Perfectness of the complements of circular complete graphs

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Abstract

For $p \geq 2q$, let $K_{p/q}$ be the graph with vertices $0, 1, 2, \ldots, p-1$ in which $i \sim j$ if $q \leq |i - j| \leq p - q$. The circular chromatic number $\chi_c(G)$ of a graph G is the minimum of those p/q for which G admits a homomorphism to $K_{p/q}$. The circular clique number $\omega_c(G)$ of G is the maximum of those p/q for which $K_{p/q}$ admits a homomorphism to G. A graph G is circular perfect if for every induced subgraph H of G we have $\chi_c(H) = \omega_c(H)$. In this paper, we characterize those rational numbers p/q for which $\overline{K_{p/q}}$ are circular perfect. We also prove that if $G(n, S)$ is a circulant graph whose generating set S has cardinality at most 3, then $G(n, S)$ is circular perfect.

Contents

Chapter 1

Introduction

All graphs considered in this paper are finite and simple, i.e., finite graphs without multiple edges and loops.

1.1 Circular chromatic number

Let $G = (V, E)$ be a graph, where V and E denote the set of vertices and edges of G, respectively. Two vertices u and v are *adjacent* if uv is an edge of G.

Suppose that $r \geq 2$ is a real number. An r-circular coloring of a graph G is a mapping $\psi : V(G) \mapsto [0, r)$ such that $1 \leq |\psi(u) - \psi(v)| \leq r - 1$ whenever $uv \in E(G)$. A graph G is called r-circular colorable if it admits an r-circular coloring. The *circular chromatic number* of G, denote by $\chi_c(G)$, is the least r such that G is r-circular colorable.

It was proved in [8] that $\chi_c(G)$ is always attained and is always equal to a rational number and $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$ for any graph G. This means that $\chi(G) = \lceil \chi_c(G) \rceil$.

Figure 1.1 below is an example graph, the 5-cycle C_5, together with a 5/2-coloring. So $\chi_c(C_5) \leq 5/2$. It is known that indeed we have $\chi_c(C_5) = 5/2$.

1.2 Circular clique number

Given a rational number p/q, the *circular complete graphs* $K_{p/q}$ has vertex set

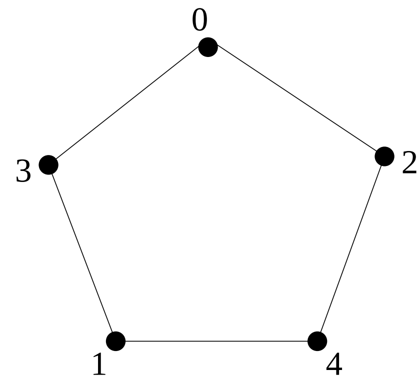

Figure 1.1: A $(5, 2)$-coloring of C_5.

$$V(K_{p/q}) = \{v_0, v_1, v_2, \ldots, v_{p-1}\},$$

and edge set

$$E(K_{p/q}) = \{v_i v_j | q \leq |j - i| \leq p - q\}.$$

Given two graphs G and H, a *homomorphism* from G to H is a mapping $f : V(G) \to V(H)$ such that for any edge xy of G, $f(x)f(y)$ is an edge of H. It is well-known and easy to see that a graph is k-colorable if and only if G admits a homomorphism to K_k. It is also known [14] that for any rational number p/q, a graph G is p/q-colorable if and only if G admits a homomorphism to $K_{p/q}$. So in the study of circular chromatic number, the circular complete graphs play the same role as the complete graphs play in the study of chromatic number.

The *clique number* $\omega(G)$ of a graph is defined to be the largest integer k for which K_k admits a homomorphism to G (or equivalently, $\omega(G)$ is the largest integer k for which K_k is a subgraph of G). As an analogue, the *circular clique number* $\omega_c(G)$ of G is defined to be the largest fraction p/q for which $K_{p/q}$ admits a homomorphism to G.

For example, the graph C_5 is isomorphic to $K_{5/2}$. Hence the circular clique number of C_5 is $5/2$.

1.3 Circular perfect graphs

A subgraph H of G is an *induced subgraph* if $E(H) = \{uv | u \in V(H), v \in V(H), uv \in E(G)\}$. A graph G is called a *perfect graph* if every induced subgraph H of G has its chromatic number $\chi(H)$ equals its clique number

$\omega(H)$. The class of perfect graphs is an very important class of graphs and has been studied extensively in the literature. Recently, the strong perfect graph conjecture has been verified in [2], which says that a graph is perfect if and only if G has no induced subgraph which is an odd cycle of length at least 5, or the complement of an odd cycle of length at least 5.

As an analogue of perfect graph, the concept of circular perfect graphs was introduced in [17, 13]. A graph is *circular perfect* if for every induced subgraph H of G we have $\chi_c(H) = \omega_c(H)$.

It follows from the definition that for any graph G,

$$\omega(G) \leq \omega_c(G) \leq \chi_c(G) \leq \chi(G).$$

Thus every perfect graph is a circular perfect graph. On the other hand, the example graph C_5 has $\chi_c(C_5) = \omega_c(C_5) = 5/2$, and for any proper induced subgraph H of C_5, we have $\chi_c(H) = \omega_c(H)$. Therefore C_5 is circular perfect. So the class of circular perfect graphs is strictly larger than the class of perfect graphs.

1.4 Previous results

Circular perfect graphs have been studied in a few papers [1, 9, 10, 11, 14, 17, 16, 15, 13]. It was proved in [13] that circular complete graphs are circular perfect.

Theorem 1.4.1. *For any fraction $p/q \geq 2$, the circular complete graph $K_{p/q}$ is circular perfect.*

A necessary condition for a graph to be circular perfect was proved in [13].

Theorem 1.4.2. *If G is a circular perfect graph, then for any vertex x, $N_G(x)$ induces a perfect subgraph of G.*

Two sufficient conditions for graphs to be circular perfect were proved in [13, 15] respectively.

Theorem 1.4.3. *Suppose G is a graph such that for each vertex x of G, $N_G(x)$ induces a perfect graph, and $G - N(x)$ induces a bipartite graph which does not contain an induced path on 5 vertices. Then G is circular perfect.*

Given an induced path $P_n = (p_0, p_1, \cdots, p_n)$ of $G - N[x]$, we say P_n is *badly-linked* with respect to x if one of the following holds:

1. There are three indices $i < j < k$ of the same parity such that $N(p_i) \cap N(x) \not\subseteq N(p_j)$ and $N(p_k) \cap N(x) \not\subseteq N(p_j)$.

2. There are three indices $i < j < k$ of the same parity such that $N(p_j) \cap N(x) \not\subseteq N(p_i)$ and $N(p_j) \cap N(x) \not\subseteq N(p_k)$.

3. There are two even indices $i < j$ and two odd indices $i' < j'$ such that $N(p_i) \cap N(x) \not\subseteq N(p_j)$ and $N(p_{i'}) \cap N(x) \not\subseteq N(p_{j'})$.

Theorem 1.4.4. *Suppose G is a triangle free graph such that for every vertex x of G, $G - N[x]$ is a bipartite graph with no induced C_n for $n \geq 6$, and there is no induced path of $G - N[x]$ which is badly-linked. Then G is circular perfect.*

A circulant graph $G[n, S]$ is called a convex circulant graph if for each vertex i of $G[n, S]$, $N(i)$ is a set of consecutive elements of $V(G)$ (here $V(G)$ is viewed to be cyclically ordered). The following result was proved in [1]:

Theorem 1.4.5. *Every convex circulant graph is circular perfect.*

Some special graphs are proved to be circular perfect, or circular imperfect in [9, 10, 11].

1.5 Results of this thesis

By the Weak Perfect Graph Theorem, a graph G is perfect if and only if its complement is perfect. Unlike the class of perfect graphs, the class of circular perfect graphs is not closed under the operation of taking the complement. This thesis first characterizes the circular complete graphs whose complements are also circular perfect.

Theorem 1.5.1. *Suppose $p/q \geq 2$. Then the complement $\overline{K_{p/q}}$ of $K_{p/q}$ is circular perfect if and only if $q = 2$ or $p = 2q$ or $p = 2q + 1$ or $q = 3$ and $p = 3k$ for some integer k.*

Then we study the circular perfectness of general circulant graphs. We prove that if circulant graphs of degree at most 3 are circular perfect.

Theorem 1.5.2. *If $G = G(n, S)$ is a circulant graph with $|S| \leq 3$, then G is circular perfect.*

Chapter 2

Complements of circular complete graphs

2.1 Circular chromatic number of the complements of circular complete graphs

Suppose p is a positive integer. Then for integer a, let $a \pmod p$ denote the unique integer a' such that $p|(a-a')$ and $0 \le a' \le p-1$.

Lemma 2.1.1. *For any rational $p/q \ge 2$, $\chi_c(\overline{K_{p/q}}) = p/\lfloor p/q \rfloor$.*

Proof. Let $d = \lfloor p/q \rfloor$. Let $f(i) = id \pmod p$ for $i = 0, 1, \cdots, p-1$. If ij is an edge of $\overline{K_{p/q}}$, then either $i-j \pmod p \le q-1$ or $j-i \pmod p \le q-1$. Assume $i-j \pmod p \le q-1$. Then $f(i)-f(j) \pmod p = (i-j)d \pmod p \ge d \pmod p$, $f(i)-f(j) \pmod p = (i-j)d \pmod p \le (q-1)d \pmod p \le p-d$. The last inequality follows from the fact that $qd \le p$. Therefore f is a (p,d)-coloring of $\overline{K_{p/q}}$. So $\chi_c(\overline{K_{p/q}}) \le p/d$. On the other hand, if X is a subset of $\{0, 1, \cdots, p-1\}$ with $|X| = d+1$, then there are two integers $i < j \in X$ such that either $j-i < q$, or $j-i > p-q$. So X is not an independent set of $\overline{K_{p/q}}$. Therefore $\alpha(\overline{K_{p/q}}) \le d$. This implies that $\chi_c(\overline{K_{p/q}}) \ge p/d$. ☐

Corollary 2.1.1. *Assume $p = kq+r$, where $0 \le r \le q-1$. Then $\chi_c(\overline{K_{p/q}}) = p/k = q + r/k$.*

Corollary 2.1.2. *Assume $p = kq+r$, where $0 \le r \le q-1$. If $k < r$, then $\chi(\overline{K_{p/q}}) \ge q+2$.*

2.2 Circular clique number of the complements of circular complete graphs

The following lemma was proved in [13].

Lemma 2.2.1. *If $\omega_c(G) = p/q$ and $(p, q) = 1$, then G contains $K_{p/q}$ as an induced subgraph.*

Lemma 2.2.2. *Assume $p = kq + r$, where $0 \leq r \leq q - 1$. Then $\omega_c(\overline{K_{p/q}}) = (tq + r')/t$, where $(t, r') = 1$ and $0 \leq r' \leq 1$. Moreover, if $r' = 1$, then either $t = 2$ or $q = 2$, and $\overline{K_{p/q}}$ is isomorphic to $K_{(tq+r')/t}$.*

Proof. Since $\omega(\overline{K_{p/q}}) = q$, it follows that $q \leq \omega_c(\overline{K_{p/q}}) < q+1$. So $\omega_c(\overline{K_{p/q}}) = (tq + r')/t$ for some t, r' such that $0 \leq r' \leq t - 1$ and $(r', t) = 1$. First we show that $r' \leq 1$. Assume to the contrary that $r' \geq 2$. By Lemma 2.2.1, $\overline{K_{p/q}}$ contains a subgraph isomorphic to $K_{(tq+r')/t}$. In $K_{(tq+r')/t}$, for $j = 0, 1, \cdots, r'$,the $X_j = \{0, t, 2t, \cdots, (q - 2)t, (q - 1)t + j\}$ induces a clique of cardinality q. Observe that $\cap_{j=0}^{r'} X_j = \{0, t, 2t, \cdots, (q - 2)t\}$ is a clique of cardinality $q - 1$. As $r' \geq 2$, so $K_{(tq+r')/t}$, which is a subgraph of $\overline{K_{p/q}}$, contains at least 3 cliques of cardinality q whose intersection is a clique of cardinality $q-1$. However, the cliques of $\overline{K_{p/q}}$ of cardinality q are of the form $\{i, i + 1, i + 2, \cdots, i + q - 1\}$. The intersection of any three of such cliques has cardinality at most $q - 2$. This is a contradiction. Therefore, we must have $r' \leq 1$.

Assume $r' = 1$. Then $t \geq 2$ and $\overline{K_{p/q}}$ contains $K_{(tq+1)/t}$ as a subgraph. In $K_{(tq+1)/t}$, each vertex has degree $tq + 1 - 2t + 1$. In $\overline{K_{p/q}}$, each vertex has degree $2q-2$. As $K_{(tq+1)/t}$ is a subgraph of $\overline{K_{p/q}}$, we have $2q-2 \geq tq-2t+2$. This implies that $t = 2$ or $q = 2$. Moreover, in both cases, $2q = tq - 2t + 2$, which implies that $\overline{K_{p/q}}$ is isomorphic to $K_{(tq+1)/q}$. □

2.3 Circular complete graphs which are circular perfect

Lemma 2.3.1. $\overline{K_{(2k+1)/2}} = C_{2k+1} \cong K_{(2k+1)/k}$.

Proof. The vertex set of $\overline{K_{(2k+1)/2}}$ is $\{0, 1, 2, \ldots, 2k\}$. The mapping $f(i) = ik \pmod{2k + 1}$ is an isomorphism from C_{2k+1} to $K_{(2k+1)/k}$. □

Lemma 2.3.2. *If $q = 2$, then $\overline{K_{p/q}}$ is circular perfect.*

Proof. If $p = 2k+1$, then $\overline{K_{(2k+1)/2}} \cong K_{(2k+1)/k}$ is circular perfect. If $p = 2k$, then $\overline{K_{2k/2}}$ is bipartite. Hence $\overline{K_{(2k+1)/2}}$ is perfect, which implies that $\overline{K_{(2k+1)/2}}$ is circular perfect. □

Lemma 2.3.3. *If $p = 2q+1$, then $\overline{K_{p/q}}$ is circular perfect.*

Proof. If $p = 2q+1$, then $\overline{K_{p/q}} \cong K_{(2q+1)/2}$. Hence $\overline{K_{p/q}}$ is circular perfect. □

Lemma 2.3.4. *If $q = 3$ and $p = 3k$, then $\overline{K_{p/q}}$ is circular perfect.*

Proof. In this case, by Lemma 2.1.1, $\chi_c(\overline{K_{p/q}}) = \chi(\overline{K_{p/q}}) = 3$. Let H be an induced subgraph of $\overline{K_{p/q}}$. If a vertex $i \in H$ has degree at least 3 in H, then either $i-1, i-2 \in V(H)$ or $i+1, i+2 \in V(H)$ (note that in $\overline{K_{p/q}}$, $N(i) = \{i-2, i-1, i+1, i+2\}$). In any case H contains a K_3, and hence $\chi_c(H) = \chi(H) = \omega(H) = \omega_c(H) = 3$. Otherwise each vertex of H has degree at most 2, and hence each component of H is either a path or a cycle. In this case we also have $\chi_c(H) = \omega_c(H)$. □

Lemma 2.3.5. *Assume $q = 3$, $p = 3k+r$, $k \geq 3$ and $r = 1$ or 2. Then $\overline{K_{p/q}}$ is not circular perfect.*

Proof. Assume $q = 3$, $p = 3k+r$, $k \geq 3$ and $r = 1$ or 2. By Lemma 2.1.1, $\chi_c(\overline{K_{p/q}}) = p/k = 3 + r/k$. By Lemma 2.2.2, $\omega(\overline{K_{p/q}}) = 3$. Therefore $\overline{K_{p/q}}$ is not circular perfect. □

The following result was proved in [6].

Lemma 2.3.6. *For any positive integers n, k, n', k', $K_{n'/k'}$ is an induced subgraph of $K_{n/k}$ if and only if $(k-1)/(k'-1) \geq n/n' \geq k/k'$.*

Lemma 2.3.7. *If $q \geq 4$, and $p \geq 2q+2$, then $\overline{K_{p/q}}$ is not circular perfect.*

Proof. Assume $p = kq+r$. If $1 \leq r \leq q-1$, then by Lemma 2.1.1, $\chi_c(\overline{K_{p/q}}) = p/k > q$, and by Lemma 2.2.2, $\omega_c(\overline{K_{p/q}}) = q$. Hence $\overline{K_{p/q}}$ is not circular perfect. Assume $r = 0$, i.e., $p = kq$, where $k \geq 3$. We shall use Lemma 2.3.6 to find a circular complete graph $K_{n/3}$ which is an induced subgraph of $K_{kq/q}$ such that $n \equiv 2 \pmod 3$. This would imply that $\overline{K_{n/3}}$ is not circular perfect. As $\overline{K_{n/3}}$ is an induced subgraph of $\overline{K_{kq/q}}$, it follows that $\overline{K_{kq/q}}$ is not circular perfect.

By Lemma 2.3.6, $K_{n/3}$ which is an induced subgraph of $K_{kq/q}$ if and only if $(q-1)/2 \geq kq/n \geq q/3$. This is equivalent to $2kq/(q-1) \leq n \leq 3k$. If $2kq/(q-1) = 2k + 2k/(q-1) \leq 3k-2$, then we would have 3 consecutive integers as the choices for n, and one of these choices has $n \equiv 2 \pmod 3$. Now $2k + 2k/(q-1) \leq 3k-2$ is equivalent to $k-2 \geq 2k/(q-1)$. By our assumption, $q \geq 4$ and $k \geq 3$. Easy calculation shows that

- If $k \geq 6$, then $k-2 \geq 2k/(q-1)$.
- If $k \geq 4$ and $q \geq 5$, then $k-2 \geq 2k/(q-1)$.
- If $k = 3$ and $q \geq 7$, then $k-2 \geq 2k/(q-1)$.

It remains to consider the cases that $k = 4, 5$ and $q = 4$, $k = 3$ and $q = 4, 5, 6$. In these cases, straightforward calculation shows that either $K_{8/3}$ or $K_{11/3}$ or $K_{14/3}$ is an induced subgraph of $K_{kq/q}$. □

Lemma 2.3.8. *If $p = 2q$, then $\overline{K_{p/q}}$ is circular perfect.*

Proof. If $p = 2q$, then $K_{p/q}$ is bipartitle , therefore $K_{p/q}$ is perfect, and $\overline{K_{p/q}}$ is perfect. We get $\overline{K_{p/q}}$ is circular perfect. □

Combining the lemmas above, we obtain the following theorem.

Theorem 2.3.1. *The graph $\overline{K_{p/q}}$ is circular perfect if and only if $p = 2q+1$ or $p = 2q$ or $q = 2$ or $q = 3$ and $p = 3k$.*

Chapter 3

Circulant graphs

3.1 Circulant graphs

Let n be a positive integer and S be a subset of $\{1, 2, \ldots, n-1\}$ such that $i \in S$ implies $n - i \in S$, where subtraction is carried out mod n. The circulant graph $G = G(n, S)$ has vertices $0, 1, 2, \ldots, n-1$ and $i \sim j$ if and only if $i - j \in S$, where subtraction is carried out mod n.

3.2 Circulant graphs with $\mid S \mid= 3$ is circular perfect

Lemma 3.2.1. *If $|S| = 3$ and $S = \{1, n-1, n/2\}$, $4 \mid n$ then $G(n, s) \cong K_{n/(n/2-1)}$.*

Proof. suppose the vertex set of $G(n, s)$ is $\{0, 1, 2, \ldots, n-1\}$, in $G(n, s)$, vertex i has edge to vertex $i+1$, vertex $i-1$, vertex $i + n/2$ mod n, $0 \leq i \leq n-1$. Suppose the vertex set of $K_{n/(n/2-1)}$ is $\{0, 1, 2, \ldots, n-1\}$, and $f(i) = (n/2-1)i$ mod n, therefore $f(i)$ is a vertex of $K_{n/(n/2-1)}$, $0 \leq i \leq n-1$. We only need to prove vertex $f(i) \in K_{n/(n/2-1)}$ has edge to vertex $f(i+1)$, vertex $f(i-1)$, vertex $f(i+n/2)$. Because $|f(i) - f(i+1)| = n/2 - 1$ mod n and $|f(i) - f(i-1)| = n/2 - 1$ mod n, $|f(i) - f(i+n/2)| = |(n/2-1)i - (n/2-1)(i+n/2)| = |-n^2/4+n/2| = n/2$ mod n, (because $4 \mid n$). Therefore vertex $f(i)$ in $K_{n/(n/2-1)}$ has edge to vertex $f(i+1)$, vertex $f(i-1)$, vertex $f(i+n/2)$. $\square$

Lemma 3.2.2. *If $|S| = 3$ and $S = \{1, n-1, n/2\}$, 4 does not divide n then $G(n, s)$ is bipartite.*

Lemma 3.2.3. *If* $|S| = 3$ *and* $S = \{p, n-p, n/2\}$, $S' = \{1, n-1, n/2\}$, *if* $(n, p) = 1$ *then* $G(n, S) \cong G(n, S')$.

Proof. Suppose the vertex set of $G(n, S')$ is $\{0, 1, 2, \ldots, n-1\}$, in $G(n, S')$, vertex i has edge to vertex $i+1$, vertex $i-1$, vertex $i + n/2 \bmod n$, $0 \leq i \leq n-1$. Suppose the vertex set of $G(n, s)$ is $\{0, 1, 2, \ldots n-1\}$, and $f(i) = pi \bmod n$, therefore $f(i)$ is a vertex of $G(n, s)$, $0 \leq i \leq n-1$. We only need to prove vertex $f(i)$ in $G(n, s)$ has edge to vertex $f(i+1)$, vertex $f(i-1)$, vertex $f(i+n/2)$. Because $|f(i) - f(i+1)| = p \bmod n$ and $|f(i) - f(i-1)| = p \bmod n$, $|f(i) - f(i+n/2)| = |n/2 \times p| = n/2 \bmod n$, (because $(p, n) = 1$). Therefore vertex $f(i)$ in $G(n, s)$ has edge to vertex $f(i+1)$, vertex $f(i-1)$, vertex $f(i+n/2)$. Therefore $G(n, s) \cong G(n, S')$. □

Lemma 3.2.4. *If* $|S| = 3$ *and* $S = \{p, n-p, n/2\}$, *if* $(n, p) = d$, n/d *is even,* $\omega_c(G) = \chi_c(G)$

Proof. In this case, G is the disjoint union of d copies of the circulant graph $H = G(n/d, \{1, n/d-1, n/(2d)\})$, therefore $\chi_c(G) = \chi_c(H) = \omega_c(H) = \omega_c(G)$. □

Lemma 3.2.5. *If* $|S| = 3$ *and* $S = \{p, n-p, n/2\}$, *if* $(n, p) = d$, n/d *is odd,* $\omega_c(G) = \chi_c(G)$.

Proof. In this case, G is the disjoint union of d copies of the graph H, H is a graph like pier, the vertex set of H is $\{0, 1, 2, 3, \ldots, 2n/d-1\}$, where $\{0, 1, 2, 3, \ldots, n/d-1\}$ is a cycle C_1, and $\{n/d, n/d+1, n/d+2, \ldots, 2n/d-1\}$ is a cycle C_2, and every vertex i has edge to vertex $i + n/d, 0 \leq i \leq n/d-1$. Because C_1 is an odd cycle and C_1 is a subgraph of H, suppose $n/d = 2p+1$, then $\omega_c(H) \geq (2p+1)/p$. $\chi_c(C_1) = (2p+1)/p$, if we use color a_i color vertex i, $0 \leq i \leq n/d$-1, and we use color $a_{i+1} (a_{n/d} = a_0)$ to color vertex $i + n/d, 0 \leq i \leq n/d$-1, then we get $\chi_c(H) = \chi_c(C_1) = (2p+1)/p$. Therefore, we get $\chi_c(G) = \chi_c(H) = (2p+1)/p = \omega_c(H) = \omega_c(G)$. □

Theorem 3.2.1. *Circulant graphs with* $\mid S \mid = 3$ *is circular perfect.*

Proof. In theorem 1.4.1, we know the graph $K_{d/k}$ is circular perfect, and bipartite graph is circular perfect $\Rightarrow$ the graph in Lemma 3.2.1, Lemma 3.2.2, Lemma 3.2.3 is circular perfect. We only need to prove the graph in Lemma 3.2.4, Lemma 3.2.5 is circular perfect. In Lemma 3.2.4, G is the disjoint union of d copies of the circulant graph H, therefore if graph H is circular perfect $\Rightarrow$ graph G is circular perfect. Because $H = G(n/d, \{1, n/d-1, n/(2d)\})$, by Lemma 3.2.1, Lemma 3.2.2, H is circular perfect $\Rightarrow$ graph G is circular perfect. In Lemma 3.2.5, G is the disjoint union of d copies of the graph H, therefore if graph H is circular perfect $\Rightarrow$ graph G is circular perfect. The vertex of H is $\{0, 1, 2, 3, \ldots, 2n/d - 1\}$, where $\{0, 1, 2, 3, \ldots, n/d - 1\}$ is a cycle C_1, and $\{n/d, n/d + 1, n/d + 2, \ldots, 2n/d - 1\}$ is a cycle C_2, and every vertex i has edge to vertex $i + n/d, 0 \leq i \leq n/d - 1$. Case1: The induced subgraph G' of H does not have odd cycle $\Rightarrow$ G' is bipartite $\Rightarrow$ $\chi_c(G') = \omega_c(G')$. Case2: The induced subgraph G' of H have odd cycle $\Rightarrow$ suppose $n/d = 2p + 1, \chi_c(G') = (2p + 1)/p = \omega_c(G')$. By caes1 and case2 we get H is circular perfect $\Rightarrow$ graph G is circular perfect, we prove Theorem 3.2.1. $\square$

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