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Path decompositions and Gallai's conjecture

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Abstract

Let G be a connected simple graph on n vertices. Gallai's conjecture asserts that the edges of G can be decomposed into $\lceil \frac{n}{2} \rceil$ paths. Let H be the subgraph induced by the vertices of even degree in G. Lovász showed that the conjecture is true if H contains at most one vertex. Extending Lovász's result, Pyber proved that the conjecture is true if H is a forest. A forest can be regarded as a graph in which each block is an isolated vertex or a single edge (and so each block has maximum degree at most 1). In this paper, we show that the conjecture is true if H can be obtained from the emptyset by a series of so-defined α-operations. As a corollary, the conjecture is true if each block of H is a triangle-free graph of maximum degree at most 3.

Keywords: Path; Decomposition; Gallai's conjecture

1. Introduction

The graphs considered here are finite, undirected, and simple (no loops or multiple edges). A graph is *triangle-free* if it contains no triangle. A *cut vertex* is a vertex whose removal increases the number of components. A connected graph is *nonseparable* if it has no cut vertex. A *block* of a graph G is a maximum nonseparable subgraph of G. The sets of vertices and edges of G are denoted by $V(G)$ and $E(G)$, respectively. The edge with ends x and y is denoted by xy. If $xy \in E(G)$, we say that xy is *incident* with x and y is a *neighbor* of x. For a subgraph H of G, $N_H(x)$ is the set of the neighbors of x which are in H, and $d_H(x) = |N_H(x)|$ is the *degree* of x in H. If $B \subseteq E(G)$, then $G \setminus B$ is the graph obtained from G by deleting all the edges of B. Let $S \subseteq V(G)$. $G - S$ denotes the graph obtained from G by deleting all the vertices of S together with all the edges with at least one end

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in S. (When $S = \{x\}$, we simplify this notation to $G - x$.) We say that H is the subgraph *induced* by S if $V(H) = S$ and $xy \in E(H)$ if and only if $xy \in E(G)$; alternatively, $H = G - (V(G) \setminus S)$. ($S$ is called an *independent set* if $E(H) = \emptyset$.) The *E-subgraph* of G is the subgraph induced by the vertices of even degree in G.

A *path-decomposition* of a graph G is a set $\{P_1, P_2, \dots, P_k\}$ of paths such that $E(G) = \cup_{i=1}^{k} E(P_i)$ and $E(P_i) \cap E(P_j) = \emptyset$ if $i \neq j$. We say that G is decomposed into k paths if G has a path-decomposition $\mathcal{D}$ with $|\mathcal{D}| = k$. A *trivial path* is one that consists of a single vertex. By the use of trivial paths, if a graph is decomposed into at most k paths, then it can be decomposed into exactly k paths.

Erdös asked what is the minimum number of paths into which every connected graph on n vertices can be decomposed. Gallai conjectured that this number is $\lceil \frac{n}{2} \rceil$. (See [4].)

Gallai's conjecture. If G is a connected graph on n vertices, then G can be decomposed into $\lceil \frac{n}{2} \rceil$ paths.

Toward a proof of the conjecture, Lovász [4] made the first significant contribution by showing that a graph G on n vertices (not necessary to be connected) can be decomposed into $\lfloor \frac{n}{2} \rfloor$ paths and circuits. Based on Lovász's result, Donald [2] showed that G can be decomposed into $\lfloor \frac{3}{4}n \rfloor$ paths, which was improved to $\lfloor \frac{2}{3}n \rfloor$ independently by Dean and Kouider [1] and Yan [7]. (An informative survey of the related topics was given by Pyber [5].) As a consequence of Lovász's theorem, G can be decomposed into $\lfloor \frac{n}{2} \rfloor$ paths if G has at most one vertex of even degree, that is, if the E-subgraph of G contains at most one vertex. Pyber [6] strengthened this result by showing that G can be decomposed into $\lfloor \frac{n}{2} \rfloor$ paths if the E-subgraph of G is a forest. A forest can be regarded as a graph in which each block is an isolated vertex or a single edge. Thus, each block of a forest has maximum degree at most 1. In this paper, we show that a graph G on n vertices (not necessary to be connected) can be decomposed into $\lfloor \frac{n}{2} \rfloor$ paths if each block of the E-subgraph of G is a triangle-free graph of maximum degree at most 3. Here, the requirement of triangle-free cannot be dropped. Consider a graph G consisting of $3k$ vertex-disjoint triangles. So $|V(G)| = 3k$ and the E-subgraph of G is G itself. Since any path-decomposition of a triangle needs at least 2 paths, we see that any path-decomposition of G needs at least $2k = \frac{2}{3}|V(G)|$ paths.

In the next section, we define a graph operation, called α-operation. In Section 3, we use Lovász's path sequence technique [4] to obtain some technical lemmas, and then, in the last section, prove a more general result: G can be decomposed into $\lfloor \frac{n}{2} \rfloor$ paths if its E-subgraph can be obtained from the emptyset by a series of α-operations.

2. α-operations and α-graphs

Definition 2.1. Let H be a graph. A pair (S, y), consisting of an independent set S and a vertex $y \in S$, is called an *α-pair* if the following holds: for every vertex $v \in S \setminus \{y\}$, if $d_H(v) \geqslant 2$, then (a) $d_H(u) \leqslant 3$ for all $u \in N_H(v)$ and (b) $d_H(u) = 3$ for at most two vertices $u \in N_H(v)$. (That is, all the neighbors of v has degree at most 3, at most two of which has degree exactly 3.) An *α-operation* on H is either (i) add an isolated vertex or (ii) pick an α-pair (S, y) and add a vertex x joined to each vertex of S, in which case the ordered triple (x, S, y) is called the *α-triple* of the α-operation.

Definition 2.2. An α-*graph* is a graph that can be obtained from the empty set via a sequence of α-operations.

Let us define the empty set to be an α-graph. Then, a graph on n vertices is an α-graph if and only if it can be obtained by an α-operation on some α-graph on $n-1$ vertices, $n \geqslant 1$. It follows that if G is an α-graph on n vertices, then the vertices of G can be ordered as $x_1 x_2 \ldots x_n$ such that if G_i denotes the subgraph induced by $\{x_1, x_2, \ldots, x_i\}$, then G_i is an α-graph obtained by an α-operation on G_{i-1}, where $1 \leqslant i \leqslant n$, $G_0 = \emptyset$, and $G_n = G$. Such an ordering $x_1 x_2 \ldots x_n$ is called an α-*ordering* of $V(G)$. Alternatively, a graph G is an α-graph if and only if $V(G)$ has an α-ordering. We note that by the definition, an α-graph is triangle-free.

Let G be an α-graph and H a subgraph of G. It is not difficult to see that the restriction of an α-ordering of $V(G)$ to $V(H)$ is an α-ordering of $V(H)$. This gives that

Proposition 2.3. *Any subgraph of an α-graph is an α-graph.*

A *subdivision* of a graph G is a graph obtained from G by replacing each edge of G with a path (inserting vertices into edges of G).

Proposition 2.4. *Any subdivision of an α-graph is an α-graph.*

Proof. It suffices to show that if H is a graph obtained from an α-graph G by replacing an edge with a path, then H is an α-graph. Suppose that $xy \in E(G)$ and H is obtained from G by replacing xy with a path $x a_1 a_2 \ldots a_k y$, where $k \geqslant 1$. We may suppose that $v_1 v_2 \ldots x v_i \ldots v_j y \ldots v_n$ is an α-ordering of $V(G)$. Then, $v_1 v_2 \ldots x v_i \ldots v_j a_1 a_2 \ldots a_k y \ldots v_n$ is an α-ordering of $V(H)$, and thus H is an α-graph. $\square$

Proposition 2.5. *Forests are α-graphs.*

Proof. Let F be a forest. If $E(F) = \emptyset$, then any ordering of $V(F)$ is an α-ordering. Suppose therefore that $E(F) \neq \emptyset$. Since F is a forest, there is $x \in V(F)$ such that $d_F(x) = 1$. Let $H = F - x$. Then H is a forest. We may use induction on the number of vertices, and thus by the induction hypothesis, H is an α-graph. Let y be the unique neighbor of x in F. Then, F is obtained from H by adding x joined to y, which is an α-operation with α-triple $(x, \{y\}, y)$. So F is an α-graph. $\square$

Let C be a circuit of length at least 4. Then C can be obtained by adding a vertex joined to the nonadjacent ends of a path P of length at least 2, which is an α-operation on P. But, by Proposition 2.5, P is an α-graph, and hence C is an α-graph. In fact, we have the following stronger result.

Proposition 2.6. *If each block of G is a triangle-free graph of maximum degree at most* 3, *then G is an α-graph.*

Proof. We use induction on $|V(G)|$. Clearly, the proposition holds if $|V(G)| = 1$. Suppose that $|V(G)| \geqslant 2$ and the proposition holds for all G' with $|V(G')| < |V(G)|$.

Let B be an end-block of G. (An *end-block* is a block that contains at most one cut vertex.) If $B = G$ (that is, if G is 2-connected), let b be any vertex of B; otherwise, let b be the unique cut vertex contained in B. Let x be a neighbor of b in B and we consider the neighbors of x. Note that $N_B(x) = N_G(x)$. Let $S = N_G(x)$ and $H = G - x$. Since B is triangle-free, we have that S is an independent set and thus b is not a neighbor of any vertex $v \in S \setminus \{b\}$, and since B has maximum degree at most 3, $d_H(u) \leqslant 3$ for all $u \in N_H(v)$. Again, since B has maximum degree at most 3, we have that $|N_H(v)| \leqslant 2$ and thus there are at most two $u \in N_H(v)$ with $d_H(u) = 3$. So G is obtained by an α-operation on H with α-triple (x, S, b). But, by the induction hypothesis, H is an α-graph, and so is G. $\square$

3. Technical lemmas

In this section, we use Lovász's path sequence technique [4] to prove some technical lemmas which are needed in the next section. First, we need some additional definitions.

Definition 3.1. Suppose that $\mathcal{D}$ is a path-decomposition of a graph G. For a vertex $v \in V(G)$, $\mathcal{D}(v)$ denotes the number of the nontrivial paths in $\mathcal{D}$ that have v as an end. (If x is a vertex of odd degree in G, then $\mathcal{D}(x) \geqslant 1$. This fact will be used frequently in the next section.)

Definition 3.2. Let a be a vertex in a graph G and let B be a set of edges incident with a. Set $H = G \setminus B$. Suppose that $\mathcal{D}$ is a path-decomposition of H. For any $A \subseteq B$, say that $A = \{ax_i : 1 \leqslant i \leqslant k\}$, we say that A is *addible* at a with respect to $\mathcal{D}$ if $H \cup A$ has a path-decomposition $\mathcal{D}^*$ such that

(a) $|\mathcal{D}^*| = |\mathcal{D}|$;
(b) $\mathcal{D}^*(a) = \mathcal{D}(a) + |A|$ and $\mathcal{D}^*(x_i) = \mathcal{D}(x_i) - 1$, $1 \leqslant i \leqslant k$;
(c) $\mathcal{D}^*(v) = \mathcal{D}(v)$ for each $v \in V(G) \setminus \{a, x_1, \ldots, x_k\}$.

We call such $\mathcal{D}^*$ a *transformation* of $\mathcal{D}$ by adding A at a. When $k = 1$, we simply say that ax_1 is addible at a with respect to $\mathcal{D}$.

Lemmas 3.3 and 3.5 below are special cases of Lemmas 4.3 and 4.6 in [3], respectively, whose proofs are rather complicated. (A path decomposition is a special case of a path covering.) To be self-contained, we present proofs without referring to [3].

Lemma 3.3. *Let a be a vertex in a graph G and let $H = G \setminus \{ax_1, ax_2 \ldots, ax_s\}$, where $x_i \in N_G(a)$. Suppose that $\mathcal{D}$ is a path-decomposition of H. Then either*

(i) *there is $x \in \{x_1, x_2, \ldots, x_s\}$ such that ax is addible at a with respect to $\mathcal{D}$; or*
(ii) $\sum_{i=1}^{s} \mathcal{D}(x_i) \leqslant |\{v \in N_H(a) : \mathcal{D}(v) = 0\}|$.

Proof. Consider the following set of pairs:

$$R = \{(x, P) : x \in \{x_1, \ldots, x_s\} \text{ and } P \text{ is a nontrivial path in } \mathcal{D} \text{ with end } x\}.$$

We note that $|R| = \sum_{i=1}^{s} \mathcal{D}(x_i)$. For each pair $(x, P) \in R$, we associate (x, P) with a sequence $b_1 P_1 b_2 P_2 \ldots$ constructed as follows.

(1) $b_1 = x$; $P_1 = P$.
(2) Suppose that P_i has been defined, $i \geqslant 1$. If P_i does not contain a, then the sequence is finished at P_i; otherwise let b_{i+1} be the vertex just before a if one goes along P_i starting at b_i.
(3) Suppose that b_i has been defined, $i \geqslant 1$. If $\mathcal{D}(b_i) = 0$, the sequence is finished at b_i; otherwise, let P_i be a path in $\mathcal{D}$ starting at b_i.

It is clear that b_{i+1} is uniquely determined by the path P_i (containing $b_{i+1}a$) and its end b_i. Such a pair (P_i, b_i) is unique since there is only one path in $\mathcal{D}$ that contains $b_{i+1}a$, and moreover, the two ends of the path are distinct. Thus, $b_i \neq b_j$ if $i \neq j$, and therefore, the sequence $b_1 P_1 b_2 P_2 \ldots$ is finite.

If the sequence is finished at a path P_t ((2) above), let $P'_i = (P_i \setminus \{b_{i+1}a\}) \cup \{b_i a\}$, $1 \leqslant i \leqslant t-1$, and $P'_t = P_t \cup \{b_t a\}$. Then $\mathcal{D}^* = (\mathcal{D} \setminus \{P_1, P_2, \ldots, P_t\}) \cup \{P'_1, P'_2, \ldots, P'_t\}$ is a path-decomposition of $H \cup \{ax\}$ such that $|\mathcal{D}^*| = |\mathcal{D}|$, $\mathcal{D}^*(a) = \mathcal{D}(a) + 1$, $\mathcal{D}^*(x) = \mathcal{D}(x) - 1$, and $\mathcal{D}^*(v) = \mathcal{D}(v)$ for each $v \in V(G) \setminus \{a, x\}$, and hence ax is addible at a with respect to $\mathcal{D}$.

In what follows, we assume that for each $(x, P) \in R$, the sequence $b_1 P_1 b_2 P_2 \ldots P_{t-1} b_t$ associated with (x, P) is finished at a vertex b_t (so $\mathcal{D}(b_t) = 0$). Let (w, P) and (z, Q) be two distinct pairs in R, associated with sequences $w_1 P_1 w_2 P_2 \ldots P_{t-1} w_t$ and $z_1 Q_1 z_2 Q_2 \ldots Q_{m-1} z_m$, respectively, where $w_1 = w$, $P_1 = P$, $z_1 = z$, $Q_1 = Q$, and $\mathcal{D}(w_t) = \mathcal{D}(z_m) = 0$.

We claim that $w_t \neq z_m$. If this is not true, suppose, without loss of generality, that $t \leqslant m$. Since the path in $\mathcal{D}$ containing $w_t a$ $(= z_m a)$ is unique, we have that $P_{t-1} = Q_{m-1}$. Now, w_{t-1} is the end of P_{t-1} with w_t between w_{t-1} and a; z_{m-1} is the end of Q_{m-1} with z_m $(= w_t)$ between z_{m-1} and a. Such an end of P_{t-1} $(= Q_{m-1})$ is unique. Thus, $w_{t-1} = z_{m-1}$. Recursively, we have that $P_1 = Q_{m-t+1}$ and $w_1 = z_{m-t+1}$. Since $w_1 = w$ and $w \in \{x_1, x_2, \ldots, x_s\}$, we have that $w_1 a \notin E(H)$, that is, $z_{m-t+1} a \notin E(H)$, which implies that $z_{m-t+1} = z_1$, and thus $m = t$. It follows that $P_1 = Q_1$ and $w_1 = z_1$. This is impossible since (w_1, P_1) and (z_1, Q_1) are two distinct pairs in R. Therefore, $w_t \neq z_m$, as claimed. Since this is true for any distinct pairs (w, P) and (z, Q) in R, we have an injection from R to $\{x \in N_H(a) : \ \mathcal{D}(x) = 0\}$, and thus,

$$\sum_{i=1}^{s} \mathcal{D}(x_i) = |R| \leqslant |\{x \in N_H(a) : \mathcal{D}(x) = 0\}|,$$

which completes the proof. □

Lemma 3.4. *Let G be a graph and $ab \in E(G)$. Suppose that $\mathcal{D}$ is a path-decomposition of $H = G \setminus \{ab\}$. If $\mathcal{D}(b) > |\{v \in N_H(a) : \ \mathcal{D}(v) = 0\}|$, then ab is addible at a with respect $\mathcal{D}$.*

Proof. This is an immediate consequence of Lemma 3.3 with $s = 1$. □

Lemma 3.5. *Let a be a vertex in a graph G and $H = G \setminus \{ax_1, ax_2, \ldots, ax_s\}$, where $x_i \in N_G(a)$. Suppose that $\mathcal{D}$ is a path-decomposition of H with $\mathcal{D}(x_i) \geqslant 1$ for each i, $1 \leqslant i \leqslant s$. Then there is $A \subseteq \{ax_1, ax_2, \ldots, ax_s\}$ such that*

(i) $|A| \geqslant \lceil \frac{s-r}{2} \rceil$, *where* $r = |\{v \in N_H(a) : \mathcal{D}(v) = 0\}|$; *and*
(ii) *A is addible at a with respect $\mathcal{D}$.*

Proof. We use induction on $s - r$. If $s - r \leqslant 0$, then take $A = \emptyset$, and the lemma holds trivially. Suppose therefore that $s - r \geqslant 1$ and the lemma holds for smaller values of $s - r$.

Since $\mathcal{D}(x_i) \geqslant 1$ for each i, $1 \leqslant i \leqslant s$, and using $s - r \geqslant 1$, we have that

$$\sum_{i=1}^{s} \mathcal{D}(x_i) \geqslant s \geqslant r + 1 = |\{v \in N_H(a) : \mathcal{D}(v) = 0\}| + 1.$$

By Lemma 3.3, there is $x \in \{x_1, x_2, \ldots, x_s\}$, say $x = x_s$, such that ax_s is addible at a with respect $\mathcal{D}$. Let $\mathcal{D}'$ be a transformation of $\mathcal{D}$ by adding ax_s at a. Let $s' = s - 1$ and $H' = H \cup \{ax_s\} = G \setminus \{ax_1, ax_2, \ldots, ax_{s'}\}$. Then $\mathcal{D}'$ is a path-decomposition of H' with $\mathcal{D}'(x_i) = \mathcal{D}(x_i) \geqslant 1$ for each i, $1 \leqslant i \leqslant s'$. Let $r' = |\{v \in N_{H'}(a) : \mathcal{D}'(v) = 0\}$. Clearly, $r' = r + 1$ or r, depending on whether $\mathcal{D}'(x_s) = 0$ or not. Thus, $s' - r' \leqslant s - r - 1$. By the induction hypothesis, there is $A' \subseteq \{ax_1, ax_2, \ldots, ax_{s'}\}$ such that
(i) $|A'| \geqslant \lceil \frac{s'-r'}{2} \rceil \geqslant \lceil \frac{(s-1)-(r+1)}{2} \rceil = \lceil \frac{s-r}{2} \rceil - 1$; and.
(ii) A' is addible at a with respect to $\mathcal{D}'$.
Set $A = A' \cup \{ax_s\}$. Then, A is addible at a with respect to $\mathcal{D}$, and moreover, $|A| = |A'| + 1 \geqslant \lceil \frac{s-r}{2} \rceil$. This completes the proof. $\square$

Lemma 3.6. *Let a be a vertex in a graph G and $H = G \setminus \{ax_1, ax_2, \ldots, ax_h\}$, where $x_i \in N_G(a)$. Suppose that $\mathcal{D}$ is a path-decomposition of H with $\mathcal{D}(v) \geqslant 1$ for all $v \in N_G(a)$. Then, for any $x \in \{x_1, x_2, \ldots, x_h\}$, there is $B \subseteq \{ax_1, ax_2, \ldots, ax_h\}$, such that*
(i) *$ax \in B$ and $|B| \geqslant \lceil \frac{h}{2} \rceil$.*
(ii) *B is addible at a with respect to $\mathcal{D}$.*

Proof. Let $W = H \cup \{ax\}$. Then $H = W \setminus \{ax\}$. Since $\mathcal{D}(v) \geqslant 1$ for all $v \in N_H(a) \cup \{x_1, x_2, \ldots, x_h\}$, we have that $\mathcal{D}(x) \geqslant 1$ and $|\{v \in N_H(a) : \mathcal{D}(v) = 0\}| = 0$. By Lemma 3.4, ax is addible at a with respect to $\mathcal{D}$. Let $\mathcal{D}'$ be a transformation of $\mathcal{D}$ by adding ax at a. Without loss of generality, we may assume that $x = x_h$. Let $s = h - 1$. Then $W = G \setminus \{ax_1, ax_2, \ldots, ax_s\}$. Set $r = |\{v \in N_W(a) : \mathcal{D}'(v) = 0\}|$. We have that $r \leqslant 1$. By Lemma 3.5, there is $A \subseteq \{ax_1, ax_2, \ldots, ax_s\}$ such that
(i) $|A| \geqslant \lceil \frac{s-r}{2} \rceil \geqslant \lceil \frac{(h-1)-1}{2} \rceil = \lceil \frac{h}{2} \rceil - 1$; and
(ii) A is addible at a with respect to $\mathcal{D}'$.
Let $B = A \cup \{ax\}$. Then B is addible at a with respect to $\mathcal{D}$ and $|B| = |A| + 1 \geqslant \lceil \frac{h}{2} \rceil$, as required by the lemma. $\square$

Lemma 3.7. *Let b be a vertex in a graph G and $H = G \setminus \{bx_1, bx_2, \ldots, bx_k\}$, where $x_i \in N_G(b)$. If H has a path-decomposition $\mathcal{D}$ such that $|\{v \in N_H(x_i) : \mathcal{D}(v) = 0\}| \leqslant m$ for each i, $1 \leqslant i \leqslant k$, and $\mathcal{D}(b) \geqslant k + m$, where m is a nonnegative integer, then G has a path-decomposition $\mathcal{D}^*$ with $|\mathcal{D}^*| = |\mathcal{D}|$.*

Proof. We use induction on k. If $k = 0$ $(H = G)$, there is nothing to prove. The lemma holds with $\mathcal{D}^* = \mathcal{D}$. Suppose therefore that $k \geqslant 1$ and the lemma holds for smaller values of k. Consider the vertex x_k. By the given condition,

$$\mathcal{D}(b) \geqslant k + m \geqslant m + 1 > |\{v \in N_H(x_k) : \mathcal{D}(v) = 0\}|.$$

By Lemma 3.4, $x_k b$ is addible at x_k with respect to $\mathcal{D}$. Let $\mathcal{D}'$ be a transformation of $\mathcal{D}$ by adding $x_k b$ at x_k. Let $H' = H \cup \{bx_k\} = G \setminus \{bx_1, bx_2, \ldots, bx_{k-1}\}$. Noting that $\mathcal{D}'(x_k) = \mathcal{D}(x_k) + 1 \geqslant 1$, we have that for each i, $1 \leqslant i \leqslant k - 1$,

$$|\{v \in N_{H'}(x_i) : \mathcal{D}'(v) = 0\}| \leqslant |\{v \in N_H(x_i) : \mathcal{D}(v) = 0\}| \leqslant m,$$

while $\mathcal{D}'(b) = \mathcal{D}(b) - 1 \geqslant (k - 1) + m$. Since $\mathcal{D}'$ is a path-decomposition of H', and by the induction hypothesis, G has a path-decomposition $\mathcal{D}^*$ with $|\mathcal{D}^*| = |\mathcal{D}'|$, which gives that $|\mathcal{D}^*| = |\mathcal{D}|$ since $|\mathcal{D}'| = |\mathcal{D}|$. This completes the proof. $\square$

4. Main theorem

As mentioned in the introduction, Pyber [6] proved that Gallai's conjecture is true for those graphs whose E-subgraph is a forest. (Recall that the E-subgraph of a graph G is the subgraph induced by the vertices of even degree in G.) As mentioned before, a forest can be regarded as a graph in which each block has maximum degree at most 1. We shall strengthen Pyber's result by showing that Gallai's conjecture is true for those graphs, each block of whose E-subgraph is a triangle-free graph of maximum degree at most 3. We first prove the following lemma.

Lemma 4.1. *Let F be the E-subgraph of a graph G. For $a \in V(F)$ and $\{x_1, x_2, \ldots, x_s\} \subseteq N_F(a)$, where s is odd and $d_F(x_i) \leqslant 3$, $2 \leqslant i \leqslant s$, if $G \setminus \{ax_1, ax_2, \ldots, ax_s\}$ has a path decomposition $\mathcal{D}$ such that $\mathcal{D}(v) \geqslant 1$ for all $v \in N_G(a) \cup \{a\}$, then G has a path decomposition $\mathcal{D}^*$ with $|\mathcal{D}^*| = |\mathcal{D}|$.*

Proof. By Lemma 3.6, there is $B \subseteq \{ax_1, ax_2, \ldots, ax_s\}$ such that
(i) $ax_1 \in B$ and $|B| \geqslant \lceil \frac{s}{2} \rceil$.
(ii) B is addible at a with respect to $\mathcal{D}$.
Let $\mathcal{D}'$ be a transformation of $\mathcal{D}$ by adding B at a. We have that

$$\mathcal{D}'(a) = \mathcal{D}(a) + |B| \geqslant |B| + 1.$$

Note that s is odd. Let $s = 2k + 1$, and by relabelling if necessary, we may assume that $B = \{ax_1, ax_2, \ldots, ax_t\}$, where $t \geqslant \lceil \frac{s}{2} \rceil = k + 1$. Let $H = G \setminus \{ax_{t+1}, ax_{t+2}, \ldots, ax_s\}$. Then $\mathcal{D}'$ is a path-decomposition of H such that

$$\mathcal{D}'(a) \geqslant t + 1 \geqslant k + 2.$$

Note that $|\{ax_{t+1}, ax_{t+2}, \ldots, ax_s\}| = s - t \leqslant k$. Let $W = F - a$. Since $d_F(x_i) \leqslant 3, 2 \leqslant i \leqslant s$, we have that for any $x \in \{x_{t+1}, x_{t+2}, \ldots, x_s\}$, $d_W(x) \leqslant 2$, and thus x has at most two neighbors of even degree in H. Therefore,

$$|\{v \in N_H(x_i) : \mathcal{D}'(v) = 0\}| \leqslant 2 \ \text{ for each } i, \quad t + 1 \leqslant i \leqslant s.$$

It follows from Lemma 3.7 with $m = 2$ that G has a path-decomposition $\mathcal{D}^*$ with $|\mathcal{D}^*| = |\mathcal{D}'| = |\mathcal{D}|$. This proves the lemma. $\square$

Main theorem. *Let G be a graph on n vertices* (*not necessarily connected*). *If the E-subgraph of G is an* α*-graph, then G can be decomposed into* $\lfloor \frac{n}{2} \rfloor$ paths.

Proof. Use induction on $|E(G)|$. If $|E(G)| = 0$, the theorem holds trivially. Suppose that $|E(G)| \geqslant 1$ and the theorem holds for all graphs G' with $|E(G')| < |E(G)|$.

Let F be the E-subgraph of G. If $E(F) = \emptyset$, then it is a special case of Pyber's result [Theorem 0, 4]. Therefore, we assume that $E(F) \neq \emptyset$. By the given condition, F is an α-graph. Let $a_1 a_2 \dots a_m$ be an α-ordering of $V(F)$. Since an isolated vertex can be put in any position of an α-ordering, we may assume that a_m is not an isolated vertex in F, that is, $d_F(a_m) \geqslant 1$. To simplify notation, let

$$a = a_m, \quad N_F(a) = \{x_1, x_2, \dots, x_s\}, \quad \text{and} \quad W = F - a,$$

where $s \geqslant 1$. By definition, F is obtained from W by adding a joined to the independent set $\{x_1, x_2, \dots, x_s\}$ with the following property: there is $y \in \{x_1, x_2, \dots, x_s\}$, say $y = x_1$, such that if $d_W(x_i) \geqslant 2$, then $d_W(u) \leqslant 3$ for all $u \in N_W(x_i)$ and there are at most two such u with $d_W(u) = 3$, where $2 \leqslant i \leqslant s$. We note that since F is the E-subgraph of G, each of $\{a, x_1, x_2, \dots, x_s\}$ has even degree in G. In what follows, we distinguish three cases.

Case 1: s is odd and $d_W(x_i) \leqslant 2$ for each i, $2 \leqslant i \leqslant s$. (We only need in fact to consider that $d_W(x_i) \leqslant 1$ here, but for the later use, we consider the more general case that $d_W(x_i) \leqslant 2$.) Let $H = G \setminus \{ax_1, ax_2, \dots, ax_s\}$. Then $F - \{a, x_1, x_2, \dots, x_s\}$ is the E-subgraph of H, which is an α-graph by Proposition 2.3. It follows from the induction hypothesis that H has a path-decomposition $\mathcal{D}$ with $|\mathcal{D}| = \lfloor \frac{n}{2} \rfloor$. Since s is odd, we have that each of $\{a, x_1, x_2, \dots, x_s\}$ has odd degree in H, and by the definition of F, each vertex of $N_H(a)$ ($= N_G(a) \setminus N_F(a)$) also has odd degree in H. Thus $\mathcal{D}(v) \geqslant 1$ for all $v \in N_G(a) \cup \{a\}$. It follows from Lemma 4.1 that G has a path-decomposition $|\mathcal{D}'| = |\mathcal{D}| = \lfloor \frac{n}{2} \rfloor$, which completes Case 1.

Case 2: s is even and $d_W(x_i) \leqslant 2$ for each i, $2 \leqslant i \leqslant s$. (As before, what we need here is to consider that $d_W(x_i) \leqslant 1$, but for the later use, we consider that $d_W(x_i) \leqslant 2$.)

Case 2.1. $d_W(x_s) = 0$. Let $H = G \setminus \{x_s a\}$. Note that x_s and a have odd degree in H. Clearly, $F - \{x_s, a\}$ is the E-subgraph of H, which is an α-graph by Proposition 2.3. By the induction hypothesis, H has a path-decomposition $\mathcal{D}$ with $|\mathcal{D}| = \lfloor \frac{n}{2} \rfloor$. But $d_W(x_s) = 0$, which implies that each neighbor of x_s has odd degree in H and thus $\mathcal{D}(v) \geqslant 1$ for all $v \in N_H(x_s)$, and using $\mathcal{D}(a) \geqslant 1$ since a has odd degree in H, it follows that

$$\mathcal{D}(a) > |\{v \in N_H(x_s) : \mathcal{D}(v) = 0\}| = 0.$$

By Lemma 3.4, $x_s a$ is addible at x_s with respect to $\mathcal{D}$, which yields a path-decomposition of G with $\lfloor \frac{n}{2} \rfloor$ paths.

Case 2.2. $d_W(x_s) = 1$. Let y be the unique neighbor of x_s in W. Set $H = G \setminus \{ax_1, ax_2, \dots, ax_{s-1}, yx_s\}$. Since $\{x_1, x_2, \dots, x_s\}$ is an independent set, we have that $y \neq x_i$, $1 \leqslant i \leqslant s$, and since s is even, it follows that each of $\{a, x_1, x_2, \dots, x_s, y\}$ has odd degree in H. As seen before, the E-subgraph of H is an α-graph, and by the induction hypothesis, H has a path-decomposition $\mathcal{D}$ with $|\mathcal{D}| = \lfloor \frac{n}{2} \rfloor$. We note that $|\{v \in N_H(x_s) : \mathcal{D}(v) = 0\}| = 0$ and $\mathcal{D}(y) \geqslant 1$. By Lemma 3.4, $x_s y$ is addible at x_s with respect to $\mathcal{D}$. Let $\mathcal{D}'$ be a transformation of $\mathcal{D}$ by adding $x_s y$ at x_s, and set $H' = H \cup \{x_s y\} = G \setminus \{ax_1, ax_2, \ldots, ax_{s-1}\}$. Then $\mathcal{D}'$ is a path-decomposition of H' with $|\mathcal{D}'| = |\mathcal{D}|$, and in particular, $\mathcal{D}'(x_s) = \mathcal{D}(x_s) + 1 \geqslant 2$. Therefore $\mathcal{D}'(v) \geqslant 1$ for all $v \in N_G(a) \cup \{a\}$. Clearly, $s-1$ is odd and $\{x_1, x_2, \ldots, x_{s-1}\} \subseteq N_F(a)$. It follows from Lemma 4.1 that G has a path-decomposition $\mathcal{D}^*$ with $|\mathcal{D}^*| = |\mathcal{D}'| = |\mathcal{D}| = \lfloor \frac{n}{2} \rfloor$, which proves Case 2. (Remark. The case that $d_W(x_s) = 2$ is included in Case 3 below.)

Case 3: There is $x \in \{x_2, \ldots, x_s\}$ such that $d_W(x) \geqslant 2$. Then, $d_W(u) \leqslant 3$ for all $u \in N_W(x)$ and there are at most two such u with $d_W(u) = 3$. Let $N_W(x) = \{u_1, u_2, \ldots, u_\ell\}$ and consider the set $S = N_F(x) = \{a, u_1, u_2, \ldots, u_\ell\}$. Since an α-graph is triangle-free, we see that S is an independent set. Let $Z = F - x$ and $H = G \setminus \{xv : v \in S\}$. Since $d_W(u_i) \leqslant 3$ for each i, $1 \leqslant i \leqslant \ell$, we have that

$$d_Z(u_i) \leqslant 2 \quad \text{for each } i, \quad 1 \leqslant i \leqslant \ell. \tag{4.1}$$

If ℓ is even, then $|S| = \ell + 1$ is odd, and by (4.1), we have Case 1. (Z and x paly here the same role as W and a there.) Suppose therefore that ℓ is odd. Then, since $\ell = d_W(x) \geqslant 2$, we have $\ell \geqslant 3$. But there are at most two u_i with $d_W(u_i) = 3$, by relabelling if necessary, we may assume that $d_W(u_\ell) \leqslant 2$, and so $d_Z(u_\ell) \leqslant 1$. Using the arguments in Case 2 with x in place of a and taking (4.1) into account, if $d_Z(u_\ell) = 0$, we have Case 2.1; if $d_Z(u_\ell) = 1$, we have Case 2.2. This proves Case 3, and so completes the proof of the theorem. □

We conclude the paper with the following corollary which is a combination of Proposition 2.6 and the Main theorem.

Corollary. *Let G be a graph on n vertices* (*not necessarily connected*). *If each block of the E-subgraph of G is a triangle-free graph with maximum degree at most* 3, *then G can be decomposed into* $\lfloor \frac{n}{2} \rfloor$ *paths*.

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