Path decompositions and Gallai’s conjecture

Genghua Fan

Department of Mathematics, Fuzhou University, Fuzhou, Fujian 350002, China

Received 23 August 2002
Available online 11 November 2004

Abstract

Let $G$ be a connected simple graph on $n$ vertices. Gallai’s conjecture asserts that the edges of $G$ can be decomposed into $\lceil \frac{n}{2} \rceil$ paths. Let $H$ be the subgraph induced by the vertices of even degree in $G$. Lovász showed that the conjecture is true if $H$ contains at most one vertex. Extending Lovász’s result, Pyber proved that the conjecture is true if $H$ is a forest. A forest can be regarded as a graph in which each block is an isolated vertex or a single edge (and so each block has maximum degree at most 1). In this paper, we show that the conjecture is true if $H$ can be obtained from the emptyset by a series of so-defined $\alpha$-operations. As a corollary, the conjecture is true if each block of $H$ is a triangle-free graph of maximum degree at most 3.

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Keywords: Path; Decomposition; Gallai’s conjecture

1. Introduction

The graphs considered here are finite, undirected, and simple (no loops or multiple edges). A graph is triangle-free if it contains no triangle. A cut vertex is a vertex whose removal increases the number of components. A connected graph is nonseparable if it has no cut vertex. A block of a graph $G$ is a maximum nonseparable subgraph of $G$. The sets of vertices and edges of $G$ are denoted by $V(G)$ and $E(G)$, respectively. The edge with ends $x$ and $y$ is denoted by $xy$. If $xy \in E(G)$, we say that $xy$ is incident with $x$ and $y$ is a neighbor of $x$. For a subgraph $H$ of $G$, $N_H(x)$ is the set of the neighbors of $x$ which are in $H$, and $d_H(x) = |N_H(x)|$ is the degree of $x$ in $H$. If $B \subseteq E(G)$, then $G \setminus B$ is the graph obtained from $G$ by deleting all the edges of $B$. Let $S \subseteq V(G)$. $G - S$ denotes the graph obtained from $G$ by deleting all the vertices of $S$ together with all the edges with at least one end.

E-mail address: fan@fzu.edu.cn (G. Fan).

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in $S$. (When $S = \{x\}$, we simplify this notation to $G - x$.) We say that $H$ is the subgraph induced by $S$ if $V(H) = S$ and $xy \in E(H)$ if and only if $xy \in E(G)$; alternatively, $H = G - (V(G) \setminus S)$. ($S$ is called an independent set if $E(H) = \emptyset$.) The E-subgraph of $G$ is the subgraph induced by the vertices of even degree in $G$.

A path-decomposition of a graph $G$ is a set $\{P_1, P_2, \ldots, P_k\}$ of paths such that $E(G) = \bigcup_{i=1}^{k} E(P_i)$ and $E(P_i) \cap E(P_j) = \emptyset$ if $i \neq j$. We say that $G$ is decomposed into $k$ paths if $G$ has a path-decomposition $\mathcal{D}$ with $|\mathcal{D}| = k$. A trivial path is one that consists of a single vertex. By the use of trivial paths, if a graph is decomposed into at most $k$ paths, then it can be decomposed into exactly $k$ paths.

Erdős asked what is the minimum number of paths into which every connected graph on $n$ vertices can be decomposed. Gallai conjectured that this number is $\lceil \frac{n}{2} \rceil$. (See [4].)

**Gallai’s conjecture.** If $G$ is a connected graph on $n$ vertices, then $G$ can be decomposed into $\lceil \frac{n}{2} \rceil$ paths.

Toward a proof of the conjecture, Lovász [4] made the first significant contribution by showing that a graph $G$ on $n$ vertices (not necessary to be connected) can be decomposed into $\lceil \frac{n}{2} \rceil$ paths and circuits. Based on Lovász’s result, Donald [2] showed that $G$ can be decomposed into $\lceil \frac{3}{4}n \rceil$ paths, which was improved to $\lceil \frac{3}{4}n \rceil$ independently by Dean and Kouider [1] and Yan [7]. (An informative survey of the related topics was given by Pyber [5].) As a consequence of Lovász’s theorem, $G$ can be decomposed into $\lceil \frac{n}{2} \rceil$ paths if $G$ has at most one vertex of even degree, that is, if the $E$-subgraph of $G$ contains at most one vertex. Pyber [6] strengthened this result by showing that $G$ can be decomposed into $\lceil \frac{n}{2} \rceil$ paths if the $E$-subgraph of $G$ is a forest. A forest can be regarded as a graph in which each block is an isolated vertex or a single edge. Thus, each block of a forest has maximum degree at most 1. In this paper, we show that a graph $G$ on $n$ vertices (not necessary to be connected) can be decomposed into $\lceil \frac{n}{2} \rceil$ paths if each block of the $E$-subgraph of $G$ is a triangle-free graph of maximum degree at most 3. Here, the requirement of triangle-free cannot be dropped. Consider a graph $G$ consisting of $3k$ vertex-disjoint triangles. So $|V(G)| = 3k$ and the $E$-subgraph of $G$ is $G$ itself. Since any path-decomposition of a triangle needs at least 2 paths, we see that any path-decomposition of $G$ needs at least $2k = \frac{3}{4}|V(G)|$ paths.

In the next section, we define a graph operation, called $\alpha$-operation. In Section 3, we use Lovász’s path sequence technique [4] to obtain some technical lemmas, and then, in the last section, prove a more general result: $G$ can be decomposed into $\lceil \frac{n}{2} \rceil$ paths if its $E$-subgraph can be obtained from the emptyset by a series of $\alpha$-operations.

### 2. $\alpha$-operations and $\alpha$-graphs

**Definition 2.1.** Let $H$ be a graph. A pair $(S, y)$, consisting of an independent set $S$ and a vertex $y \in S$, is called an $\alpha$-pair if the following holds: for every vertex $v \in S \setminus \{y\}$, if $d_H(v) \geq 2$, then (a) $d_H(u) \leq 3$ for all $u \in N_H(v)$ and (b) $d_H(u) = 3$ for at most two vertices $u \in N_H(v)$. (That is, all the neighbors of $v$ have degree at most 3, at most two of which has degree exactly 3.) An $\alpha$-operation on $H$ is either (i) add an isolated vertex or (ii) pick an $\alpha$-pair $(S, y)$ and add a vertex $x$ joined to each vertex of $S$, in which case the ordered triple $(x, S, y)$ is called the $\alpha$-triple of the $\alpha$-operation.
Definition 2.2. An $\alpha$-graph is a graph that can be obtained from the empty set via a sequence of $\alpha$-operations.

Let us define the empty set to be an $\alpha$-graph. Then, a graph on $n$ vertices is an $\alpha$-graph if and only if it can be obtained by an $\alpha$-operation on some $\alpha$-graph on $n - 1$ vertices, $n \geq 1$. It follows that if $G$ is an $\alpha$-graph on $n$ vertices, then the vertices of $G$ can be ordered as $x_1x_2 \ldots x_n$ such that if $G_i$ denotes the subgraph induced by $\{x_1, x_2, \ldots, x_i\}$, then $G_i$ is an $\alpha$-graph obtained by an $\alpha$-operation on $G_{i-1}$, where $1 \leq i \leq n$, $G_0 = \emptyset$, and $G_n = G$. Such an ordering $x_1x_2 \ldots x_n$ is called an $\alpha$-ordering of $G$. Alternatively, a graph $G$ is an $\alpha$-graph if and only if $V(G)$ has an $\alpha$-ordering. We note that by the definition, an $\alpha$-graph is triangle-free.

Let $G$ be an $\alpha$-graph and $H$ a subgraph of $G$. It is not difficult to see that the restriction of an $\alpha$-ordering of $V(G)$ to $V(H)$ is an $\alpha$-ordering of $V(H)$. This gives that

Proposition 2.3. Any subgraph of an $\alpha$-graph is an $\alpha$-graph.

A subdivision of a graph $G$ is a graph obtained from $G$ by replacing each edge of $G$ with a path (inserting vertices into edges of $G$).

Proposition 2.4. Any subdivision of an $\alpha$-graph is an $\alpha$-graph.

Proof. It suffices to show that if $H$ is a graph obtained from an $\alpha$-graph $G$ by replacing an edge with a path, then $H$ is an $\alpha$-graph. Suppose that $xy \in E(G)$ and $H$ is obtained from $G$ by replacing $xy$ with a path $xa_1a_2 \ldots a_ky$, where $k \geq 1$. We may suppose that $v_1v_2 \ldots xv_1 \ldots v_jy \ldots v_n$ is an $\alpha$-ordering of $V(G)$. Then, $v_1v_2 \ldots xv_1 \ldots v_ja_1a_2 \ldots a_ky \ldots v_n$ is an $\alpha$-ordering of $V(H)$, and thus $H$ is an $\alpha$-graph. □

Proposition 2.5. Forests are $\alpha$-graphs.

Proof. Let $F$ be a forest. If $E(F) = \emptyset$, then any ordering of $V(F)$ is an $\alpha$-ordering. Suppose therefore that $E(F) \neq \emptyset$. Since $F$ is a forest, there is $x \in V(F)$ such that $d_F(x) = 1$. Let $H = F - x$. Then $H$ is a forest. We may use induction on the number of vertices, and thus by the induction hypothesis, $H$ is an $\alpha$-graph. Let $y$ be the unique neighbor of $x$ in $F$. Then, $F$ is obtained from $H$ by adding $x$ joined to $y$, which is an $\alpha$-operation with $\alpha$-triple $(x, \{y\}, y)$. So $F$ is an $\alpha$-graph. □

Let $C$ be a circuit of length at least 4. Then $C$ can be obtained by adding a vertex joined to the nonadjacent ends of a path $P$ of length at least 2, which is an $\alpha$-operation on $P$. But, by Proposition 2.5, $P$ is an $\alpha$-graph, and hence $C$ is an $\alpha$-graph. In fact, we have the following stronger result.

Proposition 2.6. If each block of $G$ is a triangle-free graph of maximum degree at most 3, then $G$ is an $\alpha$-graph.

Proof. We use induction on $|V(G)|$. Clearly, the proposition holds if $|V(G)| = 1$. Suppose that $|V(G)| \geq 2$ and the proposition holds for all $G'$ with $|V(G')| < |V(G)|$. 
Let \( B \) be an end-block of \( G \). (An end-block is a block that contains at most one cut vertex.) If \( B = G \) (that is, if \( G \) is 2-connected), let \( b \) be any vertex of \( B \); otherwise, let \( b \) be the unique cut vertex contained in \( B \). Let \( x \) be a neighbor of \( b \) in \( B \) and we consider the neighbors of \( x \). Note that \( N_B(x) = N_G(x) \). Let \( S = N_G(x) \) and \( H = G - x \). Since \( B \) is triangle-free, we have that \( S \) is an independent set and thus \( b \) is not a neighbor of any vertex \( v \in S \setminus \{b\} \), and since \( B \) has maximum degree at most 3, \( d_H(u) \leq 3 \) for all \( u \in N_H(v) \). Again, since \( B \) has maximum degree at most 3, we have that \( |N_H(v)| \leq 2 \) and thus there are at most two \( u \in N_H(v) \) with \( d_H(u) = 3 \). So \( G \) is obtained by an \( x \)-operation on \( H \) with \( x \)-triple \((x, S, b)\). But, by the induction hypothesis, \( H \) is an \( x \)-graph, and so is \( G \). □

3. Technical lemmas

In this section, we use Lovász’s path sequence technique \([4]\) to prove some technical lemmas which are needed in the next section. First, we need some additional definitions.

**Definition 3.1.** Suppose that \( D \) is a path-decomposition of a graph \( G \). For a vertex \( v \in V(G) \), \( D(v) \) denotes the number of the nontrivial paths in \( D \) that have \( v \) as an end. (If \( x \) is a vertex of odd degree in \( G \), then \( D(x) \geq 1 \). This fact will be used frequently in the next section.)

**Definition 3.2.** Let \( a \) be a vertex in a graph \( G \) and let \( B \) be a set of edges incident with \( a \). Set \( H = G \setminus B \). Suppose that \( D \) is a path-decomposition of \( H \). For any \( A \subseteq B \), say \( \{ax_i: 1 \leq i \leq k\} \), we say that \( A \) is addible at \( a \) with respect to \( D \) if \( H \cup A \) has a path-decomposition \( D^* \) such that
\[(a) \quad |D^*| = |D|; \]
\[(b) \quad D^*(a) = D(a) + |A| \quad \text{and} \quad D^*(x_i) = D(x_i) - 1, 1 \leq i \leq k; \]
\[(c) \quad D^*(v) = D(v) \quad \text{for each} \quad v \in V(G) \setminus \{a, x_1, \ldots, x_k\}. \]
We call such \( D^* \) a transformation of \( D \) by adding \( A \) at \( a \) with respect to \( D \). When \( k = 1 \), we simply say that \( ax_1 \) is addible at \( a \) with respect to \( D \).

Lemmas 3.3 and 3.5 below are special cases of Lemmas 4.3 and 4.6 in [3], respectively, whose proofs are rather complicated. (A path decomposition is a special case of a path covering.) To be self-contained, we present proofs without referring to [3].

**Lemma 3.3.** Let \( a \) be a vertex in a graph \( G \) and let \( H = G \setminus \{ax_1, ax_2 \ldots, ax_s\} \), where \( x_i \in N_G(a) \). Suppose that \( D \) is a path-decomposition of \( H \). Then either
\[(i) \quad \text{there is} \quad x \in \{x_1, x_2, \ldots, x_s\} \quad \text{such that} \quad ax \quad \text{is addible at} \quad a \quad \text{with respect to} \quad D; \quad \text{or} \]
\[(ii) \quad \sum_{i=1}^s D(x_i) \leq |\{v \in N_H(a): D(v) = 0\}|. \]

**Proof.** Consider the following set of pairs:
\[ R = \{(x, P): x \in \{x_1, \ldots, x_s\} \quad \text{and} \quad P \quad \text{is a nontrivial path in} \quad D \quad \text{with end} \quad x\}. \]
We note that \( |R| = \sum_{i=1}^s D(x_i) \). For each pair \((x, P) \in R\), we associate \((x, P)\) with a sequence \(b_1 P_1 b_2 P_2 \ldots\) constructed as follows.
(1) $b_1 = x; P_1 = P$.
(2) Suppose that $P_i$ has been defined, $i \geq 1$. If $P_i$ does not contain $a$, then the sequence is finished at $P_i$; otherwise let $b_{i+1}$ be the vertex just before $a$ if one goes along $P_i$ starting at $b_i$.
(3) Suppose that $b_i$ has been defined, $i \geq 1$. If $D(b_i) = 0$, the sequence is finished at $b_i$; otherwise, let $P_i$ be a path in $D$ starting at $b_i$.
It is clear that $b_{i+1}$ is uniquely determined by the path $P_i$ (containing $b_{i+1}a$) and its end $b_i$. Such a pair $(P_i, b_i)$ is unique since there is only one path in $D$ that contains $b_{i+1}a$, and moreover, the two ends of the path are distinct. Thus, $b_i \neq b_j$ if $i \neq j$, and therefore, the sequence $b_1 P_1 b_2 P_2 \ldots$ is finite.

If the sequence is finished at a path $P_i$ (2), let $P'_i = (P_i \setminus \{b_{i+1}a\}) \cup \{b_i a\}$, $1 \leq i \leq t - 1$, and $P' = P_i \cup \{b_i a\}$. Then $D^* = (D \setminus \{P_1, P_2, \ldots, P_t\}) \cup \{P'_1, P'_2, \ldots, P'_t\}$ is a path-decomposition of $H \cup \{ax\}$ such that $|D^*| = |D|$, $D^*(a) = D(a) + 1$, $D^*(x) = D(x) - 1$, and $D^*(v) = D(v)$ for each $v \in V(G) \setminus \{a, x\}$, and hence $ax$ is addible at $a$ with respect to $D$.

In what follows, we assume that for each $(x, P) \in R$, the sequence $b_1 P_1 b_2 P_2 \ldots b_{t-1} b_t$ associated with $(x, P)$ is finished at a vertex $b_t$ (so $D(b_t) = 0$). Let $(w, P)$ and $(z, Q)$ be two distinct pairs in $R$, associated with sequences $w_1 P_1 w_2 P_2 \ldots P_{t-1} w_{t}$ and $z_1 Q_1 z_2 Q_2 \ldots Q_{m-1} z_m$, respectively, where $w_1 = w, P_1 = P, z_1 = z, Q_1 = Q$, and $D(w_1) = D(z_m) = 0$.

We claim that $w_t \neq z_m$. If this is not true, suppose, without loss of generality, that $t \leq m$. Since the path in $D$ containing $w_t a$ (= $z_m a$) is unique, we have that $P_{t-1} = Q_{m-1}$. Now, $w_{t-1}$ is the end of $P_{t-1}$ with $w_t$ between $w_{t-1}$ and $a$; $z_{m-1}$ is the end of $Q_{m-1}$ with $z_m$ (= $w_t$) between $z_{m-1}$ and $a$. Such an end of $P_{t-1}$ (= $Q_{m-1}$) is unique. Thus, $w_{t-1} = z_{m-1}$. Recursively, we have that $P_1 = Q_{m-t+1}$ and $w_1 = z_{m-t+1}$. Since $w_1 = w$ and $w \in \{x_1, x_2, \ldots, x_s\}$, we have that $w_1 a \notin E(H)$, that is, $z_{m-t+1} a \notin E(H)$, which implies that $z_{m-t+1} = z_1$, and thus $m = t$. It follows that $P_1 = Q_1$ and $w_1 = z_1$. This is impossible since $(w_1, P_1)$ and $(z_1, Q_1)$ are two distinct pairs in $R$. Therefore, $w_t \neq z_m$, as claimed. Since this is true for any distinct pairs $(w, P)$ and $(z, Q)$ in $R$, we have an injection from $R$ to $\{x \in N_H(a) : D(x) = 0\}$, and thus,$$
\sum_{i=1}^{s} D(x_i) = |R| \leq |\{x \in N_H(a) : D(x) = 0\}|,$$
which completes the proof. □

Lemma 3.4. Let $G$ be a graph and $ab \in E(G)$. Suppose that $D$ is a path-decomposition of $H = G \setminus \{ab\}$. If $D(b) > |\{v \in N_H(a) : D(v) = 0\}|$, then $ab$ is addible at $a$ with respect to $D$.

Proof. This is an immediate consequence of Lemma 3.3 with $s = 1$. □

Lemma 3.5. Let $a$ be a vertex in a graph $G$ and $H = G \setminus \{ax_1, ax_2, \ldots, ax_s\}$, where $x_i \in N_G(a)$. Suppose that $D$ is a path-decomposition of $H$ with $D(x_i) \geq 1$ for each $i$, $1 \leq i \leq s$. Then there is $A \subseteq \{ax_1, ax_2, \ldots, ax_s\}$ such that
(i) $|A| \geq \lceil \frac{s - r}{2} \rceil$, where $r = |\{v \in N_H(a) : D(v) = 0\}|$; and
(ii) $A$ is addible at $a$ with respect to $D$.

**Proof.** We use induction on $s - r$. If $s - r \leq 0$, then take $A = \emptyset$, and the lemma holds trivially. Suppose therefore that $s - r \geq 1$ and the lemma holds for smaller values of $s - r$.

Since $D(x_i) \geq 1$ for each $i$, $1 \leq i \leq s$, and using $s - r \geq 1$, we have that

$$
\sum_{i=1}^{s} D(x_i) \geq s \geq r + 1 = |\{v \in N_H(a) : D(v) = 0\}| + 1.
$$

By Lemma 3.3, there is $x \in \{x_1, x_2, \ldots, x_s\}$, say $x = x_s$, such that $ax_s$ is addible at $a$ with respect to $D$. Let $D'$ be a transformation of $D$ by adding $ax_s$ at $a$. Let $s' = s - 1$ and $H' = H \cup \{ax_s\} = G \setminus \{ax_1, ax_2, \ldots, ax_{s'}\}$. Then $D'$ is a path-decomposition of $H'$ with $D'(x_i) = D(x_i) \geq 1$ for each $i$, $1 \leq i \leq s'$. Let $r' = |\{v \in N_{H'}(a) : D'(v) = 0\}|$. Clearly, $r' = r + 1$ or $r$, depending on whether $D'(x_s) = 0$ or not. Thus, $s' - r' \leq s - r - 1$. By the induction hypothesis, there is $A' \subseteq \{ax_1, ax_2, \ldots, ax_{s'}\}$ such that

(i) $|A'| \geq \lceil \frac{s' - r'}{2} \rceil \geq \lceil \frac{(s-1)-(r+1)}{2} \rceil = \lceil \frac{s - r}{2} \rceil - 1$; and

(ii) $A'$ is addible at $a$ with respect to $D'$.

Set $A = A' \cup \{ax_s\}$. Then, $A$ is addible at $a$ with respect to $D$, and moreover, $|A| = |A'| + 1 \geq \lceil \frac{s - r}{2} \rceil$. This completes the proof. □

**Lemma 3.6.** Let $a$ be a vertex in a graph $G$ and $H = G \setminus \{ax_1, ax_2, \ldots, ax_h\}$, where $x_i \in N_G(a)$. Suppose that $D$ is a path-decomposition of $H$ with $D(v) \geq 1$ for all $v \in N_H(a)$. Then, for any $x \in \{x_1, x_2, \ldots, x_h\}$, there is $B \subseteq \{ax_1, ax_2, \ldots, ax_h\}$, such that

(i) $ax \in B$ and $|B| \geq \lceil \frac{h}{2} \rceil$.

(ii) $B$ is addible at $a$ with respect to $D$.

**Proof.** Let $W = H \cup \{ax\}$. Then $H = W \setminus \{ax\}$. Since $D(v) \geq 1$ for all $v \in N_H(a) \cup \{x_1, x_2, \ldots, x_h\}$, we have that $D(x) \geq 1$ and $|\{v \in N_H(a) : D(v) = 0\}| = 0$. By Lemma 3.4, $ax$ is addible at $a$ with respect to $D$. Let $D'$ be a transformation of $D$ by adding $ax$ at $a$. Without loss of generality, we may assume that $x = x_h$. Let $s = h - 1$. Then $W = G \setminus \{ax_1, ax_2, \ldots, ax_h\}$. Set $r = |\{v \in N_W(a) : D'(v) = 0\}|$. We have that $r \leq 1$.

By Lemma 3.5, there is $A \subseteq \{ax_1, ax_2, \ldots, ax_s\}$ such that

(i) $|A| \geq \lceil \frac{s - r}{2} \rceil \geq \lceil \frac{(h-1)-1}{2} \rceil = \lceil \frac{h}{2} \rceil - 1$; and

(ii) $A$ is addible at $a$ with respect to $D'$.

Let $B = A \cup \{ax\}$. Then $B$ is addible at $a$ with respect to $D$ and $|B| = |A| + 1 \geq \lceil \frac{h}{2} \rceil$, as required by the lemma. □

**Lemma 3.7.** Let $b$ be a vertex in a graph $G$ and $H = G \setminus \{bx_1, bx_2, \ldots, bx_k\}$, where $x_i \in N_G(b)$. If $H$ has a path-decomposition $D$ such that $|\{v \in N_H(x_i) : D(v) = 0\}| \leq m$ for each $i$, $1 \leq i \leq k$, and $D(b) \geq k + m$, where $m$ is a nonnegative integer, then $G$ has a path-decomposition $D^*$ with $|D^*| = |D|$.

**Proof.** We use induction on $k$. If $k = 0$ ($H = G$), there is nothing to prove. The lemma holds with $D^* = D$. Suppose therefore that $k \geq 1$ and the lemma holds for smaller values
of \( k \). Consider the vertex \( x_k \). By the given condition,
\[
\mathcal{D}(b) \geq k + m \geq m + 1 > |\{v \in N_H(x_k) : \mathcal{D}(v) = 0\}|.
\]
By Lemma 3.4, \( x_kb \) is addible at \( x_k \) with respect to \( \mathcal{D} \). Let \( \mathcal{D}' \) be a transformation of \( \mathcal{D} \) by adding \( x_kb \) at \( x_k \). Let \( H' = H \cup \{bx_k\} = G \setminus \{bx_1, bx_2, \ldots, bx_{k-1}\} \). Noting that \( \mathcal{D}'(x_k) = \mathcal{D}(x_k) + 1 \geq 1 \), we have that for each \( i \), \( 1 \leq i \leq k - 1 \),
\[
|\{v \in N_{H'}(x_i) : \mathcal{D}'(v) = 0\}| \leq |\{v \in N_H(x_i) : \mathcal{D}(v) = 0\}| \leq m,
\]
while \( \mathcal{D}'(b) = \mathcal{D}(b) - 1 \geq (k - 1) + m \). Since \( \mathcal{D}' \) is a path-decomposition of \( H' \), and by the induction hypothesis, \( G \) has a path-decomposition \( \mathcal{D}^* \) with \( |\mathcal{D}^*| = |\mathcal{D}'| \), which gives that \( |\mathcal{D}^*| = |\mathcal{D}| \) since \( |\mathcal{D}'| = |\mathcal{D}| \). This completes the proof. \( \square \)

4. Main theorem

As mentioned in the introduction, Pyber [6] proved that Gallai’s conjecture is true for those graphs whose E-subgraph is a forest. (Recall that the E-subgraph of a graph \( G \) is the subgraph induced by the vertices of even degree in \( G \).) As mentioned before, a forest can be regarded as a graph in which each block has maximum degree at most 1. We shall strengthen Pyber’s result by showing that Gallai’s conjecture is true for those graphs, each block of whose E-subgraph is a triangle-free graph of maximum degree at most 3. We first prove the following lemma.

Lemma 4.1. Let \( F \) be the E-subgraph of a graph \( G \). For \( a \in V(F) \) and \( \{x_1, x_2, \ldots, x_s\} \subseteq N_F(a) \), where \( s \) is odd and \( d_F(x_i) \leq 3 \), \( 2 \leq i \leq s \), if \( G \setminus \{ax_1, ax_2, \ldots, ax_s\} \) has a path decomposition \( \mathcal{D} \) such that \( \mathcal{D}(v) \geq 1 \) for all \( v \in N_G(a) \cup \{a\} \), then \( G \) has a path decomposition \( \mathcal{D}' \) with \( |\mathcal{D}'| = |\mathcal{D}| \).

Proof. By Lemma 3.6, there is \( B \subseteq \{ax_1, ax_2, \ldots, ax_s\} \) such that
(i) \( ax_1 \in B \) and \( |B| \geq \lceil \frac{s}{2} \rceil \).
(ii) \( B \) is addible at \( a \) with respect to \( \mathcal{D} \).

Let \( \mathcal{D}' \) be a transformation of \( \mathcal{D} \) by adding \( B \) at \( a \). We have that
\[
\mathcal{D}'(a) = \mathcal{D}(a) + |B| \geq |B| + 1.
\]
Note that \( s \) is odd. Let \( s = 2k + 1 \), and by relabelling if necessary, we may assume that \( B = \{ax_1, ax_2, \ldots, ax_t\} \), where \( t \geq \lceil \frac{s}{2} \rceil = k + 1 \). Let \( H = G \setminus \{ax_{t+1}, ax_{t+2}, \ldots, ax_s\} \). Then \( \mathcal{D}' \) is a path-decomposition of \( H \) such that
\[
\mathcal{D}'(a) \geq t + 1 \geq k + 2.
\]
Note that \( |\{ax_{t+1}, ax_{t+2}, \ldots, ax_s\}| = s - t \leq k \). Let \( W = F - a \). Since \( d_F(x_i) \leq 3 \), \( 2 \leq i \leq s \), we have that for any \( x \in \{x_{t+1}, x_{t+2}, \ldots, x_s\} \), \( d_W(x) \leq 2 \), and thus \( x \) has at most two neighbors of even degree in \( H \). Therefore,
\[
|\{v \in N_H(x_i) : \mathcal{D}'(v) = 0\}| \leq 2 \text{ for each } i, \ 1 \leq i \leq s.
\]
It follows from Lemma 3.7 with $m = 2$ that $G$ has a path-decomposition $\mathcal{D}^*$ with $|\mathcal{D}^*| = |\mathcal{D}'| = |\mathcal{D}|$. This proves the lemma. □

**Main theorem.** Let $G$ be a graph on $n$ vertices. If the $E$-subgraph of $G$ is an $x$-graph, then $G$ can be decomposed into $\left\lfloor \frac{n}{2} \right\rfloor$ paths.

**Proof.** Use induction on $|E(G)|$. If $|E(G)| = 0$, the theorem holds trivially. Suppose that $|E(G)| \geq 1$ and the theorem holds for all graphs $G'$ with $|E(G')| < |E(G)|$.

Let $F$ be the $E$-subgraph of $G$. If $E(F) = \emptyset$, then it is a special case of Pyber’s result [Theorem 0, 4]. Therefore, we assume that $E(F) \neq \emptyset$. By the given condition, $F$ is an $x$-graph. Let $a_1a_2 \ldots a_m$ be an $x$-ordering of $V(F)$. Since an isolated vertex can be put in any position of an $x$-ordering, we may assume that $a_m$ is not an isolated vertex in $F$, that is, $d_F(a_m) \geq 1$. To simplify notation, let

$$a = a_m, \quad N_F(a) = \{x_1, x_2, \ldots, x_s\}, \quad \text{and} \quad W = F - a,$$

where $s \geq 1$. By definition, $F$ is obtained from $W$ by adding $a$ joined to the independent set $\{x_1, x_2, \ldots, x_s\}$ with the following property: there is $y \in \{x_1, x_2, \ldots, x_s\}$, say $y = x_1$, such that if $d_W(x_i) \geq 2$, then $d_W(u) \leq 3$ for all $u \in N_W(x_i)$ and there are at most two such $u$ with $d_W(u) = 3$, where $2 \leq i \leq s$. We note that since $F$ is the $E$-subgraph of $G$, each of $\{a, x_1, x_2, \ldots, x_s\}$ has even degree in $G$. In what follows, we distinguish three cases.

**Case 1:** $s$ is odd and $d_W(x_i) \leq 2$ for each $i, 2 \leq i \leq s$. (We only need in fact to consider that $d_W(x_i) \leq 1$ here, but for the later use, we consider the more general case that $d_W(x_i) \leq 2$.) Let $H = G \setminus \{ax_1, ax_2, \ldots, ax_s\}$. Then $F - \{a, x_1, x_2, \ldots, x_s\}$ is the $E$-subgraph of $H$, which is an $x$-graph by Proposition 2.3. It follows from the induction hypothesis that $H$ has a path-decomposition $\mathcal{D}$ with $|\mathcal{D}| = \left\lfloor \frac{n}{2} \right\rfloor$. Since $s$ is odd, we have that each of $\{a, x_1, x_2, \ldots, x_s\}$ has odd degree in $H$, and by the definition of $F$, each vertex of $N_H(a) (= N_G(a) \setminus N_F(a))$ also has odd degree in $H$. Thus $\mathcal{D}(v) \geq 1$ for all $v \in N_G(a) \cup \{a\}$. It follows from Lemma 4.1 that $G$ has a path-decomposition $|\mathcal{D}'| = |\mathcal{D}| = \left\lfloor \frac{n}{2} \right\rfloor$, which completes Case 1.

**Case 2:** $s$ is even and $d_W(x_i) \leq 2$ for each $i, 2 \leq i \leq s$. (As before, what we need here is to consider that $d_W(x_i) \leq 1$, but for the later use, we consider that $d_W(x_i) \leq 2$.)

**Case 2.1.** $d_W(x_s) = 0$. Let $H = G \setminus \{x_s, a\}$. Note that $x_s$ and $a$ have odd degree in $H$. Clearly, $F - \{x_s, a\}$ is the $E$-subgraph of $H$, which is an $x$-graph by Proposition 2.3. By the induction hypothesis, $H$ has a path-decomposition $\mathcal{D}$ with $|\mathcal{D}| = \left\lfloor \frac{n}{2} \right\rfloor$. But $d_W(x_s) = 0$, which implies that each neighbor of $x_s$ has odd degree in $H$ and thus $\mathcal{D}(v) \geq 1$ for all $v \in N_H(x_s)$, and using $\mathcal{D}(a) \geq 1$ since $a$ has odd degree in $H$, it follows that

$$\mathcal{D}(a) > |\{v \in N_H(x_s) : \mathcal{D}(v) = 0\}| = 0.$$

By Lemma 3.4, $x_s$ is admissible at $x_s$ with respect to $\mathcal{D}$, which yields a path-decomposition of $G$ with $\left\lfloor \frac{n}{2} \right\rfloor$ paths.

**Case 2.2.** $d_W(x_s) = 1$. Let $y$ be the unique neighbor of $x_s$ in $W$. Set $H = G \setminus \{ax_1, ax_2, \ldots, ax_{s-1}, yx_s\}$. Since $\{x_1, x_2, \ldots, x_s\}$ is an independent set, we have that $y \neq x_i, 1 \leq i \leq s$, and since $s$ is even, it follows that each of $\{a, x_1, x_2, \ldots, x_s, y\}$ has odd degree in $H$. As seen before, the $E$-subgraph of $H$ is an $x$-graph, and by the induction hypothesis, $H$ has a path-decomposition $\mathcal{D}$ with $|\mathcal{D}| = \left\lfloor \frac{n}{2} \right\rfloor$. We note that $|\{v \in N_H(x_s) : \mathcal{D}(v) = 0\}| = 0$ and $\mathcal{D}(y) \geq 1$. By Lemma 3.4, $x_s$ is admissible at $x_s$ with respect to $\mathcal{D}$. Let $\mathcal{D}'$ be a transformation
of \( \mathcal{D} \) by adding \( x_s y \) at \( x_s \), and set \( H' = H \cup \{x_s y\} = G \setminus \{ax_1, ax_2, \ldots, ax_{s-1}\} \). Then \( \mathcal{D}' \)

is a path-decomposition of \( H' \) with \(|\mathcal{D}'| = |\mathcal{D}|\), and in particular, \( \mathcal{D}'(x_s) = \mathcal{D}(x_s) + 1 \geq 2 \). Therefore \( \mathcal{D}'(v) \geq 1 \) for all \( v \in N_G(a) \cup \{a\} \). Clearly, \( s - 1 \) is odd and \( \{x_1, x_2, \ldots, x_{s-1}\} \subseteq N_F(a) \). It follows from Lemma 4.1 that \( G \) has a path-decomposition \( \mathcal{D}^* \) with \(|\mathcal{D}^*| = |\mathcal{D}'| = |\mathcal{D}| = \left\lfloor \frac{n}{2} \right\rfloor \), which proves Case 2. (Remark. The case that \( d_W(x_s) = 2 \) is included in Case 3 below.)

**Case 3:** There is \( x \in \{x_2, \ldots, x_s\} \) such that \( d_W(x) \geq 2 \). Then, \( d_W(u) \leq 3 \) for all \( u \in N_W(x) \) and there are at most two such \( u \) with \( d_W(u) = 3 \). Let \( N_W(x) = \{u_1, u_2, \ldots, u_\ell\} \) and consider the set \( S = N_F(x) = \{a, u_1, u_2, \ldots, u_\ell\} \). Since an \( \alpha \)-graph is triangle-free, we see that \( S \) is an independent set. Let \( Z = F - x \) and \( H = G \setminus \{xv : v \in S\} \). Since \( d_W(u_i) \leq 3 \) for each \( i, 1 \leq i \leq \ell \), we have that

\[
d_Z(u_i) \leq 2 \quad \text{for each } i, \quad 1 \leq i \leq \ell. \tag{4.1}
\]

If \( \ell \) is even, then \( |S| = \ell + 1 \) is odd, and by (4.1), we have Case 1. (\( Z \) and \( x \) play here the same role as \( W \) and \( a \) there.) Suppose therefore that \( \ell \) is odd. Then, since \( \ell = d_W(x) \geq 2 \), we have \( \ell \geq 3 \). But there are at most two \( u_i \) with \( d_W(u_i) = 3 \), by relabelling if necessary, we may assume that \( d_W(u_\ell) \leq 3 \), and so \( d_Z(u_\ell) \leq 1 \). Using the arguments in Case 2 with \( x \) in place of \( a \) and taking (4.1) into account, if \( d_Z(u_\ell) = 0 \), we have Case 2.1; if \( d_Z(u_\ell) = 1 \), we have Case 2.2. This proves Case 3, and so completes the proof of the theorem.

We conclude the paper with the following corollary which is a combination of Proposition 2.6 and the Main theorem.

**Corollary.** Let \( G \) be a graph on \( n \) vertices (not necessarily connected). If each block of the \( E \)-subgraph of \( G \) is a triangle-free graph with maximum degree at most 3, then \( G \) can be decomposed into \( \left\lfloor \frac{n}{2} \right\rfloor \) paths.

**References**