Complexity of quasivariety lattices

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The set $Lq(\mathcal{R})$ of all subquasivarieties of a given quasivariety $\mathcal{R}$ forms a complete lattice with respect to inclusion.
A *lattice of quasivarieties* or a *quasivariety lattice* is a lattice isomorphic to $Lq(\mathcal{R})$ for some quasivariety $\mathcal{R}$.

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Since the end of nineties, the problem has become known as the Birkhoff-Maltsev problem.

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A restriction on an atomic join-semidistributive dually algebraic lattice with dual compact the least element to be a lattice of quasivarieties was found by W. Dziobiak in 1987. He proved that in a quasivariety lattice, the join of any n atoms contains at most $2^n - 1$ atoms.

Definition

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Examples.

1. Quasivariety of all semigroups (M. Sapir, 1985);
2. Quasivariety of all unars (V. I. Tumanov, 1988);
3. Quasivariety of all (modular) lattices (W. Dziobiak, 1986; V. I. Tumanov, 1981);

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A class of algebras $\mathcal{K}$ is called *irrational* (with respect to property $\mathcal{P}$) (or *unreasonable*), if the set of all finite algebras (with property $\mathcal{P}$) belonging to $\mathcal{K}$ is not computable; equivalently, if there is no algorithm to decide whether a finite algebra (with property $\mathcal{P}$) belongs to $\mathcal{K}$.

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Definition

An algebra $\mathbf{A}$ is *irrational* if the set of all its subalgebras is irrational; equivalently, if there is no algorithm to decide whether a finite algebra is a subalgebra of $\mathbf{A}$.

(C. Herrmann, 2007).

Examples.

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2. *Is a class of all algebras with a finite quasivariety lattices an irrational?*

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2. *Is a class of all quasivariety lattices an irrational?*

On complexity of quasivariety lattices.

Theorem (V. I. Tumanov, 1988)

Let $\mathcal{R}$ be a quasivariety of all algebras of signature σ. If σ contains at least one non-constant operation then $Lq(\mathcal{R})$ is Q-universal.

Theorem

For any signature σ *containing at least one non-constant operation there is continuum of non-isomorphic irrational locally finite lattices of quasivarieties of signature* σ,

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On complexity of irrationality:

Theorem

There is irrational quasivarietiy lattice $\mathbf{L}$ such that the set of all finite sublattices of $\mathbf{L}$ is recusive enumerable.

Thank you very much for your attention!