On the minimization of Horn formulas

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- Horn formulas
- Horn minimization
- Horn minimization: Steiner version
- directed hypergraphs
- hydra number

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- equivalent frameworks: closures, lattices, directed hypergraphs, functional dependencies, formal concepts, implicational systems
- Poole - Mackworth: *Artificial Intelligence: Foundations of Computational Agents*, 2010:
  - ‘uses rational computational agents and Horn clause logic as unifying threads in this vast field’

Horn formulas, entailment

- Horn clause: at most one unnegated variable, e.g. $C = \overline{a} \vee \overline{b} \vee c$, written as $a, b \to c$, $Body(C) = \{a, b\}$, $Head(C) = c$
- definite clause: exactly one unnegated variable
- (definite) formula: conjunction of (definite) Horn clauses
- Horn function: representable by a Horn formula
- entailment: $(a, b \to c) \wedge (c \to d) \models (a, b \to d)$
- implicate: $K \models C$
- prime implicate: no subclause is an implicate
- *forward chaining* - efficient

Research directions

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- dynamic problems

Horn minimization

- given a Horn formula φ and a number k, is there a Horn formula with at most k clauses equivalent to φ?
- $x_i \to y_j, \quad y_1, \ldots, y_n \to x_i \quad$ $n^2 + n$ clauses
- $x_i \to x_{i+1}, \quad x_n \to y_i, \quad y_1, \ldots, y_n \to x_1 \quad$ $2n$ clauses

Previous work: minimization

- Umans (2000): CNF minimization is Σ_2^p-complete
- Ausiello, D'Atri, Saccà (1986): Horn minimization is NP-complete
- Hammer, Kogan (1993): NP-complete if the number of literals is to be minimized; in P for quasi-acyclic formulas
- Maier (1983), Ausiello, D'Atri, Saccà (1986), Guigues, Duquenne (1986), Angluin, Frazier, Pitt (1992): minimization of the number of bodies can be done efficiently
- Boros, Čepek, Kogan (1997): iterative decomposition algorithm

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Theorem

Boros, Gruber (2011): if $P \neq NP$ then 3-Horn minimization is not efficiently

$$2^{(\log size(\varphi))^{1-o(1)}}$$

approximable.

Introducing new variables

- Flögel, Kleine Büning, Lettmann (1993)
- $x_1/y_1, \ldots, x_n/y_n \to u$ 2^n clauses
- $x_i \to z_i,\ \ y_i \to z_i,\ \ z_1, \ldots, z_n \to u$ $2n + 1$ clauses
- same set of consequences over the original variables
- $\varphi \sim_{\mathcal{R}} \psi$, where $\mathcal{R}$ is a set of variables: same set of consequences over $\mathcal{R}$
- $co - NP$-complete even in rather restricted cases, so extension is too powerful

Steiner extension

- introduce new variables in a restricted way
- $x_i \to y_j, \;\; i = 1, \ldots, n, \; j = 1, \ldots, m$ nm clauses
- $x_i \to z, \;\; z \to y_j, \;\; i = 1, \ldots, n, \; j = 1, \ldots, m$ $n + m$ clauses
- new variables can be heads, or singleton bodies with old heads
- $\varphi \sim_{\mathcal{R}} \psi$, where variables new are introduced by Steiner extension: can be decided efficiently
- minimization is MAX-SNP-hard

A $o(n)$ approximation algorithm

Theorem

There is an efficient

$$O\left(n\,\frac{\log\log n}{(\log n)^{1/4}}\right) \quad O\left(\frac{n}{\log n}\right)$$

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- find an equivalent Horn formula with the minimal number of bodies
- find a decomposition of the bipartite graph between heads and bodies

Decomposition of graphs

- given a graph, partition its edges into complete bipartite graphs

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- minimize the sum of the number of vertices

Efficient decomposition

Theorem

There is an efficient algorithm for finding

- *a decomposition of* (a, b)*-bipartite graphs* $(a \geq b)$ *into complete bipartite graphs with*

$$O\left(\frac{ab}{\log a} + a \log b\right) \quad + a$$

vertices altogether.

Directed hypergraphs

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- directed hyperedge $a, b \to c$
- directed hypergraph
  - $\{(a, b \to c), (a, c \to d), (d, e \to f)\}$

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- first-order logic

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- vertex v is reachable from S if $v \in cl_H(S)$

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- motivation: special case of the minimization of Horn formulas

Examples, single-headed graphs

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- undirected graph G is single-headed if $h(G) = |E|$

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- if G has a cut edge between with both halves having at least two vertices then G is not single-headed

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Theorem

There are single-headed graphs G_k with $\Theta(k)$ edges such that $p(L(G')) = \Theta(k)$ for every connected spanning subgraph G'

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- $h(T) = |E(T)| + 1$ *iff* T *is a caterpillar.*

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Theorem

For the complete binary tree of depth d

$$\frac{13}{12}|E(B_d)| \leq h(B_d) \leq \frac{8}{7}|E(B_d)|.$$

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