Effective bases of closure systems

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Szeged

Based on preprints:

(1) **"On the implicational bases of closure systems with the unique criticals"**
joint work with J.B.Nation (AN-2012)

(2) **" Optimum bases of convex geometries"**
(A-2012)

Outline

1. Closure spaces, lattices and implications
2. Propositional Horn logic and Horn Boolean functions
3. Types of efficient bases
4. Canonical basis of Duquenne-Guigues
5. K-basis
6. UC-closure systems
7. E-basis
8. Optimum bases in convex geometries

Closure spaces

$\langle X, \phi \rangle$ is *a closure space*, if

- X is non-empty set (finite in this talk);
- ϕ is a closure operator on X, i.e. $\phi : 2^X \to 2^X$ with
  (1) $Y \subseteq \phi(Y)$;
  (2) $Y \subseteq Z$ implies $\phi(Y) \subseteq \phi(Z)$;
  (3) $\phi(\phi(Y)) = \phi(Y)$, for all $Y, Z \subseteq X$.

Closed set: $A = \phi(A)$.

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Standard systems

If $X^* \subseteq X$, then $\phi^*(Y) = \phi(Y) \cap X^*$, $Y \subseteq X^*$, defines a closure operator on X^*.
$\langle X^*, \phi^* \rangle$ is called a *standard* closure space, if X^* is a subset of the minimal cardinality such that the closure space $\langle X^*, \phi^* \rangle$ has the same number of closed sets as $\langle X, \phi \rangle$.
Standard spaces are characterized by the property:

$$\phi^*(x) \setminus x \text{ is closed, for every } x \in X^*.$$

In particular, $\phi^*(\emptyset) = \emptyset$, and $\phi^*(x) = \phi^*(y)$ implies $x = y$.
There exists a straightforward algorithm to obtain a standard closure system from the given one.

Moore family

- Closed sets of $\langle X, \phi \rangle$ form *Moore family* $\mathcal{M}$:
  $\mathcal{M}$ is closed with respect to intersection $\cap$ and $X \in \mathcal{M}$;
- Every Moore family $\mathcal{M} \subseteq 2^X$ defines a closure operator on X:
  $\phi(Y) = \bigcap\{Z \in \mathcal{M} : Y \subseteq Z\}$, $Y \subseteq X$.
- Moore family can be turned into lattice of closed sets $Cl(X, \phi)$:
  $Y \wedge Z = Y \cap Z$,
  $Y \vee Z = \bigcap\{W \in \mathcal{M} : Y, Z \subseteq W\}$.

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Lattices and closure spaces

Proposition

Every finite lattice $(L, \vee, \wedge)$ *is the lattice of closed sets of some closure space* $\langle X, \phi \rangle$.

- Take $X = J(L)$, the set of join-irreducible elements: $j \in J(L)$, if $j \neq 0$, and $j = a \vee b$ implies $j = a$ or $j = b$;
- define $\phi(Y) = \{j \in J(L) : j \leq \bigvee Y\}$, $Y \subseteq X$.
- This closure space is *standard*.

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Example: Building a closure space associated with lattice A_{12}.
$X = J(A_{12}) = \{1, 2, 3, 4, 5, 6\}$. $\phi(\{4, 6\}) = \{1, 3, 4, 6\}$, $\phi(\{2, 4\}) = X$ etc.

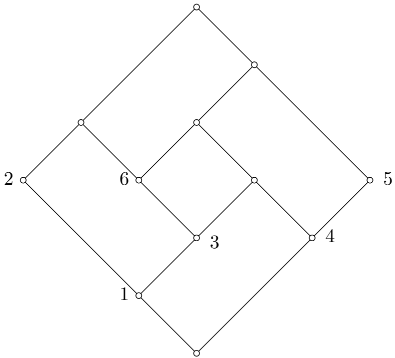

Figure: A_{12}

Closure spaces and implications

- An implication σ on X: $Y \to Z$, for $Y, Z \subseteq X$, $Z \neq \emptyset$.
- Moore family $\mathcal{M}_\sigma$ contains subsets A of X that *respects* σ: if $Y \subseteq A$, then $Z \subseteq A$.
- if Σ is a set of implications $\{\sigma_1, \dots, \sigma_k\}$, then $\mathcal{M}_\Sigma$ is Moore family of subsets A that respect all σ_j;
- the corresponding closure space is $\langle X, \phi_\Sigma \rangle$
- Every closure space $\langle X, \phi \rangle$ can be presented as $\langle X, \psi_\Sigma \rangle$, for some set Σ of implications on X.
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Implications and propositional Horn logic

- *Unit implication* σ on X: $\quad Y \to z,\ Y \subseteq X, z \in X$.
- Every implication $Y \to Z$ is equivalent to the set of unit implications $\{Y \to z, z \in Z\}$: *unit expansion*.
- Logical interpretation of unit implication σ:
  $X = \{x_1, \ldots, x_n\}, Y = \{x_1, \ldots, x_k\}, z = x_{k+1}$
  $\sigma \equiv x_1 \wedge x_2 \cdots \wedge x_k \to x_{k+1}$.
- Equivalent form without implication (a definite *Horn clause*):
  $\sigma \equiv \neg x_1 \vee \neg x_2 \cdots \vee \neg x_k \vee x_{k+1}$
- Equivalent form for the set of implications $\Sigma = \{\sigma_1, \ldots, \sigma_m\}$ (a definite *Horn formula*):

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Horn Boolean functions

For every (definite) Horn formula Σ_H of n variables $x_1, x_2, \ldots, x_n$, one can define n-ary Boolean function $f(x_1, \ldots, x_n) : \{0,1\}^n \to \{0,1\}$ such that

$$f(s_1, \ldots s_n) = 1 \text{ iff } \Sigma_H \text{ is true,}$$

under assignment $x_i := s_i$, where $s_i \in \{0,1\}$.
Σ_H corresponds to CNF of function $f(x_1, \ldots, x_n)$.

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Summarizing:

Five equivalent ways to look at closure system $\langle X, \phi \rangle$:

- Moore family;
- lattice of closed sets $Cl(X, \phi)$;
- set of implications $\Sigma(X, \phi)$;
- definite Horn formula $\Sigma_H(X, \phi)$;
- definite Horn function $f : \{0, 1\}^{|X|} \to \{0, 1\}$.

Connections to computer science fields

Closure operators given by implications or Horn formulae appear in:

- relational data bases;
- data-mining;
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- Term *a base* or *a basis* is used when the set of implications Σ' that defines the same closure system satisfies some condition of minimality.

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- A basis Σ' is *non-redundant*, if none of its implications can be removed to get another basis.
- A basis Σ' is *minimum*, if it has the minimal number of implications among all the set of implications for the same closure system.
- A basis $\Sigma' = \{X_i \to Y_i : i \leq n\}$ is called *optimum*, if number $s(\Sigma') = |X_1| + \cdots + |X_n| + |Y_1| + \cdots + |Y_n|$ is smallest among all sets of implications for the same closure system.
- A basis is called *right-side* *(left-side)* optimum basis, if the number $|Y_1| + \cdots + |Y_n|$ $(|X_1| + \cdots + |X_n|)$ is smallest among all sets of implications for the same closure system.
- The right-side optimum basis is connected to the problem of the shortest (i.e. with the minimal number of clauses) CNF-representation of a (definite) Horn function, also, minimal representations of the directed hypergraphs.

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- The right-side optimum basis is connected to the problem of the shortest (i.e. with the minimal number of clauses) CNF-representation of a (definite) Horn function, also, minimal representations of the directed hypergraphs.

Relation between bases

Theorem

[D.Maier, 1983] Optimum $\Longrightarrow$ minimum and left-side optimum $\Longrightarrow$ non-redundant.

- A basis Σ' is *minimum*, if it has the minimal number of implications among all the set of implications for the same closure system.
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Canonical basis

J.L. Guigues, V. Duquenne, *Familles minimales d'implications informatives résultant d'une tables de données binares*, Math. Sci. Hum. **95** (1986), 5–18.

- Defined *quasi-closed* and *critical* subsets of X for any given closure system $\langle X, \phi \rangle$.
- Canonical basis Σ_C is $\{A \to B : A \text{ is critical}, B = \phi(A) \setminus A\}$.
- Σ_C is a minimum basis among all the bases generating $\langle X, \phi \rangle$.
- Defined *saturation closure operator* σ associated with ϕ.
- Every other basis relates to Σ_C, via saturation operator σ.
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Binary part of canonical basis

Suppose $\phi(\{a\}) \neq \{a\}$, for some $a \in X$.

- $\{a\}$ is a critical set;
- $a \to B$ is present in the canonical basis;
- set of implications $a \to B$ from the basis are called *the binary part* of the basis;
- assuming $\langle X, \phi \rangle$ is standard, one can define *a partial order* on X: $a \geq_\phi b$ iff $a \to B$ is in the canonical basis and $b \in B$.
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4 parts of the optimum basis

Structure of the optimum basis per D. Maier.

	Binary part	Non-binary part
the left side	$a \to B$ fixed parameter	$A \to B$ $\lvert A\rvert$ is a fixed parameter
the right side	?(1)?	?(2)?

Proposition

[AN-2012] Assume that the closure system is standard.
(1) For every $a \to B$ in any optimum basis, $|B|$ is a fixed parameter.
(2) Total size $R_n = |B_1| + \ldots |B_k|$ in non-binary part is a fixed parameter.
(3) Each individual $|B_k|$ may vary.

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UC-systems

Closure system $\langle X, \phi \rangle$ has *unique criticals*, or it is *UC-system*, if $\phi(C_1) = \phi(C_2)$, for some critical sets C_1, C_2, implies $C_1 = C_2$.

Conjecture

In UC-system, for every $A \to B$ in the optimum basis, $|B|$ is a fixed parameter.

K-basis

- K-basis is inspired by $\ll$-minimal representations of elements in the lattice of closed sets.
- K-basis has the same number of implications as the canonical, i.e. it is a minimum basis.
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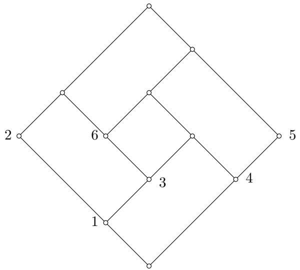
$\ll$-Minimal representations by join irreducibles

A *join-representation* of element $a \in L$ is an expression $a = j_1 \vee \cdots \vee j_k$, for some join-irreducible elements $j_1, \ldots, j_k$.

A join-representation $a = j_1 \vee \cdots \vee j_k$ is called $\ll$-*minimal*, if none of $j_1, \ldots, j_k$ can be dropped or replaced by smaller join-irreducibles to obtain another join-representation of a.

For example, the top element of A_{12} is $2 \vee 6 \vee 5$.

But $\{2, 4\} \ll \{2, 6, 5\}$ and $2 \vee 4$ is $\ll$-minimal join representation.



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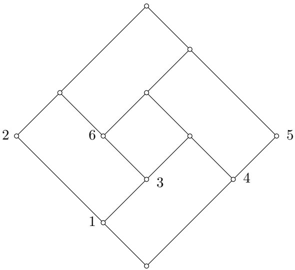

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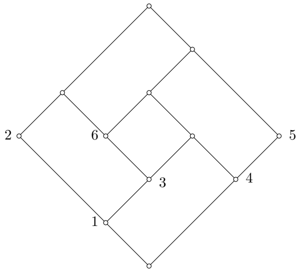

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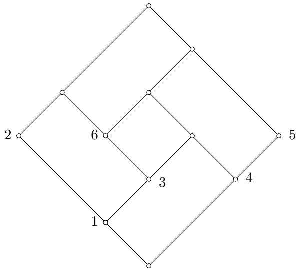

K-basis

Polynomial (in $s(\Sigma_C)$) Algorithm to obtain K-basis from the canonical.

Recall: $a \geq_\phi b$ iff $a \to B$ is in the canonical basis and $b \in B$.

Given: $A \to B$ in the canonical basis Σ_C.
Obtain: $A_K \to B_K$ in the K-basis.
Loop: (1) Find $\geq_\phi -$ max element $a \in A$,
which was not checked yet.
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When true, $A := A \setminus a$.
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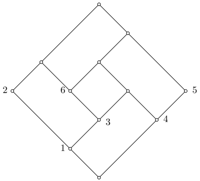

Figure: A_{12}

Canonical basis Σ_C:
$2 \to 1, 6 \to 13, 3 \to 1, 5 \to 4, 14 \to 3, 123 \to 6, 1345 \to 6, 12346 \to 5$
$s(\Sigma_C) = 27$
K-basis:
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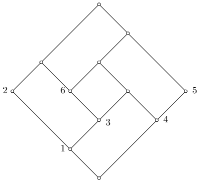

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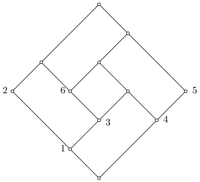

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[A. Day, 1992] Given any basis Σ' of a finite closure system, it requires time $O(s(\Sigma')^2)$ to obtain the canonical basis of Duquenne-Guigues.

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K-basis

In general, the closure space may have more than one K-basis.

A special subclass of UC-systems is defined via the lattice of closed sets.

Definition

A closure system is called semidistributive, if its closure lattice $Cl(X, \phi)$ satisfies the property:

($SD_\vee$) $\quad x \vee y = x \vee z \to x \vee y = x \vee (y \wedge z).$

Theorem

[Jónsson and Kiefer, 1962] Every element of a finite lattice has a unique minimal representation iff the lattice is semidistributive.

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Semidsitributive systems

Important subclasses of semidsitributive closure systems:

- In lattice theory: lower bounded lattices (lattices without D-cycles), free lattices (R.Freese, J.Jezek, J.B.Nation).
- In combinatorics: convex geometries and anti-matroids (P.Edelman and R. Jamison)
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Systems without *D*-cycles

Lower bounded lattices, or lattices without *D*-cycles: can be defined via *D*-relation on the set of join-irreducible elements (A.Day, 1979).

Let D^* be the binary relation on X defined via any K-basis:

aD^*b iff there exists $A^* \to B^*$ in a K-basis with $|A^*| > 1$ such that $a \in A^*$ and $b \in B^*$.

Theorem

[AN-2012]

(1) *A closure system* $\langle X, \phi\rangle$ *has the closure lattice without D-cycles iff there is no sequence* $a_1 D^* a_2 D^* \dots D^* a_n = a_1$, *with* $a_i \in X$ *and* $n > 1$.

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Bases in systems without D-cycles

This basis was introduced for the systems without D-cycles in: K.Adaricheva, J.B.Nation and R.Rand, "Ordered direct basis of a finite closure system", to appear in *Discrete Applied Mathematics*

E-basis:

	Canonical basis	$K-$basis	$E-$basis
$\|A\| > 1$	$A \to B$	$A^* \to B^*$	$A^* \to B^{**}$
$\|A\| = 1$	$a \to B$	$a \to B^*$	$a \to B^*$

$$B^{**} \subseteq B^*$$

Proposition

[AN-2012] E-basis can be obtained from K-basis via polynomial time algorithm: if $b \in B_1^*, B_2^*$, for two implications $A_1^* \to B_1^*$, $A_2^* \to B_2^*$ in the K-basis, and $\phi(A_1^*) \subset \phi(A_2^*)$, then b can be removed from B_2^*.

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E-basis

Recall: $|B_1| + |B_2| + \cdots + |B_K|$, for the right sides of non-binary part of any optimum basis, is a fixed parameter R_n.

Theorem

[AN-2012] The total right size of non-binary part of the E-basis attains the minimum R_n.

Theorem

[ANR-2011] The E-basis of a closure system without D-cycles is ordered direct.

4 parts of the optimum basis: systems without D-cycles

	Binary part	Non-binary part
the left side	$a \to B$ tractable	$A \to B$ NP
the right side	$a \to B$ NP	$A \to B$ tractable

Proposition

[AN-2012] Assume that the closure system is without D-cycles.
(1) Finding the optimum right-side in binary part of the basis is NP-complete.
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Convex geometry

A closure system $\langle X, \phi \rangle$ is called a *convex geometry*, if $\phi(\emptyset) = \emptyset$, and *anti-exchange property* holds:

For every $A = \phi(A), x, y \notin A$, if $x \in \phi(A \cup y)$, then $y \notin \phi(A \cup x)$.

$x \in A$ is called *extreme point* of A, if $x \notin \phi(A \setminus x)$. $Ex(A)$ is a set of extreme points of A.

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Assume that the closure system is a convex geometry.
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Optimum basis: convex geometries without D-cycles

	Binary part	Non-binary part
the left side	$a \rightarrow B$ tractable	$A \rightarrow B$ tractable
the right side	$a \rightarrow B$ tractable	$A \rightarrow B$ tractable

Corollary

[A-2012] If a closure system is a convex geometry without D-cycles, then optimum basis can be obtained in polynomial time.

This class properly includes the quasi-acyclic closure systems defined in [P. Hammer and A. Kogan, 1995] , which are also G-geometries in [M.Wild, 1994].

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Optimum basis: convex geometries with *n*-Carousel rule

Class of convex geometries with n-Carousel rule includes *affine* convex geometries $Co(R^n, X)$: $X \subseteq R^n$, $\phi(Y) = \text{convex hull}(Y) \cap X$.

2-Carousel Rule: $x, y \in \phi(A), A \subseteq X$, implies $x \in \phi(\{y, a_i, a_j\})$, for some $a_i, a_j \in A$.

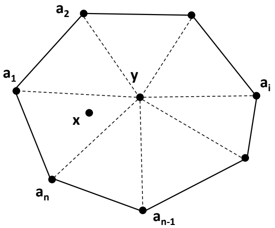

Figure: 2-Carousel rule

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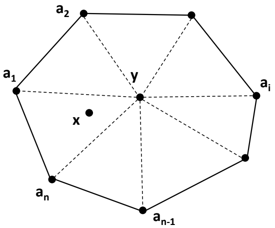

Figure: 2-Carousel rule

Optimum basis: convex geometries with n-Carousel rule

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Theorem

[A-2012] If a closure system is a convex geometry satisfying n-Carousel rule, then optimum basis can be obtained in polynomial time.

Optimum basis: general case

Another tractable subclass: component-quadratic closure systems,
E. Boros, O. Čepek, A. Kogan and P. Kucěra, RUTCOR, 2009.

Question: Can the optimum basis be found effectively, for every convex geometry?

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Conclusions

- K-basis might not be an optimum basis, but it is always the minimum basis whose size is smaller than or equal the size of the canonical basis.
- In semidistributive closure systems K-basis is unique and is a good approximation of optimum basis.
- If the closure system is without D-cycles, further refinement of the K-basis can be effectively obtained, giving right-side optimum in its non-binary part.
- If a closure system is a convex geometry, either without D-cycles or with n-Carusel rule, the optimum basis is tractable. In general, the problem is NP-complete.

Conclusions

- K-basis might not be an optimum basis, but it is always the minimum basis whose size is smaller than or equal the size of the canonical basis.
- In semidistributive closure systems K-basis is unique and is a good approximation of optimum basis.
- If the closure system is without D-cycles, further refinement of the K-basis can be effectively obtained, giving right-side optimum in its non-binary part.
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Last slide: J.B.Nation

Figure: Hiking in Catskill mountains, New York State