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On Cyclic Packing of a Tree

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Abstract. We prove that there exists a packing of $\lfloor n/2 \rfloor$ copies of a tree of size $\lceil n/2 \rceil$ into K_n. Moreover, the proof provides an easy algorithm.

Key words. Packing of graphs, Distinct lengths labelling

1. Terminology

Let G be a finite, simple graph. We will denote the order and the size of G by $|G|$ and $e(G)$, respectively. In a graph G a vertex of degree one will be called an *end-vertex*. An end-vertex in a tree is *a leaf*.

Suppose $G_1, \ldots, G_k$ are graphs of order n. We say that there is a *packing* of $G_1, \ldots, G_k$ (into the complete graph K_n) if there exist injections $\alpha_i : V(G_i) \to V(K_n), \tilde{}\, i = 1, \ldots, k$, such that $\alpha_i^*(E(G_i)) \cap \alpha_j^*(E(G_j)) = \emptyset$ for $i \neq j$, where the map $\alpha_i^* : E(G_i) \to E(K_n)$ is the one induced by α_i.

We use the following terminology: A packing of k copies of a graph G will be called a *cyclic packing* of G if there exists a permutation σ on $V(G)$ such that the graphs G, $\sigma(G)$, $\sigma^2(G)$,..., $\sigma^{k-1}(G)$ are pairwise disjoint *i.e.* they form a k-placement of G.

The main references of this paper and other packing problems are the last chapter of Bollobás's book [1], the 4th Chapter of Yap's book [10] and the survey papers [11] and [8].

2. Conjectures and Results

The main motivation of the paper is the following well-known conjecture of Bollobás and Eldridge ([1]).

Conjecture 1. *Let $G_1, \ldots, G_k$ be k graphs of order n. If $|E(G_i)| \leq n - k, i = 1, \ldots, k$, then $G_1, \ldots, G_k$ are packable into K_n.*

The case $k = 2$ (which was the origin of the conjecture) was proved by Sauer and Spencer in 1978 in [5]. The case $k = 3$ was proved recently in [3].

There are also some other results that are related to the special cases of the above conjecture. For instance, instead of k graphs we can consider k copies of the same graph or, instead of general graphs we can consider trees. The paper [9] contains a result which implies that if $e(G) \leq n - 3$ with $|G| = n$, then there exists a 3-placement of G into K_n. The analogous result for three copies of a tree (of the size $n - 2$) can be deduced from a result proved in [6]. It was shown in [7] that the above mentioned placements of two or three copies can be obtained as a cyclic packing.

The aim of this paper is to consider another special case of the Bollobás and Eldridge conjecture. First of all we put $k = \lfloor n/2 \rfloor$. Observe that in this case the total number of edges we pack into K_n is maximum (with respect to the Conjecture of Bolobás and Eldridge). Next, because of the methods we use, we consider the case of the packing of k copies of a tree. On the other hand we obtain something more than the existence of the packing. In particular, an algorithm can be easily obtained from the proof of the theorem. The main result of the paper can be formulated as follows.

Theorem 2. *Let T be a tree of size $\lceil n/2 \rceil$. Then there exists a cyclic packing of $\lfloor n/2 \rfloor$ copies of T into K_n.*

Graph labellings are well-known and used in decomposition problems such that, for instance, the conjecture of Ringel that the complete graph K_{2k+1} can be decomposed into $2k + 1$ subgraphs that are all isomorphic to a given tree with k edges (see e.g. [2])

The main tool in the proof of the above theorem is the use of a labelling we call *distinct length labelling* (DLL).

We introduce some additional terminology. Let K_k be a complete graph with vertex set $\{x_1, x_2, ..., x_k\}$. Let G be a graph of order not greater than k. *A distinct length labelling* of a graph G in K_k (shortly: DL labelling or DLL) is an injection f from the vertices of G to the set $\{1, 2, ..., k\}$ such that, when each edge uv is assigned the label $\min\{|f(u) - f(v)|, |f(v) - f(u)|\}$ modulo k, the resulting edge labels (called: *lengths*) are distinct. Moreover, if k is even we assume that the label $k/2$ does not occur (for, in this case, there are only $k/2$ edges of this length). Thus, there are exactly $\lfloor \frac{k-1}{2} \rfloor$ *possible* lengths. If we draw G in such a way that the vertex labelled i is identified with x_i, then the label of an edge is the distance between its ends on the cycle generated by $\{x_1, x_2, ..., x_k\}$. We shall assume that such an identification has been made. Let $\sigma = (x_1 x_2 ... x_k)$ be a cyclic permutation. It is easy to see that the image of an edge e has the same length as e. So, if G has a DL labelling in K_k, then the permutation σ defines a cyclic packing of k copies of G into K_k.

Remark. Observe that a DL labelling in K_{2k+1} of a tree of size k would imply the Ringel conjecture. A tree of size k with a DL labelling using only $k + 1$ labels $\{1, 2, ..., k + 1\}$ is said to be *graceful*. The well-known Ringel-Kotzig Conjecture (Graceful Tree Conjecture) says that all trees are graceful (see [2]).

Let now $K_{k,k}$ be a complete bipartite graph with vertex set partition $L = \{x_1, x_2, ..., x_k\}$ and $R = \{y_1, y_2, ..., y_k\}$. Let $e = x_i y_j$ be an edge of $K_{k,k}$. The *length* of e is given by $j - i$ modulo k. Let G be a bipartite graph of size not greater than k. *A distinct length labelling* of G in $K_{k,k}$ is an injection f from the vertices of G to the set $\{L, R\} \times \{1, 2, ..., k\}$ such that: 1. for each edge uv the first elements assigned to u and v are distinct i.e. uv can be considered as an edge of $K_{k,k}$, 2. the lengths of all edges are distinct. Let $\sigma = (x_1 x_2 ... x_k)(y_1 y_2 ... y_k)$ be a permutation on vertex set of $K_{k,k}$ having two cycles. It is easy to see that the image of an edge e has the same length as e. So, if G has a DL labelling in $K_{k,k}$, then the permutation σ defines a cyclic packing of k copies of G into $K_{k,k}$. Observe that in this case there are exactly k *admissible* lengths.

Remark. A DL labelling of a tree of size k in $K_{k,k}$ has been considered by Ringel, Llado and Serra [4] as *bigraceful labelling*. They conjectured that all trees have bigraceful labellings, which would imply that $K_{k,k}$ is decomposable into k copies of any given tree with k edges.

Let now K_{2k} be a complete graph on $2k$ vertices. We partition the vertex set of K_{2k} into two parts $L = \{x_1, x_2, ..., x_k\}$ and $R = \{y_1, y_2, ..., y_k\}$ and treat K_{2k} as a join $K_k * K_k$ of two disjoint cliques K_k.

A distinct length labelling of G into $K_k * K_k$ is an injection f from the vertices of G to the set $\{L, R\} \times \{1, 2, ..., k\}$ such that: 1. f can be considered (in a canonical way) as a DLL in K_k for the subgraph of G induced by the edges having both ends labelled by the pairs with the same first element. 2. f can be considered as a DLL in $K_{k,k}$ for the subgraph of G induced by the edges having the ends labelled by the pairs with distinct first elements.

We shall consider G as a subgraph of K_{2k} and identify the vertices labelled by (L, i) with x_i and the vertices labelled by (R, i) with y_i. This will allow us to use 'geometric' terminology such as, for instance, *crossing edge*. As above, it is easy to see that the permutation $\sigma = (x_1 x_2 ... x_k)(y_1 y_2 ... y_k)$ define a cyclic packing of k copies od G into K_{2k}.

3. Some Lemmas

Let us start with an observation how to partition trees.

Lemma 3. *Any tree of order n has a vertex u such that every component of $T - u$ has order at most $\lfloor n/2 \rfloor$.*

Proof. Let u be a vertex minimizing the order of a largest component of $T - u$. Assume that there is a component T' of $T - u$ of order $> n/2$. Then for the neighbour v of u in T', the component containing u has order $\leq n/2$, and the other components have order $< |T'|$ contradicting the choice of u. $\square$

Lemma 4. *Let T be a tree with $e(T) \leq \lfloor \frac{k+1}{3} \rfloor$. Then there exists a DLL of T in K_k.*

Proof. In order to label T we shall use an algorithm which can be roughly described as follows. If $E(T) = \{e\}$ and $e = xy$, then we label x arbitrarily and label

y in such a way that e get one of admissible lengths. If $e(T) > 1$, without loss of generality we may suppose that $T = T' + e$ where T' has a DLL f and $e = xy$ with x already labelled. Suppose that $f(x) = i$. A length l is said to be *impossible* if either l already occurs within the lengths of $E(T')$ or both labels $i + l$ and $i - l$ are already used for $V(T')$. If the number of impossible lengths is smaller than the number of admissible length, then we attribute to y a label such that e get possible length. This *a greedy* algorithm is a very well known procedure.

Suppose now that we have labelled a subtree T' of T with $e(T') = m'$ and let $e = xy$ be an edge of T such that $x \in V(T')$ and $y \in V(T) - V(T')$. We may suppose that $m' \leq \lfloor \frac{k+1}{3} \rfloor - 1$, otherwise $T' = T$ and we are done. Let $d_{T'}(x) = d$ the degree of x in T'. There are m' lengths used for m' edges of T'. Moreover, there are $m' - d$ vertices not adjacent to x. They do not allow to use $\lfloor \frac{m'-d}{2} \rfloor$ lengths. Therefore, the number of impossible lengths is at most $m' + \lfloor \frac{m'-d}{2} \rfloor$. So, the labelling of y is possible if

$$(*) \qquad \lfloor \frac{k-1}{2} \rfloor > m' + \lfloor \frac{m'-d}{2} \rfloor$$

Since $d \geq 1$ and $3m' \leq k - 2$, we have

$$m' + \lfloor \frac{m'-d}{2} \rfloor \leq m' + \frac{m'-d}{2} \leq m' + \frac{m'-1}{2} \leq \frac{3m'-1}{2} < \frac{k-3}{2}.$$

Thus, the inequality $(*)$ holds. □

We shall need the following forest version of the above lemma.

Corollary 5. *Let F be a forest having q components. If $e(F) \leq \lfloor \frac{k+1}{3} \rfloor - q + 1$, then there exists a DLL of F in K_k.*

Proof. By adding $q - 1$ edges to F we can get a tree T satisfying the assumptions of the above lemma. It suffices to observe that a DLL for T is also a DLL for F. □

Lemma 6. *Let T be a tree of size $p \leq \lfloor \frac{k-1}{2} \rfloor$. If the tree T' obtained from T by removing some of its leaves is graceful, then T has a DL labelling in K_k.*

Proof. Let K_k be a complete graph with vertex set $\{x_1, x_2, ..., x_k\}$ and let T' be a tree obtained from T by removing r of its leaves. Let f be a graceful labelling of T' with the labels $\{1, 2, ..., q + 1\}$; $q = p - r$. As usual we identify the vertices labelled by i with x_i. Denote by $x_{i_1}, ..., x_{i_m}$ the vertices adjacent to removed leaves. We have $\sum_{j=1}^{m} r_{i_j} = r$ where r_{i_j} is the number of removed vertices adjacent to x_{i_j}. We label the removed leaves in the following way: the leaves adjacent to x_{i_1} we label by $i_1 + q + 1$, $i_1 + q + 2$, ..., $i_1 + q + r_1$, the leaves adjacent to x_{i_2} by $i_2 + q + r_1 + 1$, $i_2 + q + r_1 + 2$, ..., $i_2 + q + r_1 + r_2$ and so on. So, the removed edges incident with x_{i_1} get the lengths $q + 1$, $q + 2$, ..., $q+r_1$, the edges incident with x_{i_2} get the lengths $q + r_1 + 1$, $q + r_1 + 2$,..., $q + r_1 + r_2$ and so on. It is easy to see that such a labelling is possible if the largest label does not exceed k. But the last label we use is $i_m + q + r$. We have $i_m \leq q + 1 \leq p + 1$. Thus

$$i_m + q + r \leq p + 1 + p \leq 2p + 1 \leq 2\lfloor\frac{k-1}{2}\rfloor + 1 \leq k$$

and the lemma follows. □

Corollary 7. *Let T be a tree of diameter at most six. If $e(T) < k/2$, then T has a DL labelling in K_k.*

Proof. Deleting all leaves of T we get a tree of diameter at most four. It is known (see [2]) that such a tree is graceful. Therefore, T satisfies the assumptions of Lemma 6 which completes the proof. □

Lemma 8. *Let T be a tree of size at most k. Assume that T containsa subtree T_0 such that: a) There exists a DLL f of T_0 in $K_k * K_k$. b) One of the subgraphs of T_0 induced L or R has* $\lfloor\frac{k-1}{2}\rfloor$ edges. *c) There is at least one edge in the other side. Then T has a DLL in $K_k * K_k$. The DL labelling for T can be obtained from f in greedy fashion.*

Proof. Let T' be a maximal subtree of T containing T_0 and having a DLL obtained from f in greedy fashion. Without loss of generality we may suppose that the right-hand side contains $\lfloor\frac{k-1}{2}\rfloor$ edges of T_0 and the left-hand side contains at least one edge. That means that no edge can be added to the right-hand side. Let $e = xy$ be an edge of T such that $x \in V(T')$ and $y \in V(T) - V(T')$ and let $p_1, p_2, \bar{p}$ denote the number of edges of T' in L. R and between L and R, respectively. Let n_1 and n_2 denote the number of vertices of T' different from x in L and R, respectively. We have $p = p_1 + p_2 + \bar{p}$, $p = n_1 + n_2$. We have to show that if $p \leq k - 1$ then it is possible to label y.

As in the previous lemma we shall consider two cases.

Case 1. $x \in L$.

Then, as above, the possibility to label y would be implied by

$$k + \lfloor\frac{k-1}{2}\rfloor + p_2 + \lceil\frac{n_1}{2}\rceil > 2p,$$

which is satisfied since $k + \lfloor\frac{k-1}{2}\rfloor + p_2 = k + \lfloor\frac{k-1}{2}\rfloor + \lfloor\frac{k-1}{2}\rfloor + 1 \geq 2k - 1$ and $p \leq k - 1$.

Case 2. $x \in R$.

As remarked above, y cannot get a label of the form (R, i). So, in this case we examine only the possibility tofind for y the label with respect to DLL in $K_{k,k}$. There are k admissible lengths; $\bar{p}$ of them are impossible because of the lengths used by $\bar{p}$ crossing edges of T' and n_1 of them are impossible because of n_1 vertices in L. Finally, the labelling of y is possible if

$$k > \bar{p} + n_1 = p - p_1 - p_2 + p - n_2 = 2p - (p_1 + p_2 + n_2).$$

This is equivalent to

$$k + (p_1 + p_2 + n_2) > 2p.$$

By assumption c) we have $p_1 \geq 1$. Since the subgraph of the tree T' induced by R is a forest we have $n_2 \geq p_2$. On the other hand $p \leq k-1$. So, the above inequality holds which finishes the proof of the lemma. □

4. Proof of Theorem 2

Proof of the case of n even. Let n be an even integer. Put $n = 2k$. The graph K_n will be considered as the join of two complete graphs $K_{\frac{n}{2}}$. One of these graphs will be called *left* (denoted by L), and the other *right* (denoted by R). The same notation will be used also for the corresponding vertex-sets. Let T be a tree of size k. Choose $u \in V(T)$ such that each component of $T-u$ has order $\leq k/2$ which can be done by Lemma 3. Let $F_1, \ldots, F_s$ be the components of $T-u$ and $|F_1| \leq \ldots \leq |F_s|$. Let i^* be such that $\sum_{i<i^*} |F_i| \leq k/2$ and $\sum_{i \geq i^*} > k/2$.

If $|F_{i^*}| \leq 3$ then the subtree consisting of $F_1 \cup \ldots \cup F_{i^*}$and u has diameter at most 6, and therefore T has a DLL by Corollary 7 and Lemma 8, and we are done.

So we may assume $|F_{i^*}| \geq 4$. We define two subtrees T_1 and T_2intersecting in u in the following way:

(i) If $\sum_{i<i^*} |F_i| < k/3$ then T_1 is the subtree induced by$F_1, \ldots, F_{i^*}$ and u and T_2 is the subtree induced by$F_{i^*+1}, \ldots, F_s$ and u.
(ii) If $\sum_{i<i^*} |F_i| \geq k/3$ then T_1 is the subtree inducedby $F_1, \ldots, F_{i^*-1}$ and u and T_2 is the subtree induced by $F_{i^*}, \ldots, F_s$ and u.

Let q denote the number of components of $T_2 - u$. We have

$$q \leq k/6.$$

Indeed, in case (i) we have $1 \leq q \leq 2$ and $k > 4(q+1)$, i.e. $k/6 > 2(q+1)/3 \geq q$. In case (ii) we have $4q \leq e(T_2) \leq 2k/3$. Moreover, we have

$$e(T_2) \geq \frac{k}{3} - 1 + q.$$

In case (i) this is clear for $q = 1$ and for $q = 2$ we have $e(T_2) \geq 4k/9 > k/3 + 1$ since in this case $k > 12$. In case (ii) we get $e(T_2) \geq k/2 \geq k/3 + q$ since $q \leq k/6$.

It remains to show how to find a DL labelling of T. Denote by t the index of the last component in T_1. In each component F_i, $t+1 \leq i \leq t+q$, the vertices $u_1, \ldots, u_q$ are the neighbours of u. The following procedure will give the DL labelling.

Step 1. First, we label in L a subtree of T_1 containing u and having exactly $\lfloor \frac{k+1}{3} \rfloor + 1$ vertices.

Step 2. Next, we label in R for each i, $t+1 \leq i \leq s$, a **proper** subtree of F_i containing u_i such that the total number of labelled vertices is $\lceil \frac{k}{3} \rceil$.

Step 3. For each component F_i, $i \geq t+1$, we label an unlabelled neighbour of a labelled vertex in L.

Step 4. We extend the partial labelling by labelling an unlabelled neighbour of a labelled vertex in each step until all vertices are labelled.

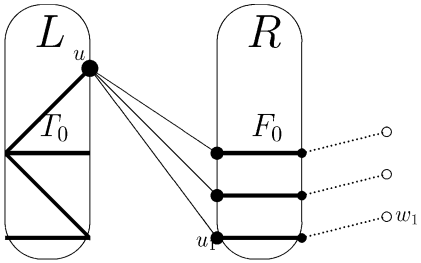

Fig. 1. Just before Step 3. T_0 is a subtree of T_1 induced by the vertices labelled in Step 1, F_0 is the forest induced by the vertices labelled in Step 2, and by w_i we denote an unlabelled neighbour of a labelled vertex of the component F_i

The structure of the tree after Step 2 is illustrated in Fig. 1.

We shall prove now the validity of the above procedure, i.e. we show that it is possible to find an admissible label in each step. Step 1 is possible by Lemma 4.

The possibility of Step 2 follows from Corollary 5 and from $e(T_2) \geq k/3$.

Step 3 can be justified as follows. Let r be the number of vertices from components F_i already labelled in L. Suppose that $r \leq q - 1$. The next vertex of a component F_i can be labelled in L if the number of admissible lengths between L and R is greater than the number of impossible lengths. There are $q + r$ crossing lengths already used and at most $r + (k+1)/3$ vertices in L. Since $q + 2r + (k+1)/3 \leq 3q - 2 + (k+1)/3 < k$, the next vertex of F_i can indeed be labelled in L.

In order to justify Step 4 we distinguish two cases. Suppose that the next vertex to label is y with x being the neighbour that is already labelled. If $x \in L$ we shall show that y can be labelled in R. We get

$$\bar{p} + n_2 = p - p_1 - p_2 + p - n_1 = 2p - (p_1 + p_2 + n_1).$$

Since $p_1 + p_2 + n_1 \geq \frac{k-1}{3} + \frac{k}{3} - q + \frac{k-1}{3} + q > k - 1 \geq p$ we conclude $\bar{p} + n_2 < k$, and hence there is an admissable label for x in R.

For $x \in R$ we have

$$p_2 + \bar{p} + \lfloor \frac{n_2}{2} \rfloor + n_1 = p - p_1 + p - \lceil \frac{n_2}{2} \rceil = 2p - (p_1 + \lceil \frac{n_2}{2} \rceil).$$

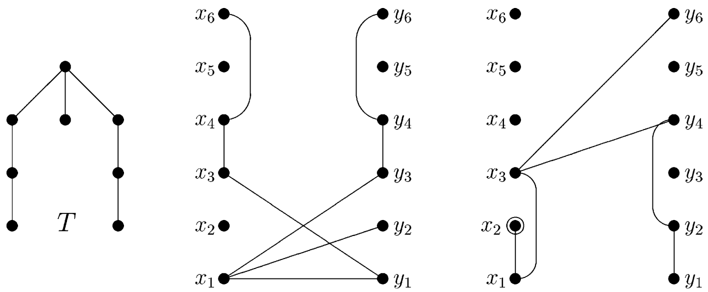

Fig. 2. A tree T of size seven, its cyclic packing in K_{12} and a cyclic packing of a subtree that cannot be extended in a greedy way

Since $p_1 + \lceil \frac{n_2}{2} \rceil \geq \frac{k-1}{3} + \frac{k}{6} > \frac{k-1}{2}$ we get $p_2 + \bar{p} + \lfloor \frac{n_2}{2} \rfloor + n_1 < \frac{3k-1}{2}$ and hence we find an admissible label for x as well. This finishes the proof of the case of n even.

Proof of the case of n odd. Let T be a tree of size $\lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor + 1$ and let z be a leaf of T. Observe that the tree $T' - z$ satisfies the assumptions of Theorem 2 with respect to K_{n-1}. Denote by σ' the cyclic packing of $\lfloor \frac{n}{2} \rfloor$ copies of T' into K_{n-1}. Since, by the proof, σ' has two cycles of length $\lfloor \frac{n}{2} \rfloor$, the permutation σ obtained by adding to σ' the fixed vertex z is a cyclic packing $\lfloor \frac{n}{2} \rfloor$ copies of T into K_n.

5. Remarks

The simple idea used in the proof of the case of odd n can be generalized. It is easy to see that the following theorem holds.

Theorem 9. *Let T be a tree of size m and having a leaves. Then there exists a cyclic packing of $m - a$ copies of T into K_{2m-a}. In particular, for even n there is a cyclic packing of $\frac{n}{2} - 1$ copies of a tree of size $\frac{n}{2} + 1$.*

Theorem 2 could be considerably improved if for instance the Graceful Tree Conjecture or Ringel-Llado-Serra Conjecture were true. In the first case we could have more copies or edges and in the second case we would have only crossing edges. However, a DL labelling used in the proof of Theorem 2 has been obtained in almost greedy fashion, and,from this point of view Theorem 2 cannot be improved as it is shown in Fig. 2.

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References

1. Bollobás, B.: Extremal Graph Theory, London: Academic Press 1978
2. Gallian, J.A.: A dynamic survey on graph labelling, www.combinatorics.org/
3. Kheddouci, H., Marshall, S., Saclé, J.-F., Woźniak, M.: On the packing of three graphs. Discrete Math. **236**, 197–225 (2001)
4. Ringel, G., Llado, A., Serra, O.: Decomposition of complete bipartite graphs into trees. DMAT Research Report Univ. Politecnica de Catalunya 11 (1996)
5. Sauer, N., Spencer, J.: Edge disjoint placement of graphs. J. Comb. Theory, Ser. B **25**, 295–302 (1978)
6. Wang, H., Sauer, N.: Packing three copies of a tree into a complete graph. Eur. J. Comb. **14**, 137–142 (1993)
7. Woźniak, M.: Packing three trees. Discrete Math. **150**, 393–402 (1996)
8. Woźniak, M.: Packing of Graphs. Diss. Math. **362**, pp.78 (1997)
9. Woźniak, M., Wojda, A.P.: Triple placement of graphs. Graphs Comb. **9**, 85–91 (1993)
10. Yap, H.P.: Some Topics in Graph Theory. Lond. Math. Soc, Lect Notes Ser, 108, Cambridge: Cambridge University Press 1986
11. Yap, H.P.: Packing of graphs — a survey. Discrete Math. **72**, 395–404 (1988)

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