CUT APPROACH TO INVARIANCE GROUPS OF LATTICE-VALUED FUNCTIONS

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Abstract. This paper deals with lattice-valued n-variable functions on a k-element domain, considered as a generalization of lattice valued Boolean functions. We investigate invariance groups of these functions i.e., the group of such permutations that leaves the considered function invariant. We show that the invariance groups of lattice-valued functions depend only on the cuts of the function. Furthermore, we construct such lattice-valued Boolean function (and its generalization), the cuts of which represent all representable invariance groups.

1. Introduction

If the permutation group G consists of those variable-permutations that leave a function invariant, then G is said to be the invariance group of the considered function. It is well known that not all subgroups of S_n arise as invariance groups of Boolean functions (see [10,13,20]). Therefore, permutation groups representable as invariance groups of functions of several variables were studied by many authors. The paper by Wnuk [20] contains some preliminary considerations on the topic, the paper of Kisielewicz [13] gives quite a detailed survey. In [10], a Galois-connection is defined, to give another aspect and result for the invariance group problem.

In this paper we investigate invariance groups of lattice valued functions and their cuts, which (cuts) are represented as characteristic functions.

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Lattice valued functions are mappings whose co-domain is a complete lattice, which may have additional properties (e.g., Boolean lattice, Heyting algebra, residuated lattice, unit real interval). These functions were investigated first in modeling classical and nonclassical (e.g., intuitionistic) logic ([5, 15]). They have been further applied to non-classical predicate logics, and also to foundations of fuzzy set theory ([1, 6, 8]). The notion of a cut set, or a p-cut is one of the basic tools in algebraic usage of lattice valued functions (see e.g., [14, 17]). Indeed, the collection of cuts characterizes the function and it is a closure system on its domain.

Here we prove that the invariance group of a lattice valued function depends only on the particular canonical representation of this function, whose co-domain is a subset of the power set of the domain. This means that the invariance groups of n-variable functions with different co-domain lattices coincide in case these functions have equal canonical representations. Further, we prove that for a fixed n, the invariance groups of Boolean functions are the invariance groups of cuts of a single lattice valued function. Finally, we prove that every subgroup of the permutation group S_n is the invariance group of a single lattice valued function.

Let us mention how the paper is organized. In preliminary section we give necessary order theoretic and lattice valued notions. We also present a construction of the canonical representation of a lattice valued function, whose co-domain is a substructure of the domain. Then we define Boolean and lattice valued functions and their connection to invariance groups. Section 3 contains the results, first some cut properties, and then representation theorems of invariance groups in term of cuts of lattice valued functions. Suitable examples are also given.

2. Preliminaries

2.1. **Order, lattices, closures.** In the paper, we mostly deal with **complete lattices** in general, and finite **Boolean lattices**, represented by all n-tuples of 0 and 1. A **closure system** on A is a collection of subsets of A, closed under set intersection and containing A. A mapping $X \mapsto \overline{X}$ on a power set $\mathcal{P}(A)$ of a set A, is a **closure operator** on A, if it fulfills: $X \subseteq \overline{X}$, $\overline{\overline{X}} = \overline{X}$, and $X \subseteq Y \Longrightarrow \overline{X} \subseteq \overline{Y}$. A subset X of A fulfilling $\overline{X} = X$ is said to be **closed** under the corresponding closure operator. We use the following known properties of closure systems and closure operators:

A closure system is a complete lattice under the set-inclusion.

The collection of all closed sets under the corresponding closure operator is a closure system.

If $\mathcal{F}$ is a closure system on A, then the mapping $X \mapsto \bigcap\{Y \in \mathcal{F} \mid X \subseteq Y\}$ is a closure operator on A.

Similarly, a **closure operator on a lattice** L is a mapping $x \mapsto \overline{x}$ on L, if it fulfills: $x \leq \overline{x}$, $\overline{\overline{x}} = \overline{x}$, and $x \leq y \Longrightarrow \overline{x} \leq \overline{y}$.

More details about posets, closures and lattices can be found in e.g., [4].

2.2. **Lattice valued functions.** Let S be a nonempty set and L a complete lattice. Every mapping $\mu\colon S \to L$ is called a **lattice valued** (**L-valued**) **function** on S.

The support of μ is the set of all elements from S having a value different from $0 \in L$ under f.

Let $p \in L$. A **cut set** of an L-valued function $\mu\colon S \to L$ (a p-cut) is a subset $\mu_p \subseteq S$ defined by:

$$x \in \mu_p \text{ if and only if } \mu(x) \geq p. \tag{2.1}$$

In other words, a p-cut of $\mu\colon S \to L$ is the inverse image of the principal filter $\uparrow p$, generated by $p \in L$:

$$\mu_p = \mu^{-1}(\uparrow p). \tag{2.2}$$

It is obvious that for every $p, q \in L$, $p \leq q$ implies $\mu_q \subseteq \mu_p$.

The collection $\mu_L = \{f \subseteq S \mid f = \mu_p, \text{ for some } p \in L\}$ of all cuts of $\mu\colon S \to L$ is usually ordered by set-inclusion. The following is known.

Lemma 2.1. *If $\mu\colon S \to L$ is an L-valued function on S, then the collection μ_L of all cuts of μ is a closure system on S under the set-inclusion.*

The following is a kind of a converse.

Proposition 2.2. *Let $\mathcal{F}$ be a closure system on a set S. Then there is a lattice L and an L-valued function $\mu\colon S \to L$, such that the collection μ_L of cuts of μ is $\mathcal{F}$.*

Proof. It is straightforward to check that a required lattice L is the collection $\mathcal{F}$ ordered by the reversed-inclusion, and that $\mu\colon S \to L$ can be defined as follows:

$$\mu(x) = \bigcap\{f \in \mathcal{F} \mid x \in f\}. \tag{2.3}$$

□

Remark 2.3. From the above proof, using the notation therein, one can straightforwardly deduce that for every $f \in \mathcal{F}$, the cut μ_f coincides with f, i.e., that $\mu_f = f$.

Throughout the paper we mostly consider p-cuts to be the corresponding characteristic functions instead of subsets. Namely, if for $p \in L$, $\mu_p \subseteq S$ is a p-cut, then we represent it as the function: $\mu_p \colon S \to \{0,1\}$, defined by $\mu_p(x) = 1$ if and only if $\mu(x) \geq p$, and $\mu_p(x) = 0$ otherwise. A cut as a characteristic function has the same notation as the cut being a subset, but this will not cause any confusion. E.g., if we write $\sigma \circ \mu_p$, where σ is a function, then it is clear that $\circ$ is a composition of functions and then μ_p is the characteristic function of the corresponding subset.

2.3. **Canonical representation of lattice valued functions.** In this part we introduce a kind of main representatives for lattice-valued functions with a fixed domain S. Namely, we show that every function from S to a complete lattice L has the same cuts as a particular function from S to a subset of the power set of S, equipped with the order dual to the set inclusion. We have been dealing with this topic in some earlier papers (see [17]), but the presentation here is adopted to our present research, and some new properties are proved.

Throughout the section, S is a nonempty set, not necessarily equipped with operations or relations.

Let $\mu \colon S \to L$ be an L-valued function and $(\mu_L, \leq)$ the poset with $\mu_L = \{\mu_p \mid p \in L\}$ (the collection of cuts of μ) and the order $\leq$ being the inverse of the set-inclusion: for $\mu_p, \mu_q \in \mu_L$,

$$\mu_p \leq \mu_q \text{ if and only if } \mu_q \subseteq \mu_p.$$

Lemma 2.4. *$(\mu_L, \leq)$ is a complete lattice and for every collection $\{\mu_p \mid p \in L_1\}$, $L_1 \subseteq L$ of cuts of μ, we have*

$$\bigcap \{\mu_p \mid p \in L_1\} = \mu_{\bigvee\{p \mid p \in L_1\}}. \tag{2.4}$$

Given an L-valued function $\mu \colon S \to L$, we define a relation $\approx$ on L: for $p, q \in L$

$$p \approx q \text{ if and only if } \mu_p = \mu_q. \tag{2.5}$$

Observe that the relation $\approx$ depends only on the range of μ.

The following is straightforward by (2.2).

Lemma 2.5. *The relation $\approx$ is an equivalence on L, and*

$$p \approx q \text{ if and only if } {\uparrow}p \cap \mu(S) = {\uparrow}q \cap \mu(S), \tag{2.6}$$

where $\mu(S) = \{r \in L \mid r = \mu(x) \text{ for some } x \in S\}$.

We denote by $L/{\approx}$ the collection of equivalence classes under $\approx$. By (2.4), each $\approx$-class contains its supremum:

$$\bigvee [p]_{\approx} \in [p]_{\approx}. \tag{2.7}$$

In particular, we have that for every $x \in S$

$$\mu(x) = \bigvee [\mu(x)]_{\approx}. \tag{2.8}$$

Lemma 2.6. *The mapping $p \mapsto \bigvee [p]_{\approx}$ is a closure operator on L.*

The quotient $L/{\approx}$ can be ordered by the relation $\leq_{L/\approx}$ defined as follows:

$$[p]_{\approx} \leq_{L/\approx} [q]_{\approx} \ \text{ if and only if } \ {\uparrow} q \cap \mu(S) \subseteq {\uparrow} p \cap \mu(S).$$

The order $\leq_{L/\approx}$ of classes in $L/{\approx}$ corresponds to the order of suprema of classes in L (we denote the order in L by $\leq_L$):

Proposition 2.7. *The poset $(L/{\approx}, \leq_{L/\approx})$ is a complete lattice fulfilling:*

(i) $[p]_{\approx} \leq_{L/\approx} [q]_{\approx}$ *if and only if* $\bigvee [p]_{\approx} \leq_L \bigvee [q]_{\approx}$.

(ii) The mapping $[p]_{\approx} \mapsto \bigvee [p]_{\approx}$ is an injection of $L/{\approx}$ into L.

Corollary 2.8. *The sub-poset $(\bigvee [p]_{\approx}, \leq_L)$ of L is isomorphic to the lattice $(L/{\approx}, \leq_{L/\approx})$ under $\bigvee [p]_{\approx} \mapsto [p]_{\approx}$.*

Next we connect the lattice $(L/{\approx}, \leq_{L/\approx})$ (Proposition 2.7) and the lattice $(\mu_L, \leq)$ of cuts of μ; recall that the latter is ordered by reversed inclusion.

Proposition 2.9. *Let $\mu \colon S \to L$ be an L-valued function on S. The lattice $(\mu_L, \leq)$ of cuts of μ is isomorphic to the lattice $(L/{\approx}, \leq_{L/\approx})$ of $\approx$-classes in L under the mapping $\mu_p \mapsto [p]_{\approx}$.*

In the following, for a function $\mu \colon S \to L$, we introduce another lattice valued function, this time from S to the lattice defined on a particular subset $\mathcal{F}$ of the power set $\mathcal{P}(S)$ of the domain S. The new function should have the same cuts as μ, moreover it should have one-element classes of the corresponding equivalence relation $\approx$. In addition, we want every $f \in \mathcal{F}$ to be equal to the corresponding cut of the newly defined lattice valued function. For the co-domain $\mathcal{F}$ we take the lattice $(\mathcal{F}, \leq)$, where $\mathcal{F} = \mu_L \subseteq \mathcal{P}(S)$ is the collection of cuts of μ, and the order $\leq$ is the dual of the set inclusion. The definition of this function follows.

Let $\widehat{\mu} \colon S \to \mathcal{F}$, where

$$\widehat{\mu}(x) \colon = \bigcap \{\mu_p \in \mu_L \mid x \in \mu_p\}. \tag{2.9}$$

Since the co-domain is constructed by the domain ($\mathcal{F} \subseteq \mathcal{P}(S)$), we say that $\widehat{\mu}$ is the **canonical representation** of μ.

By the definition, every element of the co-domain lattice of $\widehat{\mu}$ is a cut of μ. Therefore, if $f \in \mathcal{F}$, then $f = \mu_p$ for some $p \in L$, and for the cut $\widehat{\mu}_f$ of $\widehat{\mu}$, by the definition of a cut and by (2.9), we have

$$\begin{aligned}\widehat{\mu}_f &= \{x \in S \mid \widehat{\mu}(x) \geq f\} = \{x \in S \mid \widehat{\mu}(x) \subseteq \mu_p\} \\ &= \{x \in S \mid \bigcap\{\mu_q \mid x \in \mu_q\} \subseteq \mu_p\} = \mu_p = f.\end{aligned}$$

Therefore, the collection of cuts of $\widehat{\mu}$ is

$$\widehat{\mu}_{\mathcal{F}} = \{Y \subseteq S \mid Y = \widehat{\mu}_{\mu_p}, \text{ for some } \mu_p \in \mu_L\}.$$

Proposition 2.10. *The lattices of cuts of a lattice valued function μ and of its canonical representation $\widehat{\mu}$ coincide.*

Proof. Straightforward, by the above consideration. □

Observe that the lattice valued function μ defined by (2.3) in the proof of Proposition 2.2, coincides with its canonical representation, i.e., in this case we have $\mu = \widehat{\mu}$.

In the next example we present two lattice valued functions μ and ν with different co-domain lattices, having the same canonical representations.

Example 2.11. $S = \{a, b, c, d\}$

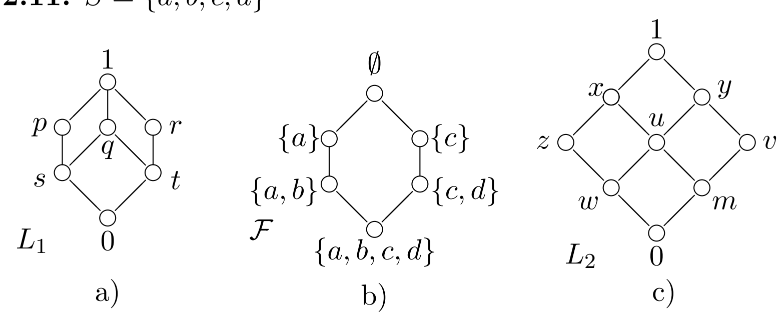

Figure 1

$\mu\colon S \to L_1$, $\mu = \begin{pmatrix} a & b & c & d \\ p & s & r & t \end{pmatrix}$. Lattice L_1 is presented in Figure 1 a).

$\mu_o = S$, $\mu_s = \{a, b\}$, $\mu_t = \{c, d\}$, $\mu_p = \{a\}$, $\mu_q = \emptyset$, $\mu_r = \{c\}$, $\mu_1 = \emptyset$. Family of these subsets under the relation inverse to inclusion is presented in Figure 1 b).

$\nu\colon S \to L_2$, $\nu = \begin{pmatrix} a & b & c & d \\ z & w & m & v \end{pmatrix}$. Lattice L_2 is presented in Figure 1 c).

$\nu_o = S$, $\nu_w = \{a, b\}$, $\nu_m = \{c, d\}$, $\nu_z = \{a\}$, $\nu_v = \{c\}$, $\nu_u = \nu_x = \nu_y = \nu_1 = \emptyset$.

$\widehat{\mu}\colon S \to \mathcal{F}$ and $\widehat{\nu}\colon S \to \mathcal{F}$, where the lattice $\mathcal{F} = \{\{a\}, \{a, b\}, \{c\}, \{c, d\}, \emptyset, \{a, b, c, d\}\}$, $(\mathcal{F}, \leq)$ is presented in Figure 1 b).

$$\widehat{\mu} = \widehat{\nu} = \begin{pmatrix} a & b & c & d \\ \{a\} & \{a,b\} & \{c\} & \{c,d\} \end{pmatrix}.$$

2.4. **Invariance group of lattice valued functions.** We recall the definition of an invariance group and we mention some related notions (see e.g., [2, 3, 10, 13, 16]).

First, we list particular n-variable functions on a finite domain, which are the subject of our study.

A **Boolean function** is a mapping $f\colon \{0,1\}^n \to \{0,1\}$, $n \in \mathbb{N}$. The subset

$$\mathrm{supp}(f) = \{a \in \{0,1\}^n \mid f(a) = 1\}$$

of $\{0,1\}^n$ is called the **support** of f.

We also deal with **lattice valued n-variable functions** on a finite domain $\{0,1,\ldots,k-1\}$:

$$f\colon \{0,1,\ldots,k-1\}^n \to L,$$

where L is a complete lattice.

In particular, for $k = 2$ we get $f\colon \{0,1\}^n \to L$, which is a **lattice valued Boolean function**.

Finally, we use also **p-cuts** of lattice valued functions as characteristic functions: for $f\colon \{0,1,\ldots,k-1\}^n \to L$ and $p \in L$, we have

$$f_p\colon \{0,1,\ldots,k-1\}^n \to \{0,1\},$$

such that $f_p(x_1,\ldots,x_n) = 1$ if and only if $f(x_1,\ldots,x_n) \geq p$.

Clearly, *a cut of a lattice valued Boolean function is (as a characteristic function) a Boolean function.*

As usual, by S_n we denote the symmetric group of all permutations over an n-element set. If f is an n-variable function on a finite domain X and $\sigma \in S_n$, then f is **invariant** under σ, symbolically $\sigma \vdash f$, if for all $(x_1,\ldots,x_n) \in X^n$

$$f(x_1,\ldots,x_n) = f(x_{\sigma(1)},\ldots,x_{\sigma(n)}).$$

If f is invariant under all permutations in $G \leq S_n$ and not invariant under any permutation from $S_n \setminus G$, then G is **the invariance group** of f, and it is denoted by $G(f)$.

Obviously, a group $G \leq S_n$ can be the invariance group of any of the above mentioned functions. Following the approach in [10], we say that $G \leq S_n$ is $(2,2)$**-representable**, if it is the invariance group of a Boolean function $f\colon \{0,1\}^n \to \{0,1\}$.

Remark 2.12. In [10], a group $G \leq S_n$ is said to be (k,m)-*representable* if there is a function $f\colon \{0,1,\ldots,k-1\}^n \to \{1,\ldots,m\}$ whose invariance group is G. If G is the invariance group of a function $f\colon \{0,1,\ldots,k-1\}^n \to \mathbb{N}$, then it is

(k, ∞)-*representable*. By Kisielewicz ([13]), $G \leq S_n$ is m-*representable* if it is the invariance group of a function $f\colon \{0,1\}^n \to \{1,\dots,m\}$; it is *representable* if it is m-representable for some $m \in \mathbb{N}$. By the above, representability is equivalent to $(2, \infty)$-representability. The notion of 2-representability is thoroughly investigated in [13] (see also [2], for invariance groups of Boolean and pseudo-Boolean functions).

According to the above definitions, a permutation group $G \leq S_n$ is (k, L)-**representable**, if there is a lattice valued function
$f\colon \{0,1,\dots,k-1\}^n \to L$, such that $\sigma \vdash f$ if and only if $\sigma \in G$.

In particular, a $(2, L)$-representable group is the invariance group of a lattice valued Boolean function $f\colon \{0,1\}^n \to L$.

The notion of $(2, L)$-representability is more general than $(2,2)$-representability. An example is the Klein 4-group: $\{id, (12)(34), (13)(24), (14)(23)\}$, which is $(2, L)$ representable (for L being a three element chain), but not $(2,2)$-representable. One can easily check that *a permutation group $G \subseteq S_n$ is L-representable if and only if it is Galois closed over* **2** ([10]). Similarly, it is easy to show that *a permutation group is (k, L)-representable if and only if it is Galois closed over the k-element domain.*

3. Results

3.1. **Cuts of composition of functions.** In this part we present some auxiliary results concerning cuts. Namely, our task is to use cuts in order to analyze lattice valued Boolean functions. To do this, we have to deal with composition of cuts represented throughout the section as characteristic functions.

Theorem 3.1. *Let L be a complete lattice, let $A \neq \emptyset$ be a set and let $\sigma\colon A \to A$, $\mu\colon A \to L$, $\psi\colon L \to L$. Then, for every $p \in L$,*
$$(\sigma \circ \mu \circ \psi)_p = \sigma \circ \mu \circ \psi_p.$$

Proof. Observe that $\sigma \circ \mu \circ \psi$ is a function from A to L, and thus $(\sigma \circ \mu \circ \psi)_p$ is a function from A to $\{0,1\}$. Further, ψ_p is a function from L to $\{0,1\}$, hence $\sigma \circ \mu \circ \psi_p$ is also a function from A to $\{0,1\}$.

For every $x \in A$, $(\sigma \circ \mu \circ \psi)_p = 1$ if and only if $(\sigma \circ \mu \circ \psi)(x) \geq p$ if and only if $(\psi(\mu(\sigma(x)))) \geq p$.

On the other hand,
$\sigma \circ \mu \circ \psi_p(x) = 1$ if and only if $\psi_p(\mu(\sigma(x))) = 1$ if and only if $(\psi(\mu(\sigma(x)))) \geq p$.

Therefore, we proved that $(\sigma \circ \mu \circ \psi)_p = \sigma \circ \mu \circ \psi_p$. $\square$

Corollary 3.2. *Let L be a complete lattice, let $A \neq \emptyset$ and let $\mu\colon A \to L$. Then the following holds.*

(i) $\mu_p = \mu \circ (\mathcal{I}_L)_p$, *where* $\mathcal{I}_L$ *is the identity mapping* $\mathcal{I}_{\mathcal{L}} \colon L \to L$.
(ii) $(\sigma \circ \mu)_p = \sigma \circ \mu_p$, *for* $\sigma \colon A \to A$.
(iii) $(\mu \circ \psi)_p = \mu \circ \psi_p$, *where* ψ *is a map* $\psi \colon L \to L$.

Proof. (i) Using Theorem 3.1, where σ is the identity function $\mathcal{I}_{\mathcal{A}}$ on A, we get:

$$\mu_p = (\mathcal{I}_A \circ \mu \circ \mathcal{I}_L)_p = \mathcal{I}_A \circ \mu \circ (\mathcal{I}_L)_p = \mu \circ (\mathcal{I}_L)_p$$

(ii) By Theorem 3.1, where ψ is the identity function $\mathcal{I}_L$ on L, we have:

$$(\sigma \circ \mu \circ \mathcal{I}_L)_p = \sigma \circ \mu \circ (\mathcal{I}_L)_p.$$

By (i), we get

$$(\sigma \circ \mu)_p = (\sigma \circ \mu \circ \mathcal{I}_L)_p = \sigma \circ \mu \circ (\mathcal{I}_L)_p = \sigma \circ \mu_p.$$

(iii) Similarly as in (i):
$(\mathcal{I}_A \circ \mu \circ \psi)_p = \mathcal{I}_A \circ \mu \circ \psi_p$, i.e., $(\mu \circ \psi)_p = \mu \circ \psi_p$. □

3.2. **Main topic: Invariance groups of lattice valued Boolean functions via cuts.** Here we analyze invariance groups of Boolean functions in the framework of cuts of lattice valued Boolean functions. We prove that the invariance group of a lattice valued function depends only on the canonical representation of this function. Further, we prove that the (2,2)-representable groups, subgroups of S_n for a fixed n, are the invariance groups of cuts of a single lattice valued Boolean function.

Proposition 3.3. *Let* $f \colon \{0, \ldots, k-1\}^n \to L$ *and* $\sigma \in S_n$. *Then*

$$\sigma \vdash f \;\text{ if and only if for every }\; p \in L,\ \sigma \vdash f_p.$$

Proof. Observe that every permutation $\sigma \in S_n$ uniquely determines a function $\overline{\sigma} \colon \{0, \ldots, k-1\}^n \to \{0, \ldots, k-1\}^n$, as follows:

$$\overline{\sigma}(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots x_{\sigma(n)}).$$

Now we apply Corollary 3.2 (ii) to functions $\overline{\sigma}$ and f, and we obtain

$$(\overline{\sigma} \circ f)_p = \overline{\sigma} \circ f_p.$$

Next, we suppose that $\sigma \vdash f$, i.e., that for all $x_1, \ldots, x_n \in \{0, 1, \ldots, k-1\}^n$,

$$f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

By the definition of $\overline{\sigma}$, this is further equal to $f(\overline{\sigma}(x_1, \ldots, x_n))$, hence we have $f(x_1, \ldots, x_n) = \overline{\sigma} \circ f(x_1, \ldots, x_n)$.

$(\overline{\sigma} \circ f)_p = \overline{\sigma} \circ f_p$ gives that this is equivalent to

$$\sigma \vdash f_p, \;\text{ for every }\; p \in L.$$

□

Since by Proposition 2.10, an L-valued function and its canonical representation have equal cuts, Proposition 3.3 yields that *the invariance group of a lattice valued function f depends only on the canonical representation of f*. Therefore, the following theorem is a straightforward corollary of Proposition 3.3.

Theorem 3.4. *If $f_1\colon \{0,\dots,k-1\}^n \to L_1$ and $f_2\colon \{0,\dots,k-1\}^n \to L_2$ are two n-variable lattice valued functions on the same domain, then $\widehat{f_1} = \widehat{f_2}$ implies $G(f_1) = G(f_2)$.*

In the following we prove a *representation theorem* for representable subgroups of S_n, for a given n: there is a lattice L and an L-valued Boolean function $\{0,1\}^n \to L$, such that every subgroup representable by a Boolean function is also representable by a cut of this lattice valued Boolean function.

Theorem 3.5. *For every $n \in \mathbb{N}$, there is a lattice L and a lattice valued Boolean function $F\colon \{0,1\}^n \to L$ satisfying the following: If $G \leq S_n$ and $G = G(f)$ for an n-variable Boolean function f, then $G = G(F_p)$, for a cut F_p of F.*

Remark 3.6. This theorem states that every (2,2)-representable subgroup of S_n is the invariance group of a cut of a single lattice valued Boolean function.

Proof. Let $n \in \mathbb{N}$. Let $\{G_i \mid i \in I\}$ be the family of all 2-representable subgroups of S_n. Let $\{f_i \mid i \in I\}$ be a family of Boolean functions $f_i\colon \{0,1\}^n \to \{0,1\}$, such that G_i is 2-representable by f_i for every $i \in I$. Now we take lattice L to be a Boolean lattice with $2^n - 2$ (co-)atoms. Let $c_1,\dots,c_{2^n-2}$ be co-atoms of L. Now, we define a Boolean function $F\colon \{0,1\}^n \to L$ such that $F(0,\dots,0) = 0$, $F(1,\dots,1) = 0$ and values of all other elements from $\{0,1\}^n$ are different co-atoms. Now, if a group G is representable by a Boolean function $f\colon \{0,1\}^n \to \{0,1\}$, we consider the support $\operatorname{supp}(f)$.

Let S be a set of co-atoms, such that $c \in S$ if and only if there is an $x \in \operatorname{supp}(f)$ such that $F(x) = c$. Let $s = \bigwedge S$.

Now, it is straightforward that the cut F_s is the support of a Boolean function which represents G. Indeed, $F_s(x) = 1$ if and only if $F(x) \geq s$, and $F(x) \geq s$ for all $x \in S$.

The group S_n is represented by the 0-cut. □

The function F constructed in the proof of Theorem 3.5 is not unique, since the value of F for n-tuples $(0,\dots,0)$ and $(1,\dots,1)$ can be different (and cuts of F would still represent the observed groups).

It is a part of the folklore (first appeared probably in [20]), that every subgroup of S_n is an invariance group of a function $\{0,\dots,k-1\}^n \to \{0,\dots,k-1\}$

if and only if $k \geq n$. The following proposition shows that if $k \geq n$, then more is true, namely we have:

Proposition 3.7. *If $k \geq n$, then for every subgroup G of S_n there exists a function $f\colon \{0,\ldots,k-1\}^n \to \{0,1\}$ such that the invariance group of f is exactly G.*

Proof. Let G be a subgroup of S_n. Let $(y_1,\ldots,y_n) = (0,\ldots,n-1)$. We define a required function f, as follows: for every $\sigma \in G$, let $f(y_{\sigma(1)},\ldots,y_{\sigma(n)}) = 1$ and let $f(x_1,\ldots,x_n) = 0$ otherwise. $\square$

Using Proposition 3.7, we prove the following theorem which is a generalization of Theorem 3.5.

Theorem 3.8. *For $k, n \in \mathbb{N}$ and $k \geq n$, there is a lattice L and a lattice valued function $F\colon \{0,\ldots,k-1\}^n \to L$ such that the following holds: If $G \leq S_n$, then $G = G(F_p)$ for a cut F_p of of F.*

Remark 3.9. The theorem states that every subgroup of S_n is the invariance group of a cut of a single lattice valued function.

Proof. Let $k, n \in \mathbb{N}$ and $k \geq n$. Let $\{G_i \mid i \in I\}$ be the family of all subgroups of S_n. Let $\{f_i \mid i \in I\}$ be a family of functions $f_i\colon \{0,\ldots,k-1\}^n \to \{0,1\}$, such that G_i is representable by f_i for every $i \in I$. We take the functions defined as in Proposition 3.7. For a subgroup G_i of S_n and $(y_1,\ldots,y_n) = (0,\ldots,n-1)$, the function f_i is defined as follows: for every $\sigma \in G_i$, $f_i(y_{\sigma(1)},\ldots,y_{\sigma(n)}) = 1$ and $f_i(x_1,\ldots,x_n) = 0$ otherwise.

Now we take lattice L to be a Boolean lattice with $n!$ co-atoms $c_1,\ldots,c_{n!}$. We define a function $F\colon \{0,\ldots,k-1\}^n \to L$ such that for every permutation π of elements $0,\ldots,n-1$ the value $F(\pi(0),\ldots,\pi(n-1))$ is a co-atom in L, different permutations determining different values (co-atoms); for all the remaining n-tuples $(d_1,\ldots,d_n) \in \{0,\ldots,k-1\}^n$ let $F(d_1,\ldots,d_n) = 0 \in L$.

Now, we take an arbitrary subgroup G_i of S_n and prove that it is the invariance group of a cut of lattice valued function F.

Let G_i be representable by a function f_i as above, and let O_i be a set of co-atoms, such that $c \in O_i$ if and only if there is an $x \in \operatorname{supp}(f_i)$ fulfilling $F(x) = c$. Let $o_i = \bigwedge O_i$.

As in Theorem 3.5, we deduce that the cut F_{o_i} is the support of a Boolean function which represents G_i. Namely, $F_{o_i}(x) = 1$ if and only if $F(x) \geq o_i$. We have that $F(x) \geq o_i$ for all $x \in O_i$. Finally, S_n is represented by the 0-cut. $\square$

Example 3.10.

Let us demonstrate the proof of Theorem 3.8, for $n = k = 3$. Following this proof, we define 6 functions from the set $\{0, 1, 2\}$ into $\{0, 1\}$, corresponding to the 6 permutation subgroups of S_3:

If G_1 is identity group $\{I\}$ (consisting only of the identity function I), then the corresponding function f_1 is defined by: $f_1(0, 1, 2) = 1$ and $f_1(x, y, z) = 0$, otherwise.

If G_2 is the alternating group (consisting of three elements), then the corresponding function f_2 is defined by: $f_2(0, 1, 2) = f_2(1, 2, 0) = f_2(2, 0, 1) = 1$ and $f_2(x, y, z) = 0$, otherwise.

If G_3 is the group $\{I, (1, 2)\}$ then the corresponding function f_3 is defined by: $f_3(0, 1, 2) = f_3(1, 0, 2) = 1$ and $f_3(x, y, z) = 0$, otherwise.

If G_4 is the group $\{I, (1, 3)\}$ then the corresponding function f_4 is defined by: $f_4(0, 1, 2) = f_4(2, 1, 0) = 1$ and $f_4(x, y, z) = 0$, otherwise.

If G_5 is the group $\{I, (2, 3)\}$ then the corresponding function f_5 is defined by: $f_5(0, 1, 2) = f_5(0, 2, 1) = 1$ and $f_5(x, y, z) = 0$, otherwise.

The lattice L is a Boolean lattice with six co-atoms. We define the function $F\colon \{0, 1, 2\}^3 \to L$, such that for every permutation π of $\{0, 1, 2\}$, the value $F(\pi(0), \pi(1), \pi(2))$ is a co-atom in L, different permutations determining different co-atoms. All other values of F are equal to the bottom, $0 \in L$. By the construction, every subgroup of S_3 is the invariance group of a cut of the function F.

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