminus \{m_1, ..., m_{k-1}\} \subseteq B \cap I(D_k) = B_k$. On the other hand, (5) implies $\{m_1, ..., m_{k-1}\} \cap I(D_k) = \emptyset$, whence we get $B_k \subseteq B \setminus \{m_1, ..., m_{k-1}\}$, proving our claim. Hence $D_k = \max(B_k)$, and $D_{k+1} = \max\ (B \setminus \{m_1, ..., m_{k-1}, m_k\}) = \max(B_k \setminus \{m_k\})$.

Now, by applying Lemma 1.7, we obtain that $D_{k+1} \prec D_k$ holds in $\mathcal{D}(I(D_k))$. Finally, observe that any $S \in \mathcal{D}(P)$ with $S \leqslant D_k$ is also a disjoint set in $(I(D_k), \leq)$ according to (A). Moreover, since $D_{k+1} \prec D_k$ holds in $\mathcal{D}(I(D_k))$, $D_{k+1} \leqslant S \leqslant D_k$ implies either $S = D_k$ or $S = D_{k+1}$. This means that $D_{k+1} \prec D_k$ holds in $\mathcal{D}(P)$, too.

Thus we conclude by induction that the chain $D_1 \succ ... \succ D_k \succ ...$ can be continued as long as the condition $|B \setminus \{m_1, ..., m_{k-1}\}| \geq 2$ is still valid. Since P is finite, the process stops after finite - let say $n-1$ steps, when $|B \setminus \{m_1, ..., m_{n-1}\}| = 1$, and the last set is $D_n = B \setminus \{m_1, ..., m_{n-1}\}$. As $0 \in B$, and since $0 \notin \max(X)$ whenever $|X| \geq 2$, we get $\{0\} = B \setminus \{m_1, ..., m_{n-1}\} = D_n$. As D_1 is a maximal element and $D_n = \{0\}$ is the least element in $\mathcal{D}(P)$, $D_1 \succ ... \succ D_n$ is a maximal chain in $\mathcal{D}(P)$. Since $B = \{m_1, ..., m_{n-1}, 0\}$, we obtain $|B| = n$.

To prove the second part of Theorem 1.5, assume that the disjoint sets $D_1, ..., D_m$ form a maximal chain $\mathcal{C}$:

$$D_1 \prec ... \prec D_m$$

in $\mathcal{D}(P)$. Then $D_1 = \{0\}$. Let $D = \bigcup_{i=1}^{m} D_i$. First, we prove that the set D is CD-independent. Indeed, take any $x, y \in D$, i.e. $x \in D_i$ and $y \in D_j$ for some $1 \leq i \leq j \leq m$. Then $x \leq z$ for some $z \in D_j$ by (A). Assume that x and y are not comparable. Then $z \neq y$, and $z \perp y$ implies $x \perp y$ by (1). This means that D is CD-independent.

Now, assume that D is not a CD-base. Then there is an $x \in P \setminus D$ such that $D \cup \{x\}$ is CD-independent. Next, consider the set

$$\mathcal{E} = \{D_i \in \mathcal{C} \mid x \nleq d \text{ for all } d \in D_i\}.$$

Clearly, $D_1 = \{0\} \in \mathcal{E}$ since $x \nleq 0$. Let $D_i \in \mathcal{E}$. Then $d \perp x$ or $d < x$ holds for each $d \in D_i$ because $D \cup \{x\}$ is CD-independent. Thus $T_i := \{x\} \cup \{d \in D_i \mid d \nless x\}$ is a disjoint set, and $d < x$ or $d \in T_i$ holds for all $d \in D_i$. Hence

$$D_i < T_i, \qquad (6)$$

in view of (A) and $x \notin D_i$. Observe that $D_m \notin \mathcal{E}$ since $D_m < T_m$ is not possible because $\mathcal{C}$ is a maximal chain. Thus, there exists a $k \leq m-1$ such that $D_k \in \mathcal{E}$ but $D_{k+1} \notin \mathcal{E}$. This means that $x \nleq d$ for all $d \in D_k$, and $x \leq z$ holds for some $z \in D_{k+1}$. Then $T_k = \{x\} \cup \{d \in D_k \mid d \nless x\} \in \mathcal{D}(P)$ satisfies $D_k < T_k$ in virtue of (6). Since $T_k \setminus \{x\} \subseteq D_k < D_{k+1}$ and $x \leq z$, for each $t \in T_k$ there is a $v \in D_{k+1}$ with $t \leq v$. In view of (A) we get $D_k < T_k < D_{k+1}$ because $x \notin D_{k+1} \subseteq D$. Since this fact contradicts $D_k \prec D_{k+1}$, we conclude that D is a CD-base.

Further, in view of (4), it follows that any set $D_i \setminus D_{i-1}$, $2 \leq i \leq m$ contains exactly one element, let say, a_i. Observe also that

$$D = \bigcup_{i=1}^{m} D_i = D_1 \cup \bigcup_{i=2}^{m} (D_i \setminus D_{i-1}).$$

Since $D_1 = \{0\}$ and $D_i \setminus D_{i-1} = \{a_i\}$, we get $D = \{0, a_2, ..., a_m\}$. We prove that all the elements $0, a_2, ..., a_m$ are different: Clearly, $0 \notin \{a_2, ..., a_m\}$. Take any $i, j \in \{2, ..., m\}$, $i < j$. Then $D_i \leqslant D_{j-1} \prec D_j$. As $a_i \in D_i$, there is a $b \in D_{j-1}$ with $0 < a_i \leq b$ by (A). As $a_j \in D_j \setminus D_{j-1}$, $b < a_j$ or $b \perp a_j$ holds by (2). Since both facts imply $a_i \neq a_j$, we conclude that D contains m different elements. □

The length $l(P)$ of a poset $\mathbb{P}$ is defined as the supremum of $|C| - 1$ where C is a chain in $\mathbb{P}$. The poset $\mathbb{P}$ is called *graded* if all its maximal chains have the same cardinality. In this case, $l(P) = |C_m| - 1$ for any maximal chain C_m in $\mathbb{P}$. It is known that any principal ideal (principal filter) of a finite graded poset is also graded. The next assertion is a direct consequence of Theorem 1.5.

Corollary 1.8. *Let $\mathbb{P} = (P, \leq)$ be a finite poset. Then the CD-bases of $\mathbb{P}$ have the same number of elements if and only if the poset $\mathcal{D}(P)$ is graded.*

Corollary 1.9. *Let $\mathbb{P} = (P, \leq)$ be a finite poset.*
(i) *Let $B \subseteq P$ be a CD-base of $\mathbb{P}$. Let $(B, \leq)$ be the poset under the restricted ordering. Then any maximal chain $\mathcal{C} = \{D_i\}_{1 \leq i \leq m}$ in $\mathcal{D}(B)$ is also a maximal chain in $\mathcal{D}(P)$.*
(ii) *If D is a disjoint set in $\mathbb{P}$ and the CD-bases of $\mathbb{P}$ have the same number of elements, then the CD-bases of the subposet $(I(D), \leq)$ also have the same number of elements.*

Proof. (i). Since all D_i $(1 \leq i \leq n)$ are antichains with $D_i \subseteq B$, they are disjoint sets in $(P, \leq)$, too. Thus $\mathcal{C}$ is a chain in $\mathcal{D}(P)$, and hence it is contained in a maximal chain $\mathcal{M}$ of $\mathcal{D}(P)$. In view of Theorem 1.5, $B' = \bigcup\{D \mid D \in \mathcal{M}\}$ is a CD-base in $\mathcal{D}(P)$, and we have $B' \supseteq \bigcup_{i=1}^{n} D_i = B$. As B is also a CD-base, we get $B' = B$ and this implies $D \subseteq B$ for all $D \in \mathcal{M}$. Hence $\mathcal{M}$ is a chain in $\mathcal{D}(B)$, and since $\mathcal{C}$ is maximal in $\mathcal{D}(B)$, we obtain $\mathcal{C} = \mathcal{M}$. Therefore, $\mathcal{C}$ is a maximal chain in $\mathcal{D}(P)$.
(ii). We claim that $\mathcal{D}(I(D))$ coincides with the principal ideal generated by D in $\mathcal{D}(P)$. Indeed, any $S \in \mathcal{D}(I(D))$ is also a disjoint set in $\mathbb{P}$, and satisfies $S \leqslant D$ in $\mathcal{D}(P)$ by (A). Conversely, if $S \leqslant D$ holds in $\mathcal{D}(P)$, then (A) implies $S \in \mathcal{D}(I(D))$. As any principal ideal of a graded poset is graded, in view of Corollary 1.8, the CD-bases of $I(D)$ have the same number of elements. □

Remark 1.10. Let $\mathbb{P} = (P, \leq)$ be a poset with 0. Let $A(P)$ be the set of *atoms* of $\mathbb{P}$, i.e. of all $a \in P$ with $0 \prec a$. Then $x \perp y$ or $x \geq y$ holds for all

$x \in P$ and $y \in A(P) \cup \{0\}$, and hence $A(P) \cup \{0\}$ is a subset of any CD-base of $\mathbb{P}$.

A disjoint set $D \neq \{0\}$ of a poset $(P, \leq)$ is called *complete,* if there is no $p \in P \setminus D$ such that $D \cup \{p\}$ is also a disjoint set.

Lemma 1.11. *Let $\mathbb{P} = (P, \leq)$ be a finite poset with 0. A disjoint set $D \neq \{0\}$ of $\mathbb{P}$ is complete if and only if $A(P) \leqslant D$* in *$\mathcal{D}(P)$.*

Proof. Let D be a complete disjoint set and $a \in A(P)$. Then there is an $x \in D$ with $a \leq x$, otherwise $a \perp x$ for all $x \in D$ would imply that $D \cup \{a\}$ is a disjoint set, a contradiction. Hence $A(P) \leqslant D$ according to (A). Conversely, assume that $A(P) \leqslant D$ holds for some $D \in \mathcal{D}(P) \setminus \{0\}$, and take $p \in P \setminus D$. In view of Remark 1.1(i), $D \cup \{p\}$ is not a disjoint set for $p = 0$. Let $p \neq 0$. Since $\mathbb{P}$ is finite, there is an atom $a \in P$ with $a \leq p$. As $A(P) \leqslant D$, we get $a \leq x$ for some $x \in D$ by (A). Since $x \perp p$ is not satisfied, $D \cup \{p\}$ is not a disjoint set. Thus D is complete. □

This result means that the complete disjoint sets of $\mathbb{P}$ coincide with the principal filter $[A(P))$ in $\mathcal{D}(P)$. Their subposet $([A(P)), \leqslant)$ will be denoted by $\mathcal{DC}(P)$. Its importance is shown by the following assertion:

Proposition 1.12. *Let $\mathbb{P} = (P, \leq)$ be a finite poset with 0. Then the following conditions are equivalent:*
(i) *The CD-bases of $\mathbb{P}$ have the same number of elements.*
(ii) *$\mathcal{D}(P)$ is graded.*
(iii) *$\mathcal{DC}(P)$ is graded.*

Proof. (i)$\Leftrightarrow$(ii) is just Corollary 1.8, and (ii)$\Rightarrow$(iii) follows from the fact that any pricipal filter of a finite graded poset is also graded.
(iii)$\Rightarrow$(ii). Let l be the length of $\mathcal{DC}(P)$ and $A(P) = \{a_1, a_2, ..., a_{k-1}\}$. We will show that any maximal chain $\mathcal{C} : D_1 \prec ... \prec D_n$ in $\mathcal{D}(P)$ has the length $k + l - 1$. Indeed, in virtue of Theorem 1.5, $B = \bigcup_{i=1}^{n} D_i$ is a CD-base in $\mathbb{P}$ with $|B| = n$. Thus $A(P) \cup \{0\} \subseteq B$ by Remark 1.10. By Lemma 1.6

$$\mathcal{A} : \{0\} \prec \{a_1\} \prec \{a_1, a_2\} \prec ... \prec \{a_1, ..., a_{k-1}\}$$

is a maximal chain in the interval $[\{0\}, A(P)]$. Consider the subposet $(B, \leq)$ of $\mathbb{P}$. As $\mathcal{A}$ is a chain in $\mathcal{D}(B)$, it is contained in a maximal chain $\mathcal{M}$: $\{0\} = D'_1 \prec ... \prec D'_m$ of $\mathcal{D}(B)$. Then $D'_k = A(P)$, and since B is the only CD-base in $(B, \leq)$, by Theorem 1.5 we get $B = \bigcup_{i=1}^{m} D'_i$ and $m = |B| = n$. By Corollary 1.9(i) , $\mathcal{M}$ is also a maximal chain in $\mathcal{D}(P)$, moreover, $D'_k, ..., D'_n$ are complete disjoint sets, according to Lemma 1.11. Therefore, $D'_k \prec ... \prec D'_n$ is a maximal chain in $\mathcal{DC}(P)$, and hence its length $n - k$ is equal to l. Then $n - 1 = k + (n - k - 1) = k + l - 1$, i.e. the length $n - 1$ of $\mathcal{C}$ equals to $k + l - 1$. □

Now, we will show that under some weak conditions, $\mathcal{D}(P)$ graded implies that $\mathbb{P}$ itself is graded. A poset with least element 0 and greatest element 1 is called *bounded*. A lattice L with 0 is called 0-*modular* if for any $a, b, c \in L$

$$(\mathrm{M}_0) \qquad a \leq b \text{ and } b \wedge c = 0 \text{ imply } b \wedge (a \vee c) = a$$

Equivalently, L has no pentagon sublattice N_5 that contains $0 = 0_L$. If (M_0) is satisfied under the assumptions that a is an atom and $c \prec b \vee c$, then L is called *weakly* 0-*modular*. L is *lower-semimodular* if for any $a, b, c \in L$, $b \prec c$ implies $a \wedge b \preceq a \wedge c$. It is easy to see that any lower-semimodular lattice and any 0-modular lattice is weakly 0-modular. We say that a poset $\mathbb{P}$ with 0 is *weakly* 0-*modular* if the above weak form of (M_0) holds whenever $\sup\{a, c\}$ and $\sup\{b, c\}$ exist in $\mathbb{P}$.

Proposition 1.13. *Let $\mathbb{P}$ be a finite bounded poset.*
(a) *If all the principal ideals* $(a]$ of $\mathbb{P}$ *are weakly* 0-*modular, then* $A(P) \cup C$ *is a CD-base for every maximal chain* C *in* $\mathbb{P}$.
(b) *If* $\mathbb{P}$ *has weakly* 0-*modular principal ideals and* $\mathcal{D}(P)$ *is graded, then* $\mathbb{P}$ *is also graded, and any CD-base of* $\mathbb{P}$ *contains* $|A(P)| + l(P)$ *elements.*

Proof. (a) Let C be a maximal chain. Clearly, C has the form: $0 = c_0 \prec c_1 \prec \ldots \prec c_n = 1$ and $A(P) \cup C$ is CD-independent. Now let $y \in P \setminus C$ such that $C \cup \{y\}$ is CD-independent; we will prove $y \in A(P)$. Since $y < 1$ and $y \neq 0$, there exists an element $c_i \in C \setminus \{0\}$ such that $y < c_i$ and $y \nleq c_{i-1}$. Since $y \geq c_{i-1}$ does not hold, we get $y \perp c_{i-1}$. Let a be an atom under y: then $a \leq c_i$, and $a \perp c_{i-1}$ by (1). As c_i is the unique upper cover of c_{i-1} in the subposet $(c_i]$, it is also the least upperbound for $\{y, c_{i-1}\}$ and $\{a, c_{i-1}\}$ in $(c_i]$. Hence $a \vee c_{i-1} = y \vee c_{i-1} = c_i$ holds in $(c_i]$. Since $((c_i], \leq)$ is weakly 0-modular, $0 \prec a \leq y$, $y \wedge c_{i-1} = \inf\{y, c_{i-1}\} = 0$ and $c_{i-1} \prec y \vee c_{i-1}$ imply $a = y \wedge (a \vee c_{i-1}) = y \wedge c_i = y$. Thus $y \in A(P)$, hence $A(P) \cup C$ is a CD base.
(b) In view of Corollary 1.8, if $\mathcal{D}(P)$ is graded, then any CD-base B of $\mathbb{P}$ has the same number of elements as $A(P) \cup C$, i.e. $|B| = |A(P)| + |C| - 1$. Consequently, if C_1 and C_2 are two maximal chains in $\mathbb{P}$, then $|A(P)| + |C_1| - 1 = |A(P)| + |C_2| - 1$, i.e. $|C_1| = |C_2|$.
Thus $\mathbb{P}$ is graded and $l(P) = |C| - 1$. The remaining part is clear. □

2. CD-bases in semilattices and lattices

Lemma 2.1. *Let $\mathbb{P}$ be a poset with* 0. *Let D_k be disjoint sets in $\mathbb{P}$ for any $k \in K$, where K is a nonempty set. If the meet $\bigwedge_{k \in K} a^{(k)}$ of any system of elements $a^{(k)} \in D_k$, $k \in K$, exist in $\mathbb{P}$, then $\bigwedge_{k \in K} D_k$ also exists in $\mathcal{D}(P)$. In particular, for $K = \{1, 2\}$ and $D_1 = \{a_i \mid i \in I\}$, $D_2 = \{b_j \mid j \in J\} \in \mathcal{D}(P,)$*

$$D_1 \wedge D_2 = \begin{cases} M := \{a_i \wedge b_j \neq 0 \mid i \in I, j \in J\}, \ \textit{if } M \neq \emptyset; \\ \{0\}, \ \textit{otherwise.} \end{cases} \tag{7}$$

Proof. Since $\{0\}$ is the least element in $\mathcal{D}(P)$, we have $\{0\} = \bigwedge_{k\in K} D_k$, whenever $\{0\}$ belongs to $\{D_k \mid k \in K\}$. Hence we may assume that $D_k \neq \{0\}$, $k \in K$. Now, for all possible systems of elements $a^{(k)} \in D_k$, $k \in K$, form the set M of their nonzero meets $\bigwedge_{k\in K} a^{(k)}$ in $\mathbb{P}$. If $M \neq \emptyset$, then define $S := M$, otherwise let $S := \{0\}$. We show that S is a disjoint set. This is clear for $S = \{0\}$. If $S \neq \{0\}$, then for any elements $\bigwedge_{k\in K} a^{(k)} \neq \bigwedge_{k\in K} b^{(k)}$ of S, there exists a $k_0 \in K$ such that $a^{(k_0)} \neq b^{(k_0)}$. As $a^{(k_0)}, b^{(k_0)} \in D_{k_0}$, we get $a^{(k_0)} \perp b^{(k_0)}$, and this fact implies $\left(\bigwedge_{k\in K} a^{(k)}\right) \perp \left(\bigwedge_{k\in K} b^{(k)}\right)$ by (1). This result means that $S \in \mathcal{D}(P)$. Next, we prove $S = \bigwedge_{k\in K} D_k$. The case $M = \emptyset$ is clear since then $S = 0$ is the only lower bound of the D_k, $k \in K$. Hence we can assume that $M \neq \emptyset$. As for each $\bigwedge_{k\in K} a^{(k)} \in S$ we have $\bigwedge_{k\in K} a^{(k)} \leq a^{(k)} \in D_k$, $k \in K$, we get $S \leqslant D_k$ for all $k \in K$. Let $T = \{t_\lambda \mid \lambda \in \Lambda\} \in \mathcal{D}(P)$, such that $T \leqslant D_k$, $k \in K$. If $T = \{0\}$, then $T \leqslant S$. If $T \neq \{0\}$, then $t_\lambda \neq 0$ for all $\lambda \in \Lambda$, and in view of (A), for each $k \in K$ there is an element $a_\lambda^{(k)} \in D_k$, such that $t_\lambda \leq a_\lambda^{(k)}$. Since by our assumption, all $\bigwedge_{k\in K} a_\lambda^{(k)}$ exist in $\mathbb{P}$, we get $0 < t_\lambda \leq \bigwedge_{k\in K} a_\lambda^{(k)}$, $\lambda \in \Lambda$. As $\bigwedge_{k\in K} a_\lambda^{(k)} \in S$, we obtain $T \leqslant S$ by (A). This proves $S = \bigwedge_{k\in K} D_k$. The remaining part is clear. □

Let $\mathbb{P} = (P, \wedge)$ be a semilattice with 0. Now, for any $a, b \in P$ the relation $a \perp b$ means that $a \wedge b = 0$. Hence, a set $\{a_i \mid i \in I\}$ of nonzero elements is a disjoint set if and only if $a_i \wedge a_j = 0$ for all $i, j \in I$, $i \neq j$. A pair $a, b \in P$ with least upperbound $a \vee b$ in $\mathbb{P}$ is called a *distributive pair* if $(c \wedge a) \vee (c \wedge b)$ exists in $\mathbb{P}$ for any $c \in P$, and $c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$. We say that $(P, \wedge)$ is *dp-distributive* (*distributive with respect to disjoint pairs*) if any pair $a, b \in P$ with $a \wedge b = 0$ is a distributive pair.

Theorem 2.2. (i) *If $\mathbb{P} = (P, \wedge)$ is a semilattice with 0, then $\mathcal{D}(P)$ is a dp-distributive semilattice. If, in addition, $D_1, D_2 \in \mathcal{D}(P)$ such that $D_1 \cup D_2$ is a CD-independent set, then D_1, D_2 is a distributive pair in $\mathcal{D}(P)$.*
(ii) *If $\mathbb{P}$ is a complete lattice, then $\mathcal{D}(P)$ is a dp-distributive complete lattice.*

Proof. (i) Let $D_1 = \{a_i \mid i \in I\}, D_2 = \{b_j \mid j \in J\} \in \mathcal{D}(P)$. By applying Lemma 2.1 (with $K = \{1, 2\}$) we get that $D_1 \wedge D_2 \in \mathcal{D}(P)$ always exists, and it is given by (7). Thus $\mathcal{D}(P)$ is a semilattice with 0.

¿From now on, suppose that $D_1 \cup D_2$ is a CD-independent set. Since D_1, D_2 are antichains, any chain in $D_1 \cup D_2$ has at most two elements (one in D_1 and the other in D_2). Thus $\max(D_1 \cup D_2) \neq \emptyset$, and for any $d \in D_1 \cup D_2$ there exists an $m \in \max(D_1 \cup D_2)$ with $d \leq m$. Since $\max(D_1 \cup D_2)$ is an antichain in a CD-independent set, it is a disjoint set, and $D_1, D_2 \leqslant \max(D_1 \cup D_2)$ by (A). We show that

$$\max(D_1 \cup D_2) = D_1 \vee D_2 \text{ in } \mathcal{D}(P). \tag{8}$$

Indeed, take any $T \in \mathcal{D}(P)$, with $D_1, D_2 \leqslant T$. Then, in view of (A), for any $d \in D_1 \cup D_2$, there is a $t \in T$ with $d \leq t$. As $\max(D_1 \cup D_2) \subseteq D_1 \cup D_2$, we get $\max(D_1 \cup D_2) \leqslant T$ by (A). Thus $\max(D_1 \cup D_2) = D_1 \vee D_2$.

Further, we prove that for any $D_3 = \{c_q \mid q \in Q\} \in \mathcal{D}(P)$ we have

$$(D_1 \vee D_2) \wedge D_3 = (D_1 \wedge D_3) \vee (D_2 \wedge D_3). \tag{9}$$

Since the inequality $(D_1 \wedge D_3) \vee (D_2 \wedge D_3) \leqslant (D_1 \vee D_2) \wedge D_3$ holds whenever both of its sides exist, it is enough to show its converse. Clearly, we may assume $(D_1 \vee D_2) \wedge D_3 \neq \{0\}$. Then, by applying (8) and (7), we obtain:

$$(D_1 \vee D_2) \wedge D_3 = \{m \wedge c_q \neq 0 \mid m \in \max(D_1 \cup D_2), q \in Q\}.$$

In view of (8), $(D_1 \wedge D_3) \vee (D_2 \wedge D_3)$ exists in $\mathcal{D}(P)$, whenever $(D_1 \wedge D_3) \cup (D_2 \wedge D_3)$ is CD-independent. This holds if $D_1 \wedge D_3 = \{0\}$ or $D_2 \wedge D_3 = \{0\}$. Otherwise, by (7), $D_1 \wedge D_3 = \{a_i \wedge c_q \neq 0 \mid i \in I, q \in Q\} \in \mathcal{D}(P)$ and $D_2 \wedge D_3 = \{b_j \wedge c_q \neq 0 \mid j \in J, q \in Q\} \in \mathcal{D}(P)$. If $(a_i \wedge c_q) \wedge (b_j \wedge c_{q'}) \neq 0$ for some $i \in I, j \in J$ and $q, q' \in Q$, then $c_q \wedge c_{q'} \neq 0$, $a_i \wedge b_j \neq 0$, hence we get $c_q = c_{q'}$, and $a_i \leq b_j$ or $b_j \leq a_i$, because $c_q, c_{q'} \in D_3$, $a_i, b_j \in D_1 \cup D_2$ and $D_1 \cup D_2$ is CD-independent. This implies $a_i \wedge c_q \leq b_j \wedge c_{q'}$ or $b_j \wedge c_{q'} \leq a_i \wedge c_q$, proving that $(D_1 \wedge D_3) \cup (D_2 \wedge D_3)$ is CD-independent.

Now, consider an $x \in (D_1 \vee D_2) \wedge D_3$. Since $\{0\} \neq (D_1 \vee D_2) \wedge D_3 \in D(P)$, $x \neq 0$. By (8) and Lemma 2.1, there are $i \in \{1, 2\}$, $d_i \in D_i$ and $d_3 \in D_3$ such that $x = d_i \wedge d_3$. (A) together with $d_i \wedge d_3 \in D_i \wedge D_3 \leqslant (D_1 \wedge D_3) \cup (D_2 \wedge D_3)$ give a $y \in (D_1 \wedge D_3) \cup (D_2 \wedge D_3)$ such that $x = d_i \wedge d_3 \leq y$. Hence $(D_1 \vee D_2) \wedge D_3 \leqslant (D_1 \wedge D_3) \vee (D_2 \wedge D_3)$ by (A) since x was arbitrary. This proves (9).

Finally, let $D_1 = \{a_i \mid i \in I\} \in \mathcal{D}(P)$, $D_2 = \{b_j \mid j \in J\} \in \mathcal{D}(P)$ such that $D_1 \wedge D_2 = \{0\}$. Then, in view of (7) we have $a_i \wedge b_j = 0$ for all $i \in I$ and $j \in J$. Thus $D_1 \cup D_2$ is a CD-independent set, and hence $D_1 \vee D_2$ exists in $\mathcal{D}(P)$. Therefore, D_1, D_2 is a distributive pair in $\mathcal{D}(P)$, according to (9). This result means that $(\mathcal{D}(P), \wedge)$ is dp-distributive.

(ii) As $\mathbb{P}$ is a complete lattice, it has a 1 element, and $\{1\}$ is the greatest element of $\mathcal{D}(P)$. Since by Lemma 2.1, $\bigwedge X$ exists for all $X \subseteq \mathcal{D}(P)$, $\mathcal{D}(P)$ is complete lattice. In view of (i), $\mathcal{D}(P)$ is dp-distributive. □

Let $(P, \leq)$ be a poset and $A \subseteq P$. $(A, \leq)$ is called a *sublattice* of $(P, \leq)$, if $(A, \leq)$ is a lattice such that for any $a, b \in A$ the infimum and the supremum of $\{a, b\}$ are the same in the subposet $(A, \leq)$ and in $(P, \leq)$. If $x \prec_A y$ implies $x \prec_P y$ for all $x, y \in A$, then we say that $(A, \leq)$ is a *cover-preserving subposet* of $(P, \leq)$.

Theorem 2.3. *Let $\mathbb{P} = (P, \leq)$ be a poset with* 0 *and let B be a CD-base of it. Then $(\mathcal{D}(B), \leqslant)$ is a distributive cover-preserving sublattice of the poset $(\mathcal{D}(P), \leqslant)$. If $\mathbb{P}$ is a $\wedge$-semilattice, then for any $D \in \mathcal{D}(P)$ and $D_1, D_2 \in \mathcal{D}(B)$ we have $(D_1 \vee D_2) \wedge D = (D_1 \wedge D) \vee (D_2 \wedge D)$ in $\mathcal{D}(P)$.*

Proof. Observe that any $D \in \mathcal{D}(B)$ is also a disjoint set in $\mathbb{P}$, since D is an antichain within the CD-base B. Hence $\mathcal{D}(B) \subseteq \mathcal{D}(P)$. Take any $x, y \in B$. Since we have $x \leq y$ or $y \leq x$ or $x \perp y$, we get $x \wedge y = x$, or $x \wedge y = y$, or $x \wedge y = 0$; so $x \wedge y \in B$ exists in all possible cases. Thus $(B, \leq)$ is a $\wedge$-semilattice with 0. Take $D_1, D_2 \in \mathcal{D}(B)$. In view of (7), $D_1 \wedge D_2 = \{0\} \subseteq B$ or $D_1 \wedge D_2 = \{x \wedge y \neq 0 \mid x \in D_1, y \in D_2\} \subseteq B$. Hence $D_1 \wedge D_2$ is the same both in $\mathcal{D}(B)$ and $\mathcal{D}(P)$. As $D_1 \cup D_2 \subseteq B$, $D_1 \cup D_2$ is CD-independent. Then, in virtue of (8), $D_1 \vee D_2$ exists in $\mathcal{D}(P)$ and $D_1 \vee D_2 = \max(D_1 \cup D_2) \subseteq B$. Thus $D_1 \vee D_2$ in $\mathcal{D}(B)$ is the same as in $\mathcal{D}(P)$. Hence $(\mathcal{D}(B), \leqslant)$ is sublattice of $(\mathcal{D}(P), \leqslant)$. Let $D \in \mathcal{D}(B)$. Since $(\mathcal{D}(B), \leqslant)$ is a lattice and $D_1 \cup D_2$ is CD-independent, in view of Theorem 2.2 we get $(D_1 \vee D_2) \wedge D = (D_1 \wedge D) \vee (D_2 \wedge D)$. Thus $\mathcal{D}(B)$ is a distributive lattice. Finally, suppose that $D \prec S$ holds in $\mathcal{D}(B)$ for some $D, S \in \mathcal{D}(B)$. Then $D \prec S$ is contained in a maximal chain $\mathcal{C}$ of $\mathcal{D}(B)$. Since by Corollary 1.9(i), $\mathcal{C}$ is also a maximal chain in $\mathcal{D}(P)$, $D \prec S$ holds in $\mathcal{D}(P)$, too.

Let $\mathbb{P}$ be a $\wedge$-semilattice and $D_1, D_2 \in \mathcal{D}(B)$. Since $D_1 \cup D_2$ is CD-independent, $(D_1 \vee D_2) \wedge D$ exists for any $D \in \mathcal{D}(P)$, and in view of Theorem 2.2, $(D_1 \vee D_2) \wedge D = (D_1 \wedge D) \vee (D_2 \wedge D)$. ☐

3. CD-bases in particular lattice classes

In this section we investigate CD-bases in two particular lattice classes. The properties of the first class generalize the properties of tolerance lattices of majority algebras. It was proved in [7] and [3] that the tolerance lattice of any majority algebra is a pseudocomplemented, 0-modular and dp-distributive lattice. These properties are not independent, we will show for instance that dp-distributivity implies 0-modularity.

A lattice L with 0 is called *pseudocomplemented* if for each $x \in L$ there exists an element $x^* \in L$ such that for any $y \in L$, $y \wedge x = 0 \Leftrightarrow y \leq x^*$. It is known that an algebraic lattice L is pseudocomplemented if and only if it is 0-*distributive*, that is, for any $a, b, x \in L$, $x \wedge a = 0$ and $x \wedge b = 0$ imply $x \wedge (a \vee b) = 0$. We say that L is *weakly* 0-*distributive* if this implication holds under the condition $a \wedge b = 0$. Clearly, any 0-distributive lattice is weakly 0-distributive. If D is a disjoint set in a weakly 0-distributive lattice and $|D| \geq 2$, then it is easy to see that replacing two different elements $d_1, d_2 \in D$ by their join $d_1 \vee d_2$, we obtain again a disjoint set.

Lemma 3.1. *Let L be a finite weakly* 0*-distributive lattice and D a dual atom in $\mathcal{D}(L)$. Then either $D = \{d\}$ for some $d \in L$ with $d \prec 1$, or D consist of two different elements $d_1, d_2 \in L$ with $d_1 \vee d_2 = 1$.*

Proof. Assume that $D \prec \{1\}$ holds in $\mathcal{D}(L)$. If there exists $d_1, d_2 \in D$, $d_1 \neq d_2$, then $D' = \{d_1 \vee d_2\} \cup (D \setminus \{d_1, d_2\})$ is disjoint set and $D < D'$. Hence $D \prec \{1\}$ implies $D' = \{1\}$, and this is possible only when $d_1 \vee d_2 = 1$ and $D = \{d_1, d_2\}$. If $D = \{d\}$ for some $d \in L$, then $d \prec 1$, otherwise $d < x < 1$ for some $x \in L$ would imply $D < \{x\} < \{1\}$, a contradiction. □

Let L be a graded lattice, and $0, a \in L$. Then the *height* of a is the length of the interval $[0, a]$, denoted by $l(a)$ (In literature, it is also denoted by $h(a)$.)

Remark 3.2. A graded lattice L with 0 is 0-modular, whenever $l(a) + l(b) = l(a \vee b)$ holds for all $a, b \in L$ with $a \wedge b = 0$: Indeed, if L is not 0-modular, then in view of Varlet's result [19] it has an N_5 sublattice containing 0, thus there exist $a, b, c \in L$ such that $c > b$ and $a \wedge b = a \wedge c = 0$, $a \vee b = a \vee c$. Hence by our assumption $l(a) + l(b) = l(a \vee b) = l(a \vee c) = l(a) + l(c)$. Thus we obtain $l(b) = l(c)$, in contradiction with $b < c$.

Theorem 3.3. *Let L be a finite, weakly 0-distributive lattice. Then the following are equivalent:*
(i) *L is graded, and $l(a) + l(b) = l(a \vee b)$ holds for all $a, b \in L$ with $a \wedge b = 0$.*
(ii) *L is 0-modular, and the CD-bases of L have the same number of elements.*

Proof. (i)⇒(ii). In view of Remark 3.2, (i) implies that L is 0-modular. Further, denote by $\mathcal{T}$ the class of finite, weakly 0-distributive lattices satisfying condition (i). We prove via induction on the length l of the lattices $L \in \mathcal{T}$ that any CD-base of them has $|A(L)| + l$ elements. If $l = 1$, then L is a chain $0 \prec a$, and our assertion holds trivially, since L itself is a CD-base. Let $L \in \mathcal{T}$ have length $l \geq 2$, and suppose that the assertion is true for any $K \in \mathcal{T}$, with length $l(K) \leq l - 1$. Take any CD-base B of L; then $\{0, 1\} \cup A(L) \subseteq B$, $\max(B) = \{1\}$ is the greatest element in $\mathcal{D}(L)$, and $1 \notin A(L)$. Let $N = \max(B \setminus \{1\})$. In virtue of Lemma 1.7, N is a dual atom in $\mathcal{D}(L)$. Clearly, $A(L) \subseteq B \setminus \{1\} \subseteq I(N)$. Since L is finite and weakly 0-distributive, Lemma 3.1 yields either $N = \{d\}$ for some $d \prec 1$, or $N = \{d_1, d_2\}$ with $d_1 \vee d_2 = 1$.

In the first case, $A(L) \subseteq B \setminus \{1\} = B \cap (d]$, $l(d) = l - 1$, and clearly, the lattice $(d]$ belongs to the class $\mathcal{T}$. In view of Proposition 1.4, $B \cap (d]$ is a CD-base in $(d]$, hence by applying the induction hypothesis to $(d]$, we get $|B| - 1 = |A(L)| + l - 1$, i.e. $|B| = |A(L)| + l$.

In the second case $A(L) \subseteq B \setminus \{1\} \subseteq (d_1] \cup (d_2]$, and since N is a disjoint set, $d_1 \wedge d_2 = 0$. Hence the single common element of $(d_1]$ and $(d_2]$ is 0, and since $B \setminus \{1\} = (B \cap (d_1]) \cup (B \cap (d_2])$ and $A(L) \subseteq B \setminus \{0, 1\}$, we obtain $|B| - 1 = |B \cap (d_1]| + |B \cap (d_2]| - 1$, and $|A(L)| = |A((d_1])| + |A((d_2])|$. In view of Proposition 1.4, $B \cap (d_1]$ and $B \cap (d_2]$ is a CD-base in $(d_1]$, $(d_2]$, respectively. It is obvious that $(d_1], (d_2] \in \mathcal{T}$ and $l(d_1), l(d_2) \leq l - 1$, hence the induction hypothesis implies

$$|B \cap (d_1]| + |B \cap (d_2]| = |A((d_1])| + l(d_1) + |A((d_2])| + l(d_2) =$$

$= |A(L)| + l(d_1) + l(d_2)$.

As $d_1 \wedge d_2 = 0$, (i) implies $l(d_1) + l(d_2) = l(d_1 \vee d_2) = l$. Thus we obtain $|B| = |B \cap (d_1]| + |B \cap (d_2]| = |A(L)| + l$, which proves that (i) implies (ii).

(ii)⇒(i). Since the CD-bases of L have the same cardinality, in virtue of Corollary 1.8 and Proposition 1.13(b), L is graded. Hence any principal ideal $(p]$ of L is a graded lattice, and by Corollary 1.9(ii) the CD-bases of $(p]$ have the same number of elements. As all the principal ideals in $(p]$ are 0-modular, by Proposition 1.13(b) this number is $|A((p])| + l(p)$. (10)

Now, let $a, b \in L$, $a \wedge b = 0$. Clearly, to prove (i) it is enough to consider the case $a \neq 0$, $b \neq 0$. Then a and b are incomparable, since $a \wedge b \notin \{a, b\}$. Consider the principal ideal $I = (a \vee b]$. Since $\{a, b\}$ is a CD-independent set in I, there exists a CD-base B_I of I containing $\{a, b\}$. As $l(I) = l(a \vee b)$, B_I has $|A(I)| + l(a \vee b)$ elements by (10).

Further, we prove that a and b are maximal elements in $B_I \setminus \{a \vee b\}$. Indeed, $a, b < a \vee b$ because a, b are incomparable. Suppose that $a \leq x$ for some $x \in B_I$, $x < a \vee b$. Then $a \vee b \leq x \vee b \leq a \vee b$ implies $x \vee b = a \vee b$. Observe that b and x are incomparable; indeed, $b \leq x$ is not possible since it yields $x = a \vee b$. Furthermore, $x \leq b$ would imply $b = a \vee b$, i.e. $a \leq b$, hence it should also be excluded. Thus we obtain $x \wedge b = 0$ because B is CD-independent. Since L is 0-modular, by using (M_0) we get $x = x \wedge (x \vee b) = x \wedge (a \vee b) = a$. Therefore, a is a maximal element in $B_I \setminus \{a \vee b\}$. Similarly, we can prove $b \in \max(B_I \setminus \{a \vee b\})$. Next, let $N := \max(B_I \setminus \{a \vee b\})$. Since $\max(B_I) = \{a \vee b\}$, in view of Lemma 1.7 we get that N is a disjoint set and $N \prec \{a \vee b\}$ in $\mathcal{D}(I)$. Since $a \neq b$, $a, b \in N$, and I is finite and weakly 0-distributive, by applying Lemma 3.1 we obtain $N = \{a, b\}$.

Now, we can repeat the argument in the proof of (i)⇒(ii) (with $d_1 = a$, $d_2 = b$ and $l = l(a \vee b) \geqq 2$), and by using (10) we get $|B_I| = |B_I \cap (a]| + |B_I \cap (b]| = |A(I)| + l(a) + l(b)$. Thus we deduce $l(a) + l(b) = l(a \vee b)$, and our proof is completed. ☐

We say that two elements $a, b \in L$ form a *modular pair* in the lattice L and we write $(a, b)M$ if for any $x \in L$, $x \leq b$ implies $x \vee (a \wedge b) = (x \vee a) \wedge b$. a, b is called a *dually modular pair* if for any $x \in L$, $x \geq b$ implies $x \wedge (a \vee b) = (x \wedge a) \vee b$. This is denoted by $(a, b)M^*$. Clearly, if a, b is a distributive pair, then $(a, b)M^*$ is satisfied. By the mean of modular pairs the 0-modularity condition can be reformulated as follows (see [17]): For any $a, b \in L$,

$$a \wedge b = 0 \Longrightarrow (a, b)M. \tag{11}$$

Lemma 3.4. ([[17]; Lemma 1.9.15]) *In a graded lattice of finite length, $(a, b)M$ implies $l(a) + l(b) \leq l(a \wedge b) + l(a \vee b)$.*

Proposition 3.5. *If L is a lattice with 0 such that $(a, b)M^*$ holds for all $a, b \in L$ with $a \wedge b = 0$, then L is 0-modular. If, in addition, L is a graded lattice of finite length, then $l(a \vee b) = l(a) + l(b)$ holds for all $a, b \in L$ with $a \wedge b = 0$.*

Proof. If L is not 0-modular, then in view of Varlet's result, it has an N_5 sublattice containing 0, i.e. there exist elements $a, b, c \in L$ such that $c > b$ and $a \wedge b = a \wedge c = 0$, $a \vee b = a \vee c$. Since $(a, b)M^*$ by our assumption, we obtain $c = c \wedge (a \vee b) = (c \wedge a) \vee b = (a \wedge b) \vee b = b$, a contradiction. Thus L is 0-modular. Now, suppose that in addition L is graded and has a finite length l, and let $a, b \in L$, $a \wedge b = 0$. Since L is 0-modular, we get $l(a \vee b) \geq l(a) + l(b)$ by (11) and Lemma 3.4. Now consider the lattice $L^{(d)}$ dual to L. Then $(a, b)M^*$ in L implies $(a, b)M$ in the lattice $L^{(d)}$. Clearly, $L^{(d)}$ is also a graded lattice with length l and the height $l^{(d)}(x)$ of any element x in $L^{(d)}$ is equal to $l - l(x)$. (where $l(x)$ is the height of x in L). Since $(a, b)M$ holds in $L^{(d)}$, by using again Lemma 3.4 we get $l^{(d)}(a) + l^{(d)}(b) \leq l^{(d)}(a \wedge b) + l^{(d)}(a \vee b)$, i.e. $(l - l(a)) + (l - l(b)) \leq l + l - l(a \vee b)$ because $l^{(d)}(a \wedge b) = l$. Hence $l(a) + l(b) \geq l(a \vee b)$, and this proves $l(a \vee b) = l(a) + l(b)$. □

Corollary 3.6. (i) *Let L be a finite, weakly 0-distributive lattice such that for each $a, b \in L$ with $a \wedge b = 0$, condition $(a, b)M^*$ holds. Then the CD-bases of L have the same number of elements if and only if L is graded.*
(ii) *If L is a finite, pseudocomplemented and modular lattice, then the CD-bases of L have the same number of elements.*

Proof. (i) follows directly from Proposition 3.5 and Theorem 3.3.
(ii) Since any pseudocomplemented lattice is weakly 0-distributive, and any finite modular lattice L is graded, moreover, $(a, b)M^*$ holds for all $a, b \in L$, (ii) is an immediate consequence of (i). □

As any dp-distributive lattice L is weakly 0-distributive, and $(a, b)M^*$ holds for all $a, b \in L$ with $a \wedge b = 0$ since a, b is a distributive pair, we obtain

Corollary 3.7. (i) *Any dp-distributive lattice is 0-modular. If L is a dp-distributive graded lattice with finite length, then $l(a \vee b) = l(a) + l(b)$ holds for all $a, b \in L$ with $a \wedge b = 0$.*
(ii) *The CD-bases in a finite dp-distributive lattice L have the same number of elements if and only if L is graded.*

The second lattice class mentioned in Introduction generalizes the properties of the lattice of closed sets of a so-called interval system. An *interval system* $(V, \mathcal{I})$ is an algebraic closure system satisfying the axioms (cf. [12]):

(I_0) $\{x\} \in \mathcal{I}$ for all $x \in V$, and $\emptyset \in \mathcal{I}$;
(I_1) $A, B \in \mathcal{I}$ and $A \cap B \neq \emptyset$ imply $A \cup B \in \mathcal{I}$;
(I_2) For any $A, B \in \mathcal{I}$ the relations $A \cap B \neq \emptyset$, $A \nsubseteq B$ and $B \nsubseteq A$ imply $A \setminus B \in \mathcal{I}$ (and $B \setminus A \in \mathcal{I}$).

The *modules* (*X-sets, or autonomous sets*) of an undirected graph $G = (V, E)$ (see [16]), the *intervals* of an n-ary relation $R \subsetneq V^n$ on the set V for $n \geq 2$ (cf. [12]) – in particular, the usual intervals of a linearly ordered set $(V, \leq)$ (cf. [18]) – form interval systems. Clearly, $\cap$ is the meet operation in the lattice $(\mathcal{I}, \subseteq)$ of closed sets of $(V, \mathcal{I})$, and condition (I_0) implies that

$(\mathcal{I}, \subseteq)$ is an atomistic lattice with 0-element $\emptyset$. Moreover, for any $A, B \in \mathcal{I}$ with $A \cap B \neq \emptyset$, $A \vee B = A \cup B$ by (I_1) (see e.g. [16]). Hence condition (I_1) yields that for any $A, B, C \in \mathcal{I}$ the implication

$$A \wedge B \neq 0 \Longrightarrow C \wedge (A \vee B) = (C \wedge A) \vee (C \wedge B) \tag{12}$$

holds in this lattice, i.e. every $A, B \in \mathcal{I}$ with $A \wedge B \neq 0$ is a distributive pair in $(\mathcal{I}, \subseteq)$. From here one can deduce that for any $X \in \mathcal{I}$, $X \neq \emptyset$, the principal filter $[X)$ is a distributive sublattice of $(\mathcal{I}, \subseteq)$ (although $(\mathcal{I}, \subseteq)$ in general is not distributive). Let us consider now the condition:

($\mathcal{I}$) If $a \wedge b \neq 0$, then ($x \leq a \vee b$ and $x \wedge a = 0$) $\Rightarrow x \leq b$ for all $a, b, x \in L$

Observe that ($\mathcal{I}$) is satisfied whenever each pair $a, b \in L$ with $a \wedge b \neq 0$ is a distributive pair. Indeed, for any $x \leq a \vee b$ now we obtain $x = x \wedge (a \vee b) = (x \wedge a) \vee (x \wedge b)$, and hence $x \wedge a = 0$ implies $x = x \wedge b$, i.e. $x \leq b$. Clearly, the converse is not true; ($\mathcal{I}$) does not imply that every $a, b \in L$ with $a \wedge b \neq 0$ is a distributive pair. Hence lattices with 0 satisfying condition ($\mathcal{I}$) and with the property that $[a)$ is a modular lattice for any $a \in L$, $a \neq 0$, can be considered as a generalization of the lattice $(\mathcal{I}, \subseteq)$ of an interval system $(V, \mathcal{I})$. To study their CD-bases, first we prove:

Lemma 3.8. *Let L be an atomic lattice satisfying condition ($\mathcal{I}$), $D \in \mathcal{D}(L)$ and*

$$S_D = \big\{ s \in L \setminus (D \cup \{0\}) \mid d \wedge s = 0 \text{ or } d < s, \text{ for all } d \in D \big\}. \tag{13}$$

Then for any $b, c \in S_D$ with $b \wedge c \neq 0$ and any $d \in D$, $d \wedge (b \vee c) \neq 0$ if and only if $0 < d < b$ or $0 < d < c$ holds.

Proof. Assume that $b \wedge c \neq 0$, and take a $d \in D$ such that $d \wedge (b \vee c) \neq 0$. Then $d \neq 0$, and since L is an atomic lattice, there exists an $a \in A(L)$ such that $a \leq d$ and $a \leq b \vee c$. Since L satisfies condition ($\mathcal{I}$) and $b \wedge c \neq 0$, in the case $a \wedge b = 0$ we get $a \leq c$. Hence $a \leq b$ or $a \leq c$ must hold. This implies $d \wedge b \neq 0$ or $d \wedge c \neq 0$. As $b, c \in S_D$, in view of (13) we obtain $0 < d < b$ or $0 < d < c$. The converse implication is obvious. □

Remark 3.9. Let L be a finite lattice and $D = \{d_j \mid j \in J\} \in \mathcal{DC}(L)$. If $D \prec D'$ for some $D' \in D(L)$, in view of Lemma 1.6, there is a minimal element $a \in S_D$ such that $D' = \{a\} \cup \{d_j \in D \setminus \{0\} \mid d_j \wedge a = 0\}$. We claim that there exists a set $K \subseteq J$ such that

$$K = \{j \in J \mid d_j < a\} \neq \emptyset \text{ and } D' = \{a\} \cup \{d_j \mid j \in J \setminus K\}. \tag{14}$$

Indeed, as D is complete, $D \neq \{0\}$, and hence $0 \notin D$. Then $K \neq \emptyset$, otherwise $a \wedge d_j = 0$ for all $j \in J$ would imply that $D \cup \{a\}$ is a disjoint set, in contrary with $D \in \mathcal{DC}(L)$. By the definition of K and S_D we have: $d_j \wedge a = 0 \Leftrightarrow j \in J \setminus K$. Hence $D' = \{a\} \cup \{d_j \mid j \in J \setminus K\}$.

It is well-known that a finite lattice L is semimodular if and only if it satisfies *Birkhoff's condition*, namely, for any $a, b \in L$

$$(Bi) \qquad a \wedge b \prec a, b \text{ implies } a, b \prec a \vee b.$$

We also say that a pair $a, b \in L$ satisfies Birkhoff's condition, if the above implication (Bi) is valid for a, b. It is known that any distributive pair $a, b \in L$ satisfies condition (Bi) (see e.g. [15]).

Theorem 3.10. *Let L be a finite lattice satisfying condition $(\mathcal{I})$. Assume, in addition, that every proper principal filter of L is a modular lattice. Then $\mathcal{DC}(L)$ is a semimodular lattice.*

Proof. In view of Theorem 2.2(ii) and Lemma 1.11, $\mathcal{DC}(L)$ is a lattice. We will show that it satisfies Birkhoff's condition, i.e. $D \prec D_1$ and $D \prec D_2$ imply $D_1, D_2 \prec D_1 \vee D_2$ for any $D = \{d_j \mid j \in J\}, D_1, D_2 \in \mathcal{DC}(L)$, $D_1 \neq D_2$. By Remark 3.9 and (14), there are some minimal elements $b_1 \neq b_2$ in

$S_D = \{s \in L \setminus (D \cup \{0\}) \mid s \wedge d_j = 0 \text{ or } d_j < s \text{ for all } d_j \in D\}$

and some nonempty sets $K_1 = \{j \in J \mid d_j < b_1\}, K_2 = \{j \in J \mid d_j < b_2\}$ such that $D_1 = \{b_1\} \cup \{d_j \mid j \in J \setminus K_1\}$, $D_2 = \{b_2\} \cup \{d_j \mid j \in J \setminus K_2\}$. If $b_1 \wedge b_2 = 0$, then $\bigvee\limits_{j \in K_1} d_j \leq b_1$, $\bigvee\limits_{j \in K_2} d_j \leq b_2$ imply $d_j \wedge d_{j'} = 0$ for all $j \in K_1$ and $j' \in K_2$, hence $D_1 \cup D_2$ is a CD-independent set. Then, in view of Theorem 2.2, D_1, D_2 is a distributive pair. Hence $D_1, D_2 \prec D_1 \vee D_2$ by (14).

Now, suppose that $b_1 \wedge b_2 \neq 0$. Since $D = D_1 \wedge D_2$, by using (7), we obtain $b_1 \wedge b_2 \in D$. Hence $b_1 \wedge b_2 = d_{j_0} > 0$ for some $j_0 \in J$, and we have $D \neq \{0\}$. In view of Lemma 3.8, $d_j \wedge (b_1 \vee b_2) \neq 0$ for $j \in J$ implies $0 < d_j < b_1$ or $0 < d_j < b_2$, whence we get $j \in K_1 \cup K_2$. Thus we have either $d_j \wedge (b_1 \vee b_2) = 0$ for all $j \in J \setminus (K_1 \cup K_2)$, or $J \setminus (K_1 \cup K_2) = \emptyset$. Therefore, $T = \{b_1 \vee b_2\} \cup \{d_j \mid j \in J \setminus (K_1 \cup K_2)\}$ is a disjoint set. Since $\bigvee\{d_j \mid j \in K_1 \cup K_2\} \leq b_1 \vee b_2$, we obtain $D_1, D_2 \leqslant T$ by (A).

Next, we prove $T = D_1 \vee D_2$: Take any $X \in \mathcal{D}(L)$ with $D_1, D_2 \leqslant X$. Then, $X \neq \{0\}$ and we obtain in virtue of (A) that $b_1 \leq x_1$, $b_2 \leq x_2$ for some $x_1, x_2 \in X$. Moreover, if $J \setminus (K_1 \cup K_2) \neq \emptyset$, then for any $j \in J \setminus (K_1 \cup K_2)$ there exist an $x^{(j)} \in X$ with $d_j \leq x^{(j)}$. If $x_1 \neq x_2$, then $d_{j_0} = b_1 \wedge b_2 \leq x_1 \wedge x_2 = 0$, a contradiction. Hence $x_1 = x_2$, and $b_1 \vee b_2 \leq x_1 = x_2$. Thus we deduce $T \leqslant X$, proving $T = D_1 \vee D_2$.

Further, assume for a contradiction that there exists a $D_3 \in \mathcal{DC}(L)$ with $D_1 < D_3 < D_1 \vee D_2 = \{b_1 \vee b_2\} \cup \{d_j \mid j \in J \setminus (K_1 \cup K_2)\}$. In view of (A), then there exist $d^* \in D_3$ and $d \in D_1 \vee D_2$ with $0 < b_1 \leq d^* \leq d$. Notice, that $d^* = d_j$ for some $j \in J \setminus (K_1 \cup K_2)$ is not possible, since it implies $0 < b_1 = b_1 \wedge d_j$, however $b_1 \wedge d_j = 0$ for all $j \in J \setminus K_1$. Hence $d^* \leq b_1 \vee b_2$. We are going to prove that $\{b_1 \wedge b_2, b_1, d^*, b_2, b_1 \vee b_2\}$ is a sublattice of L isomorphic to N_5. First, we show $d^* \wedge b_2 = b_1 \wedge b_2$ and $b_1 < d^* < b_1 \vee b_2$. Clearly $d^* \wedge b_2 \neq 0$, since $d^* \wedge b_2 \geq b_1 \wedge b_2 \neq 0$. Observe that $D_1 < D_3 < D_1 \vee D_2$ is in contradiction with $D_2 \leqslant D_3$, so $D_3 \wedge D_2 \neq D_2$. Now $D = D_1 \wedge D_2 \leq D_3 \wedge D_2 < D_2$ and $D \prec D_2$ imply $D_3 \wedge D_2 = D$. As $d^* \wedge b_2 \neq 0$, by applying (7) to $D_3 \wedge D_2$, we get $d^* \wedge b_2 \in D$. Since $b_1 \wedge b_2 \in D$, $b_1 \wedge b_2 \leq d^* \wedge b_2$ implies $b_1 \wedge b_2 = d^* \wedge b_2$. As both b_1 and b_2 are minimal elements in S_D and $b_1 \neq b_2$, they are incomparable. Hence $d^* \neq b_1 \vee b_2$, since otherwise $b_1 \wedge b_2 = d^* \wedge b_2 = (b_1 \vee b_2) \wedge b_2 = b_2$ would impy $b_2 \leq b_1$. Thus $d^* < b_1 \vee b_2$. In order to prove $b_1 < d^*$, first notice that for each $u \in D_3 \setminus \{d^*\}$ we have $b_1 \wedge u \leq d^* \wedge u = 0$. Now, let us show that $u \leq d_j$ for some $d_j \in D$: Indeed, if $u \nleq b_1 \vee b_2$, then $D_3 < D_1 \vee D_2$ yields $u \leq d_j$ for some $j \in J \setminus (K_1 \cup K_2)$ by (A) and we are done. If $u \leq b_1 \vee b_2$, then $b_1 \wedge b_2 \neq 0$ and $b_1 \wedge u = 0$ by condition $(\mathcal{I})$ imply $u \leq b_2$. As $u \neq 0$, by (7) it follows $u = u \wedge b_2 \in D_3 \wedge D_2 = D$, i.e. $u = d_j$ for some $d_j \in D$. This result proves $D_3 \setminus \{d^*\} \leqslant D \leqslant D_1$, according to (A). Thus any $u \in D_3 \setminus \{d^*\}$ is less than or equal to some $y \in D_1$. Hence $d^* = b_1 \in D_1$ would imply $D_3 \leqslant D_1$, a contradiction. This proves $b_1 < d^*$. Finally, $b_1 \leq d^* \leq b_1 \vee b_2$ implies $b_1 \vee b_2 \leq d^* \vee b_2 \leq b_1 \vee b_2$, whence we obtain $d^* \vee b_2 = b_1 \vee b_2$.

Now, it is easy to see that $Q = \{b_1 \wedge b_2, b_1, d, b_2, b_1 \vee b_2\}$ is a sublattice of L isomorphic to N_5. Clearly, $Q \subseteq [b_1 \wedge b_2)$. However, this is a contradiction, since $[b_1 \wedge b_2)$ is a modular sublattice of L, because $b_1 \wedge b_2 \neq 0$. Therefore, we conclude that there is no $D_3 \in \mathcal{DC}(L)$ with $D_1 < D_3 < D_1 \vee D_2$, i.e. $D_1 \prec D_1 \vee D_2$ holds. Symmetrically, we can prove $D_2 \prec D_1 \vee D_2$. □

Corollary 3.11.(i) *If L is a finite distributive lattice, then $\mathcal{DC}(L)$ is a semimodular lattice.*
(ii) *If L is a finite lattice which satisfies the conditions in Theorem 3.10, then its CD-bases have the same number of elements.*

Proof. (i) Clearly, any distributive lattice satisfies the conditions in 3.10.
(ii) Since now $\mathcal{DC}(L)$ is a finite semimodular lattice, it is graded. Hence (ii) is proved by applying Proposition 1.12.

By applying Corollary 3.11(ii) to interval systems we obtain:

Corollary 3.12. *If $(V, \mathcal{I})$ is a finite interval system, then the CD-bases of the lattice $(\mathcal{I}, \subseteq)$ contain the same number of elements.*

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