LATTICES AND ISLANDS

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A rectangular $m \times n$ board consisting of square cells is given, with a positive integer (height) associated to each cell. A rectangular island is a rectangle in the board such that the height of its cells are greater than the heights of all neighboring cells.

The notion of an island comes from information theory. The characterization of the lexicographical length sequences of binary maximal instantaneous codes in [6] uses the notion of *full segments*, which are one-dimensional islands. Several generalizations of this notion gave interesting combinatorial problems. In two dimensions, Czédli [3] has determined the maximum number of rectangular islands; for the maximum number of rectangular islands on the rectangular board of size $m \times n$ he obtained $f(m,n) = \lfloor (mn + m + n - 1)/2 \rfloor$. Pluhár [17] gave upper and lower bounds in higher dimensions. E.K. Horváth, Z. Németh and G. Pluhár determined upper and lower bounds for the maximum number of triangular islands on a triangular grid in [7]. Some further results on triangular islands can be found in [12]. The number of square islands is a similar problem to the triangular case and it is treated in [8] and in [13]. Some proving methods for the maximum number of islands as well as exact formulas for some further island-problems are summarized in [1]. The problem of minimum cardinality of maximal systems of rectangular islands is treated in [11]. The investigations on islands motivated further research on independence properties in lattices, see [4] and [5].

We investigate the "horizontal cuts" of the height function and the connections of the obtained properties with the rectangular islands (and island-systems) determined by the given height function. Using the cut technics of the height function, we show that the minimum cardinality of maximal systems of rectangular islands – determined by Lengvárszky in [11] – is equal to the maximum number of different cuts of a standard rectangular height function. In addition, the realizing standard rectangular height functions have the same islands as those height functions that realize the minimum cardinality of maximal systems of rectangular islands. We show also that if the height function gives maximally many islands i.e., if $f(m,n) = \left\lfloor \frac{mn+m+n-1}{2} \right\rfloor$ as shown by Czédli, then the number of essentially different cuts is at least $\lceil log_2(m+1) \rceil + \lceil log_2(n+1) \rceil - 1$ and it is at most $\left\lfloor \frac{(m+n+3)}{2} \right\rfloor$.

We compare combinatorial and lattice theoretical proofs for some underlying lemmas.

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