Distance of Closures

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1 Introduction

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A notion of *distance* between databases is needed.

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Two database are considered the same if they have the same number of attributes and the system of functional dependencies are identical. The distance is introduced only between two databases having the same number of attributes.

2 Definition of distance of databases/closures

Closures

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$$\begin{aligned} &A \subseteq \boldsymbol{\ell}(\mathbf{r})(A) \\ &A \subset B \Longrightarrow \boldsymbol{\ell}(\mathbf{r})(A) \subseteq \boldsymbol{\ell}(\mathbf{r})(B) \\ &\boldsymbol{\ell}(\mathbf{r})(\boldsymbol{\ell}(\mathbf{r})(A)) = \boldsymbol{\ell}(\mathbf{r})(A). \end{aligned} \tag{1}$$

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Attribute set A is *closed* if $A = \ell(\mathbf{r})(A)$. Since constant columns are not really interesting, we assume that $\ell(\mathbf{r})(\emptyset) = \emptyset$.

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In $\mathbf{r_2}$ there are more subsets of attributes that only determine attributes inside them, that is in $\mathbf{r_2}$ we need more attributes to determine some attribute, so $\mathbf{r_2}$ conveys "more information" in the sense that the values of tuples are more abitrary.

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ℓ_2 *covers* ℓ_1, if $\ell_1 \leq \ell_2$ and for all ℓ' such that $\ell_1 \leq \ell' \leq \ell_2$ either $\ell' = \ell_1$ or $\ell' = \ell_2$.

Proposition (BDK)

ℓ_2 covers ℓ_1 iff $\mathcal{F}(\ell_1) \subseteq \mathcal{F}(\ell_2)$ and $|\mathcal{F}(\ell_2) \setminus \mathcal{F}(\ell_1)| = 1$.

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The poset of all closures over a given schema $\mathcal{R}$, $\mathbf{P}(\mathcal{R})$, is *ranked*: its elements are distributed in *levels* and if ℓ_2 covers ℓ_1, then ℓ_2 is in the next level above ℓ_1's one.

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Let $|\mathcal{R}| = n$. Since $\emptyset$ and $\mathcal{R}$ are both closed for any closure considered, the *height* of $\mathbf{P}(\mathcal{R})$ is $2^n - 2$.

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Definition

Let $\mathbf{r}$ and $\mathbf{r}'$ be two instances of schema $\mathcal{R}$. Their distance $d(\mathbf{r}, \mathbf{r}')$ is defined to be the graph theoretic distance of $\ell(\mathbf{r})$ and $\ell(\mathbf{r}')$ in the Hasse diagram of $\mathbf{P}(\mathcal{R})$. That is, the length of the shortest path between points $\ell(\mathbf{r})$ and $\ell(\mathbf{r}')$ using only covering edges.

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Satisfies the triangle inequality.
Since every closure $\mathcal{L}$ on a finite underlying set is in the form $\mathcal{L} = \ell(\mathbf{r})$, the *distance of closures* is defined.

3 Possible applications

Approximate semantic sampling of existing databases

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Approximation of the problem: Given a database relation $\mathbf{r}$ and an upper bound b, find a subrelation $\mathbf{r}' \subseteq \mathbf{r}$ such that $|\mathbf{r}'| \leq b$ and $d(\mathbf{r}, \mathbf{r}') = min\{d(\mathbf{r}, \mathbf{r}''): |\mathbf{r}''| \leq b,\ \mathbf{r}'' \subseteq \mathbf{r}\}$.

Approximate integrity enforcement

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The usual update operations on a database system only allow updates on a given relation $\mathbf{r}$ that result in a modified relation $\mathbf{r}'$ that satisfies the set Σ of functional dependencies specified over the given relation schema. Instead of this, we may allow more updates that keep the integrity constraints "approximately intact", as follows.

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Given $\mathbf{r}$, a time-window t and an upper bound b, only allow updates within the time-window t that result in a modified relation $\mathbf{r}'$ whose distance $d(\mathbf{r}', \mathbf{r})$ from $\mathbf{r}$ is at most b. Here it is an important special case if we enforce that $\mathbf{r}'$ must satisfy the same keys as $\mathbf{r}$.

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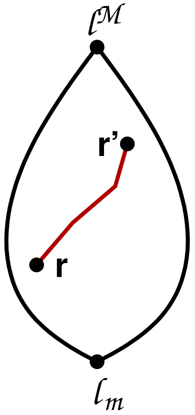
We want to match the two schemata as much as possible, so we want to find a bijection β between $\mathcal{R}$ and $\mathcal{R}'$ such that
$d(\beta(\Sigma), \Sigma') = \min\{d(\beta'(\Sigma), \Sigma') \colon \beta' \text{ is a bijection between} \mathcal{R} \text{ and } \mathcal{R}'\}$.
Here $\beta(\Sigma) = \{\beta(a_1)...\beta(a_n) \to \beta(a) \colon a_1 \ldots a_n \to a \in \Sigma\}$.

4 General observations on the distance of closures

Maximum distance

Proposition

Let $|\mathcal{R}| = n$. Then $d(\mathbf{r}, \mathbf{r}') \leq 2^n - 2$ for any two instances of the schema $\mathcal{R}$.



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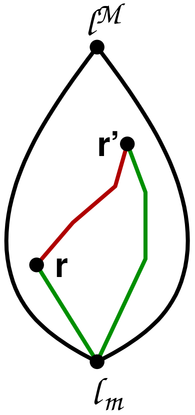

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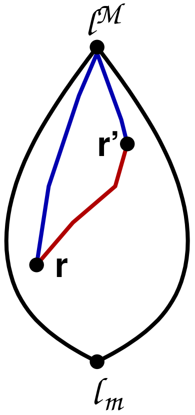

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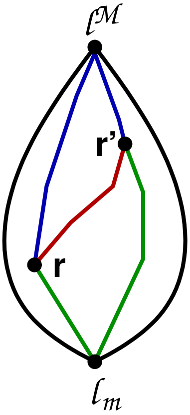
Thus,

$$2\, d(\mathbf{r}, \mathbf{r}') \leq d(\mathbf{r}, \boldsymbol{\ell}_m) + d(\mathbf{r}, \boldsymbol{\ell}^M) + d(\boldsymbol{\ell}_m, \mathbf{r}') + \\ + d(\boldsymbol{\ell}^M, \mathbf{r}') = 2^n - 2 + 2^n - 2.$$



Distance of two instances

Theorem

For any two instances $\mathbf{r}$ *and* $\mathbf{r}'$ *of the schema* $\mathcal{R}$ *we have*

$$d(\mathbf{r},\mathbf{r}') = |\mathcal{F}(\ell(\mathbf{r})) \,\Delta\, \mathcal{F}(\ell(\mathbf{r}'))|, \tag{2}$$

where $A \,\Delta\, B$ *denotes the symmetric difference of the two sets, i.e.,* $A \,\Delta\, B = A \setminus B \cup B \setminus A$.

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Theorem (Demetrovics, Katona)

A collection $\mathcal{F}$ *of subsets of* $\mathcal{R}$ *is the collection of closed sets of some closure* $\ell(\mathbf{r})$ *for an appropriate instance* $\mathbf{r}$ *of* $\mathcal{R}$ *iff* $\emptyset, \mathcal{R} \in \mathcal{F}$ *and* $\mathcal{F}$ *is closed under intersection.*

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$|\mathcal{F}(\ell(\mathbf{r}))|$ changes by one traversing along a covering edge in the Hasse diagram of $\mathbf{P}(\mathcal{R})$, so $LHS \geq RHS$.

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Add successively a minimal element of $\mathcal{F}(\boldsymbol{\ell}(\mathbf{r}')) \setminus \mathcal{F}(\boldsymbol{\ell}(\mathbf{r}))$ with respect to set containment.

3 Diameter of collection of databases with the same set of minimal keys

Keys, minimal keys

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$\mathcal{K}(\mathbf{r})$ does not determine $\ell(\mathbf{r})$.

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$$\mathbf{r_2} = \begin{array}{cccc} a & b & c & d \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 3 \\ 4 & 0 & 4 & 4 \\ 5 & 5 & 0 & 5 \\ 6 & 6 & 6 & 0 \end{array}$$

Antikeys

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In the previous example $\mathcal{K} = \{\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}\}$ and $\mathcal{K}^{-1} = \{\{a, b\}, \{c, d\}\}$.

The theorem

For a set system $\mathcal{A}$ of subsets of $\mathcal{R}$ let $\mathcal{A}\!\downarrow = \{B \subseteq \mathcal{R} \colon \exists A \in \mathcal{A} \text{ with } B \subseteq A\} \cup \{\mathcal{R}\}$ and let $\mathcal{A}_\cap = \{B \subseteq \mathcal{R} \colon \exists i \geq 1, A_1, A_2, \ldots A_i \in \mathcal{A} \text{ with } B = A_1 \cap A_2 \cap \ldots \cap A_i\} \cup \{\mathcal{R}\} \cup \{\emptyset\}$.

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Let $\mathcal{K}$ be an inclusion-free family of subsets of $\mathcal{R}$. Then the closures whose minimal key system is $\mathcal{K}$ form an interval in the poset of closures $\mathbf{P}(\mathcal{R})$ whose smallest element is the closure with closed sets $\mathcal{K}_{\cap}^{-1}$ and largest element is the closure with closed sets $\mathcal{K}^{-1}{\downarrow}$

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Corollary

The diameter, that is the largest distance between any two elements of the collection of closures with given key system $\mathcal{K}$ is $|\mathcal{K}^{-1}\downarrow| - |\mathcal{K}_\cap^{-1}|$.

Proof (or something like that) of the Theorem

Let A be a maximal antikey. For any $b \in \mathcal{R} \setminus A$, $A \cup \{b\}$ is a key, thus $A \cup \{b\} \to \mathcal{R}$ holds.

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If $\boldsymbol{\ell}(\mathbf{r})(X) = X$ holds for some $X \subsetneq \mathcal{R}$, then X is contained in maximal antikey $A \supset X$, hence $X \in \mathcal{K}^{-1}\downarrow$.

4.1 Non-uniform minimal key system

The Theorem

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Let $\mathcal{M}$ be a non-empty, inclusion-free family. Define

$$\mathcal{D}(\mathcal{M}) = \{H : \exists M \in \mathcal{M} \text{ such that } H \subseteq M\}, \tag{3}$$

$$\mathcal{U}(\mathcal{M}) = \{H : \exists M \in \mathcal{M} \text{ such that } H \supseteq M\}. \tag{4}$$

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Theorem

Let $\mathcal{K}$ be a non-empty inclusion-free family of subsets of $[n]$, where $|\mathcal{K}| \geq n$ is fixed. Furthermore, let $S(\mathcal{K})$ denote the set of all closures in which the family of minimal keys is exactly $\mathcal{K}$. Then

$$\mathrm{diam}(S(\mathcal{K})) \leq 2^n - |\mathcal{U}(\mathcal{K}^*)|, \tag{5}$$

where $\mathcal{K}^*$ consists of some lexicographically last sets of size s and all the $s+1$-element sets not containing the selected s-element ones, for some $0 \leq s \leq n-2$ and $|\mathcal{K}^*| = |\mathcal{K}|$.

Tools of the proof

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Define the (r, ℓ)-*shadow* of a family of r-element sets $\mathcal{A} \subseteq \binom{[n]}{r}$ for $\ell < r$:

$$\sigma_{r,\ell}(\mathcal{A}) = \{H : \ |H| = \ell, \exists A \in \mathcal{A} \text{ such that } H \subset A\}. \tag{6}$$

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Theorem (Shadow Theorem, Kruskal, Katona)

If $\mathcal{A} \subseteq \binom{[n]}{r}, |\mathcal{A}| = m$ then $|\sigma_{r,\ell}(\mathcal{A})|$ is at least as large as the (r,ℓ)-shadow of the family of the lexicographically first m members of $\binom{[n]}{r}$, that is, the size of the (r,ℓ)-shadow attains its minimum for the lexicographically first r-element sets.

Tools of the proof II

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For $\mathcal{A} \subseteq 2^{[n]}$ let

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Lemma

Let $\mathcal{M}$ be a non-empty inclusion-free family of subsets of $[n]$ with fixed $|\mathcal{M}| \geq n$. Then $|\mathcal{D}(\mathcal{M})|$ attains its minimum for a family satisfying the following conditions with some $2 \leq r \leq n$.

$$p_n = \ldots = p_{r+1} = p_{r-2} = \ldots = p_1 = p_0 = 0, \tag{8}$$

$$\mathcal{M}_r \text{ consists of the lexicographically first } p_r \ r-\text{element subsets,} \tag{9}$$

$$\mathcal{M}_{r-1} = \binom{[n]}{r-1} \setminus \sigma_{r,r-1}(\mathcal{M}_r). \tag{10}$$

4.2 Uniform minimal key system

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The family of closed sets $\mathcal{F}$ satisfies the following conditions.

$$\text{If } F \supseteq \mathcal{R} \setminus A \text{ then } F \in \mathcal{F}, \tag{11}$$

$$\text{if } F \supseteq A, F \neq R \text{ then } F \notin \mathcal{F}, \tag{12}$$

$$\emptyset \in \mathcal{F}. \tag{13}$$

Why not 1-element keys?

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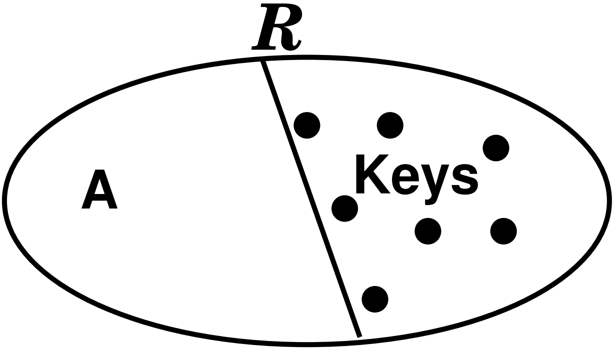
If all keys are one-element sets, then $\mathcal{K}^{-1}$ consists of a single set A, thus $\mathcal{K}^{-1}\downarrow$ consists of all subsets of A and $\mathcal{R}$, while $\mathcal{K}_{\cap}^{-1}$ consists of three sets, $\emptyset$, A and $\mathcal{R}$, i.e. the diameter is $2^{|A|} - 2$.



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$$s_2(e) = \max_{\{G=([n],E):\ |E|=e\}} \mathrm{diam} S_2(G). \tag{14}$$

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Theorem

If $e=\binom{t}{2}+r$*, where* $0<r\leq t$*, then*

$$\operatorname{diam}S_2(G)\leq\begin{cases}2^t+2^r-4 \ \textit{if}\ r<t\\ 2^{t+1}-2\ \textit{if}\ r=t\end{cases} \tag{15}$$

for a graph G whose connected components are isolated vertices except for one component. Furthermore, this bound is sharp.

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Follows from the following theorem of Erdős.

Theorem (Erdős, 1962)

Let $G = (V, E)$ be a connected graph of e edges. Assume, that $e = \binom{t}{2} + r$, where $0 < r \leq t$. Then the number of complete k-subgraphs $C_k(G)$ of G is at most

$$C_k(G) \leq \binom{t}{k} + \binom{r}{k-1}. \tag{17}$$

r-uniform key system

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Let D be a closure whose minimal keys have exactly $r(\geq 2)$ elements. $H = ([n], \mathcal{E})$ be the hypergraph where $\mathcal{K} = \binom{[n]}{r} \setminus \mathcal{E}$, with $|\mathcal{E}| = e$. The set of closures having $\binom{[n]}{r} \setminus \mathcal{E}$ as the set of minimal keys will be denoted by $S_r(H)$.

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Theorem

If $e \leq \binom{a}{r}$ *then* $\mathrm{diam}(S_r(H)) \leq 2^a + e2^r$.

Proof

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$\mathcal{K} = \binom{[n]}{r} \setminus \mathcal{E}$ implies that
$\mathcal{K}^{-1} = \{B \subset [n] \colon \binom{B}{r} \subset \mathcal{E} \wedge \forall B' \supsetneq B \ \binom{B'}{r} \setminus \mathcal{E} \neq \emptyset\}$. These are called the *(hyper)cliques* of H.

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$r < i$. Let $A_1, \ldots, A_m$ be the family of i-element subsets, whose all r-element subsets are in $\mathcal{E}$. If $m > \binom{a}{i}$ then by the Shadow Theorem (Lovász' version) the number of r-element subsets (hyperedges) is $> \binom{a}{r} \geq e$, so $m \leq \binom{a}{i}$.

Proof sketch

$\mathcal{K} = \binom{[n]}{r} \setminus \mathcal{E}$ implies that
$\mathcal{K}^{-1} = \{B \subset [n]\colon \binom{B}{r} \subset \mathcal{E} \land \forall B' \supsetneq B \; \binom{B'}{r} \setminus \mathcal{E} \neq \emptyset\}$. These are called the *(hyper)cliques* of H.
Need: the number of sets of the vertices of H which are subsets of at least one hyperclique and are not intersections of those is at most $2^a + e2^r$. $|\mathcal{K}^{-1}\!\downarrow| - |\mathcal{K}_{\cap}^{-1}| \le 2^a + e2^r$.
We show that $|\mathcal{K}^{-1}\!\downarrow| \le 2^a + e2^r$.
For $0 < i \le r$ let I be an i-element subset of a hyperclique.
$\Rightarrow \exists |R| = r\colon I \subseteq R \in \mathcal{E}$. number of such I's is at most $e\binom{r}{i}$.
$r < i$. Let $A_1, \ldots, A_m$ be the family of i-element subsets, whose all r-element subsets are in $\mathcal{E}$. If $m > \binom{a}{i}$ then by the Shadow Theorem (Lovász' version) the number of r-element subsets (hyperedges) is $> \binom{a}{r} \ge e$, so $m \le \binom{a}{i}$.
Add up these maximums:

$$e \sum_{i=1}^{r} \binom{r}{i} + \sum_{i=r+1}^{a} \binom{a}{i} \le 2^a + e2^r.$$

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We have given good upper bounds in the case of ordinary graphs.